

Stochastic processes - Exam

23rd october 2025

Duration : 3h

The use of any documents (lecture notes,...), mobile phones, calculators, ... , is FORBIDDEN.

Recommendations :

Read the text carefully and write your answers as *succinctly* and as *clearly* as possible.

Many questions are independent $\Rightarrow$ DON'T GET STUCK! Check the **appendices at the end.**

Do not forget to **reread yourself.**

1 Two questions from the list... ($\sim 15\text{mn} + 15\text{mn} = 30\text{mn}$)

Q11 Dissipation of the energy in the Langevin equation.— We consider the Langevin equation for the velocity of a particle in a fluid

$$m \frac{d}{dt} v(t) = -\gamma v(t) + \xi(t) \quad (1)$$

where $\xi(t)$ is a *Gaussian* white noise such that $\langle \xi(t) \rangle = 0$ and $\langle \xi(t) \xi(t') \rangle = 2\gamma k_B T \delta(t - t')$.

a) Solve (1). Deduce the correlator between the velocity and the Langevin force $\langle v(t) \xi(t') \rangle$.

b) Consider now the kinetic energy $E_{\text{kin}}(t) = \frac{1}{2} m v(t)^2$. Using (1), deduce $\frac{dE_{\text{kin}}}{dt}$ and interpret the two terms.

c) Deduce a differential equation for $\langle E_{\text{kin}}(t) \rangle$.

Hint : you should need $\langle v(t) \xi(t) \rangle$ at coinciding times, which requires $\theta_H(0) = 1/2$ (Heaviside function at the origin).

d) Solve the differential equation for $\langle E_{\text{kin}}(t) \rangle$ and discuss the result.

Q18 Compound Poisson process.— We consider a Poisson jump process $X(t)$, who's distribution $P_t(x)$ obeys the master equation

$$\frac{\partial P_t(x)}{\partial t} = \int dy W(x|y) P_t(y) \quad (2)$$

for a translation invariant kernel $W(x|y) = \lambda w(x - y) - \lambda \delta(x - y)$; here λ is the rate of jumps and $w(\eta) \geq 0$ the (normalised) distribution of the jump amplitudes.

a) Discuss conservation of probability.

b) Introduce the Fourier transforms $\hat{P}_t(k) = \int dx e^{-ikx} P_t(x)$ and $\hat{w}(k) = \int d\eta e^{-ik\eta} w(\eta)$. Deduce a differential equation for $\hat{P}_t(k)$ and express the solution $P_t(x)$ for initial condition $P_0(x) = \delta(x)$ under the form of an integral.

c) Assuming that we can replace $\hat{w}(k)$ by its $k \rightarrow 0$ expansion $\hat{w}(k) \simeq 1 - \frac{1}{2} c k^2$ in the integral representation of $P_t(x)$ obtained previously, compute explicitly $P_t(x)$. What is the meaning of the parameter c ? How would you qualify the process in this limit ?

d) Same question when $\hat{w}(k) \simeq 1 - c|k|$ for $k \rightarrow 0$.

2 Wealth distribution in a mean field model of economy (~55mn)

We consider a simple model of economy with N agents, each having the wealth $W_i(t)$ at time t . The rate of growth has a fixed part r , a fluctuating part $\eta_i(t)$ (noise) and depends also on exchanges between agents, with positive couplings K_{ij} :

$$\frac{dW_i(t)}{dt} = [r + \eta_i(t)] W_i(t) + \sum_{j(\neq i)} (K_{ij}W_j(t) - K_{ji}W_i(t)) \quad \text{for } i = 1, \dots, N. \quad (3)$$

The noises are independent Gaussian white noises such that $\langle \eta_i(t)\eta_j(t') \rangle = 2D\delta_{ij}\delta(t-t')$. These SDE are understood in the **Stratonovich** sense, which will be more convenient below.

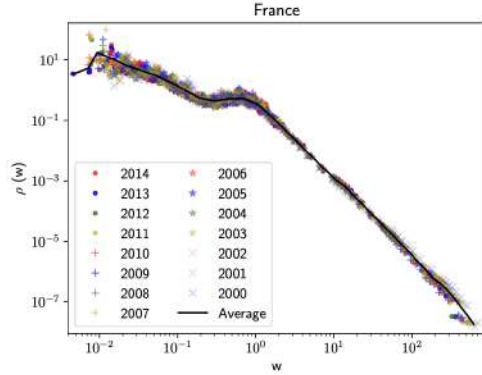


Figure 1: *Normalised histogram of the French relative wealths ($w = W/\bar{W}$) ; tail $\rho(w) \propto w^{-\theta}$ is characterized by the exponent $\theta = 2.68$. Note that the wealth can be negative (debts) : this part of the distribution is not shown here. From I. Gere et al., Physica A **581**, 126194 (2021).*

1/ Consider one single agent (set $N = 1$ or $K_{ij} = 0$). Deduce $\langle W_i(t) \rangle$ (integrate the Stratonovich SDE or use Itô calculus).

2/ **Mean field model.**— We assume that all agents interact in the same way, $K_{ij} = K/N$.

a) Rewrite the Stratonovich SDE (3) in terms of the wealth averaged over agents

$$\bar{W}(t) \stackrel{\text{def}}{=} \frac{1}{N} \sum_{i=1}^N W_i(t). \quad (4)$$

b) Deduce the Itô SDE for $\bar{W}(t)$ (for that, it is convenient to write $\eta_i(t) = \sqrt{2D} dB_i(t)/dt$, where $B_i(t)$ is a normalised Wiener process). Argue that the noise of the Itô SDE for $\bar{W}(t)$ has the form $(d\bar{W})_{\text{noise}} = \frac{(\dots)}{N} \sum_i W_i dB_i$. Compute $\langle (d\bar{W})_{\text{noise}} \rangle$ and $\langle (d\bar{W})_{\text{noise}}^2 \rangle$.

c) Justify that the noise for $\bar{W}(t)$ becomes negligible in the large N limit, i.e. that

$$\frac{d\bar{W}(t)}{dt} = (r + D) \bar{W}(t) \quad \text{for } N \rightarrow \infty. \quad (5)$$

3/ Inequalities between agents can be studied by considering the *relative* wealth $w_i \stackrel{\text{def}}{=} W_i/\bar{W}$. Using (5), show that it obeys

$$\frac{dw_i(t)}{dt} = K + [\eta_i(t) - K - D] w_i(t) \quad (\text{Stratonovich}). \quad (6)$$

4/ Show that the forward generator for the distribution of w_i can be written under the form

$$\mathcal{G}^\dagger = \partial_w \sqrt{\mathcal{D}(w)} [\partial_w + U'(w)] \sqrt{\mathcal{D}(w)} \quad (7)$$

where $\mathcal{D}(w) = D w^2$ and $U(w) = \mu \ln w + \frac{w_0}{w}$. Express μ and w_0 in terms of K and D .

5/ Find the stationary solution $P^*(w)$ of the FPE $\partial_t P_t(w) = \mathcal{G}^\dagger P_t(w)$. Is it an equilibrium or a NESS ? Discuss the large w behaviour. Show that small w are cut off on a scale w_c ; express w_c in terms of the parameters of the model. Comment (how D and K control the $w \rightarrow 0$ and $w \rightarrow \infty$ behaviours of the distribution).

6/ The distribution of the relative wealths in France is plotted in Figure 1. Comment.

3 Quantum non destructive (QND) measurements and quantum trajectories ($\sim 1\text{h}25\text{mn}$)

Introduction.— In quantum mechanics, measurement has usually a dramatic effect on the state of the system due to the wave function collapse. It is however possible to minimize the interaction of the measurement apparatus with the system, which allows to repeat the measurement with high frequency and leads to the concept of "quantum trajectory". A famous example of such protocol was realized in the group of S. Haroche (Nobel prize 2012), M. Brune & J.-M. Raimond : the system is the electromagnetic (EM) field trapped in a resonant cavity made of two superconducting mirrors with high quality factor. A beam of Rydberg atoms crosses the cavity, and become entangled with the EM field before being measured out of the cavity, allowing to perform a "non destructive" measurement (the photons are detected without being absorbed). This is repeated with high frequency (in 2007, Haroche and collaborators have recorded up to 3000 measurements over the lifetime of the few photons trapped in the cavity !).

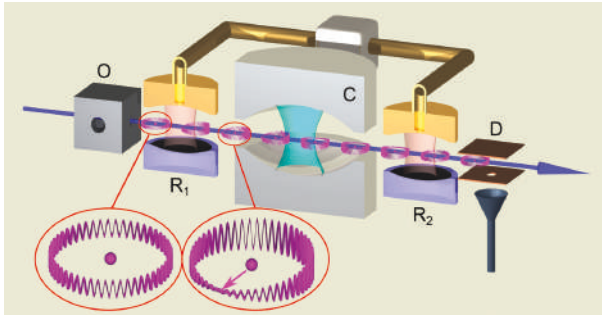


FIGURE 10. The Cavity QED Ramsey interferometer set-up. The insets show sketches of the circular atom in an energy eigenstate (left) and in a superposition state after interaction with the microwave pulse in R_1 (right). (Reprinted with permission from [49], © American Physical Society).

Figure 2: *Electromagnetic field is trapped in a cavity. A beam of Rydberg atoms is sent through the cavity (C), and measured after the cavity (D). Figure from Haroche's Nobel lecture.*

In order to simplify the description, we will assume that the cavity contains a **two level system** (for example if only two EM states are occupied, $|\text{vacuum}\rangle \equiv |g\rangle$ and $|1\text{photon}\rangle \equiv |e\rangle$). The state of the cavity at time t can then be described by a 2×2 density matrix

$$\hat{\rho} = \begin{pmatrix} 1 - X & c \\ c^* & X \end{pmatrix} \quad \text{where } X = \langle e | \hat{\rho} | e \rangle \in [0, 1] \quad (8)$$

is the probability to occupy state $|e\rangle$ while $1 - X$ is the probability to occupy $|g\rangle$. The off-diagonal coefficient $c = \langle g | \hat{\rho} | e \rangle \in \mathbb{C}$ is the "coherence" (which distinguishes between a pure state, $\hat{\rho} = |\psi\rangle\langle\psi|$ and a statistical mixture $\hat{\rho} = (1 - X)|g\rangle\langle g| + X|e\rangle\langle e|$). The detector measures the state of the probe (the Rydberg atom), which gives $\eta_t = \pm 1$ at time t . Measurements are repeated every δt ; the results being random, the "signal" $Y(t) = \sqrt{\delta t} \sum_{n=1}^t \eta_n$ is a random process; in the continuum limit, $\delta t \rightarrow 0$, it obeys the SDE

$$dY(t) = \sqrt{\gamma} \left(X(t) - \frac{1}{2} \right) dt + dW(t) \quad (9)$$

where γ is the rate of the weak measurements. The recorded signal depends on the quantum state (the drift of $Y(t)$ is related to the state $\hat{\rho}$ of the system through X) and some noise $dW(t)$, where $W(t)$ is a Wiener process, which accounts for the stochastic nature of the measurement in quantum mechanics. The feedback of the weak measurements on the system is

$$dX(t) = \sqrt{\gamma} X(t) (1 - X(t)) dW(t) \quad (\text{It}\hat{o}) \quad (10)$$

$$dc(t) = -\frac{\gamma}{8} c(t) dt - \sqrt{\gamma} \left(X(t) - \frac{1}{2} \right) c(t) dW(t) \quad (\text{It}\hat{o}) \quad (11)$$

(these equations describe the evolution of the quantum state under the action of the repeated measurements : the quantum probability $X(t)$ is a stochastic process ; note that the standard Hamiltonian time evolution is neglected here).

- 1/ *Decoherence.* — Give a differential equation for the averaged coherence $\langle c(t) \rangle$ and solve it.
- 2/ We introduce $\theta(t) \stackrel{\text{def}}{=} \sqrt{X(t)(1-X(t))}$. Using Itô calculus, give the Itô SDE for $\theta(t)$. Deduce a differential equation for $\langle \theta(t) \rangle$ and solve it. What can you conclude for the evolution of the probability $X(t)$ of occupation of the state $|e\rangle$?
- 3/ The distribution $P_x(t)$ of $X(t)$ deserves a detailed study. Write the related FPE

$$\partial_t P_t(x) = \partial_x^2 [D(x)P_t(x)] \quad \text{for } x \in [0, 1] \quad (12)$$

explicitely (give $D(x)$). Give the corresponding probability current $J_t(x)$.

- 4/ Solve $\partial_x^2 [D(x)P^*(x)] = 0$. Is there a normalisable solution on the interval $[0, 1]$? Interpret.
- 5/ Rewrite explicitly the FPE (12), with the $D(x)$ found above, under the form

$$\partial_t P_t(x) = -\partial_x [F(x)P_t(x)] + \partial_x \left[\sqrt{D(x)} \partial_x [\sqrt{D(x)} P_t(x)] \right] \quad \text{for } x \in [0, 1] \quad (13)$$

and give the drift $F(x)$. Comment the form of (13).

- 6/ In order to clarify the behaviour of $P_t(x)$ at large time, we perform a change of variable

$$Z(t) = \Psi(X(t)) \quad \text{with } \Psi(x) \stackrel{\text{def}}{=} \int_{1/2}^x \frac{dx'}{\sqrt{2D(x')}} \quad (14)$$

Compute the function $\Psi(x)$. On which interval is mapped $]0, 1[$? (i.e. to which interval does $Z(t)$ belong ?) Plot *neatly* $z = \Psi(x)$.

- 7/ Show that the FPE for the distribution $Q_t(z)$ of $Z(t)$ has a free diffusion term

$$\partial_t Q_t(z) = -\partial_z [\phi(z)Q_t(z)] + \frac{1}{2} \partial_z^2 Q_t(z) \quad \text{for } \phi(z) = \frac{1}{2} \sqrt{\gamma} \tanh(\sqrt{\gamma}z/2) \quad (15)$$

- 8/ What is the potential $U(z) = -\int_0^z dz' \phi(z')$ related to the new drift ? Plot *neatly* $U(z)$.
- 9/ What is the expected behaviour of $Z(t)$, depending on $Z(0) < 0$ or $Z(0) > 0$ ("far" from zero) ? Deduce an *approximate* form for the distribution $Q_t(z)$ in these two situations.
- 10/ Without any calculation, deduce $\langle X(t) \rangle$ from (10), for $X(0) = x_0$ (non random). With this remark and the previous analysis, describe the evolution of $P_t(x)$ at large time.
- 11/ Four quantum trajectories have been simulated numerically (Fig. 3). Comment the simulation at the light of the above analysis.

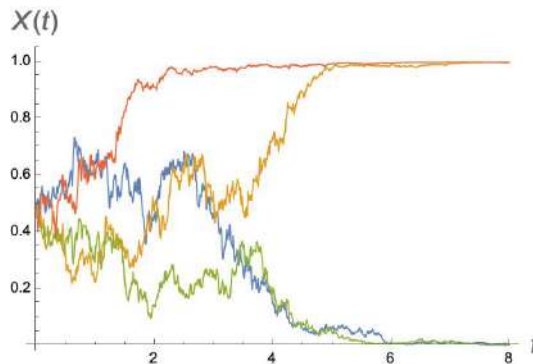


Figure 3: *Numerical simulation of four quantum trajectories $X(t)$ for $x_0 = 1/2$.*

12/ BONUS :

- a) What is the discrete recurrence which has allowed to generate the four trajectories in the numerical simulation of Fig. 3 ?
- b) *From the signal to the quantum state.*— Write the Stratonovich SDE related to (10). Deduce that $X(t)$ (the state of the system) can be expressed as a function of $Y(t)$ (the measured signal) and x_0 .

Proofreading (~10mn)

Appendix

$$\int_{\mathbb{R}} dx e^{-\frac{1}{2}ax^2 + ikx} = \sqrt{\frac{2\pi}{a}} e^{-\frac{1}{2a}k^2}$$

Doblin-Itô calculus.— If $X(t)$ obeys a Itô SDE $dX(t) = a(X)dt + b(X)dW(t)$, the Itô formula is deduced from $d\varphi(X) = \varphi'(X)dX + \frac{1}{2}\varphi''(X)(dX)^2$ where one should use $dW(t)^2 = dt$.

Itô/FPE and Stratonovich/FPE.— Relation between SDE and FPE :

$$\begin{aligned} dX(t) &= a(X)dt + b(X)dW(t) \quad (\text{Itô}) & \leftrightarrow & \quad \partial_t P_t(x) = -\partial_x[a(x)P_t(x)] + \frac{1}{2}\partial_x^2[b(x)^2 P_t(x)] \\ dX(t) &= F(X)dt + b(X)dW(t) \quad (\text{Strato.}) & \leftrightarrow & \quad \partial_t P_t(x) = -\partial_x[F(x)P_t(x)] + \frac{1}{2}\partial_x[b(x)\partial_x[b(x)P_t(x)]] \end{aligned}$$

so that $a(x) = F(x) + \frac{1}{2}b(x)b'(x)$.

SOLUTIONS WILL BE AVAILABLE AT http://www.lptms.universite-paris-saclay.fr/christophe_texier/