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Bose-Einstein Condensation in Quasi-2D Trapped Gases

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We discuss Bose-Einstein condensation (BEC) in quasi-2D trapped gases and find that well below the transition temperature T_c the equilibrium state is a true condensate, whereas at intermediate temperatures $T < T_c$ one has a quasicondensate (condensate with fluctuating phase). The mean-field interaction in a quasi-2D gas is sensitive to the frequency ω_0 of the (tight) confinement in the "frozen" direction, and one can switch the sign of the interaction by changing ω_0. Variation of ω_0 can also reduce the rates of inelastic processes. This offers promising prospects for tunable BEC in trapped quasi-2D gases.

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The influence of dimensionality of the system of bosons on the presence and character of Bose-Einstein condensation (BEC) and superfluid phase transition has been a subject of extensive studies in spatially homogeneous systems. In 2D a true condensate can exist only at $T = 0$, and its absence at finite temperatures follows from the Bogolyubov k^{-2} theorem and originates from long-wave fluctuations of the phase (see, e.g., [1,2]). However, as was first pointed out by Kane and Kadanoff [3] and then proved by Berezinskii [4], there is a superfluid phase transition at sufficiently low T. Kosterlitz and Thouless [5] found that this transition is associated with the formation of bound pairs of vortices below the critical temperature $T_{\mathrm{KT}} = (\pi\hbar^2/2m)n_s$ (m is the atom mass, and n_s the superfluid density just below T_{KT}). Earlier theoretical studies of 2D systems have been reviewed in [2] and have led to the conclusion that below the Kosterlitz-Thouless transition (KTT) temperature the Bose liquid (gas) is characterized by the presence of a quasicondensate, that is a condensate with fluctuating phase (see [6]). In this case the system can be divided into blocks with a characteristic size greatly exceeding the healing length but smaller than the radius of phase fluctuations. Then, there is a true condensate in each block but the phases of different blocks are not correlated with each other.

The KTT has been observed in monolayers of liquid helium [7]. The only dilute atomic system studied thus far was a 2D gas of spin-polarized atomic hydrogen on liquid-helium surface (see [8] for review). Recently, the observation of KTT in this system has been reported [9].

BEC in trapped 2D gases is expected to be qualitatively different. The trapping potential introduces a finite size of the sample, which sets a lower bound for the momentum of excitations and reduces the phase fluctuations. Moreover, for an ideal 2D Bose gas in a harmonic potential Bagnato and Kleppner [10] found a macroscopic occupation of the ground state of the trap (ordinary BEC) at temperatures $T < T_c \approx N^{1/2}\hbar\omega$, where N is the number of particles, and ω the trap frequency. Thus, there is a question of whether an interacting trapped 2D gas supports the ordinary BEC or the KTT type of a crossover to the BEC regime [11]. However, the critical temperature will be always comparable with T_c of an ideal gas: On approaching T_c from above, the gas density is $n_c \sim N/R_{T_c}^2$, where $R_{T_c} \approx \sqrt{T_c/m\omega^2}$ is the thermal size of the cloud, and hence the KTT temperature is $\sim\hbar^2 n_c/m \sim N^{1/2}\hbar\omega \approx T_c$.

The discovery of BEC in trapped alkali-atom clouds [12] stimulated a progress in optical cooling and trapping of atoms. Present facilities allow one to tightly confine the motion of trapped particles in one direction and to create a (quasi-)2D gas. In other words, kinematically the gas is 2D, and in the "frozen" direction the particles undergo zero point oscillations. This requires the frequency of the tight confinement ω_0 to be much larger than the gas temperature T and the mean-field interparticle interaction $n_0 g$ (n_0 is the gas density, and g the coupling constant). Recent experiments [13–15] indicate a realistic possibility of creating quasi-2D trapped gases and achieving the regime of quantum degeneracy in these systems. The character of BEC will be similar to that in purely 2D trapped gases, and the main difference is related to the sign and value of the coupling constant g.

In this Letter, we discuss BEC in quasi-2D trapped gases and arrive at two key conclusions. First, well below T_c the phase fluctuations are small, and the equilibrium state is a true condensate. At intermediate temperatures $T < T_c$ the phase fluctuates on a distance scale smaller than the Thomas-Fermi size of the gas, and one has a quasicondensate (condensate with fluctuating phase). Second, in quasi-2D the coupling constant g is sensitive to the frequency of the tight confinement ω_0 and, for a negative 3D scattering length a, one can switch the mean-field interaction from attractive to repulsive by increasing ω_0. Variation of ω_0 can also reduce the rates of inelastic processes. These findings are promising for tunable BEC.

In a weakly interacting Bose-condensed gas the correlation (healing) length $\hbar/\sqrt{mn_0 g}$ ($g > 0$) should greatly exceed the mean interparticle separation. In (quasi)-2D the latter is $\sim 1/\sqrt{2\pi n_0}$, and we obtain a small parameter of the theory, $(mg/2\pi\hbar^2) \ll 1$ (see [6]).

We first analyze the character of BEC in a harmonically trapped 2D gas with repulsive interparticle interaction, relying on the calculation of the one-particle density matrix. Similarly to the spatially homogeneous case [1,2], at sufficiently low temperatures only phase fluctuations are relevant. Then the field operator can be written as $\hat{\Psi}(\mathbf{R}) = n_0^{1/2}(\mathbf{R}) \exp\{i\hat{\phi}(\mathbf{R})\}$, where $\hat{\phi}(\mathbf{R})$ is the operator of the phase fluctuations, and $n_0(\mathbf{R})$ the condensate density at $T = 0$. The one-particle density matrix takes the form [2]

$$\langle \hat{\Psi}^\dagger(\mathbf{R})\hat{\Psi}(0)\rangle = \sqrt{n_0(\mathbf{R})n_0(0)}\, \exp\{-\langle[\delta\hat{\phi}(\mathbf{R})]^2\rangle/2\}. \tag{1}$$

Here $\delta\hat{\phi}(\mathbf{R}) = \hat{\phi}(\mathbf{R}) - \hat{\phi}(0)$, and $\mathbf{R} = 0$ at the trap center. For a trapped gas the operator $\hat{\phi}(\mathbf{R})$ is given by

$$\hat{\phi}(\mathbf{R}) = \sum_\nu [4n_0(\mathbf{R})]^{-1/2} f_\nu^+ \hat{a}_\nu + \text{H.c.}, \tag{2}$$

where $\hat{a}_\nu$ is the annihilation operator of an elementary excitation with energy ε_ν, and $f_\nu^\pm = u_\nu \pm v_\nu$ are the Bogolyubov u, v functions of the excitations. In the Thomas-Fermi (TF) regime the density $n_0(\mathbf{R})$ has the well-known parabolic shape, with the maximum value $n_{0m} = n_0(0) \approx (Nm/\pi g)^{1/2}\omega$, and the radius $R_{TF} \approx (2\mu/m\omega^2)^{1/2}$. The chemical potential is $\mu = n_{0m} g \gg \hbar\omega$, and the ratio $T_c/\mu \approx (\pi\hbar^2/mg)^{1/2} \gg 1$.

For calculating the mean-square fluctuations of the phase, we explicitly found the (discrete) spectrum and wave functions $f_\nu^\pm$ of excitations with energies $\varepsilon_\nu \ll \mu$ by using the method developed for 3D trapped condensates [16]. For excitations with higher energies we used the WKB approach. Then, in the TF regime at distances $R \gg \lambda_T$, where λ_T is the wavelength of thermal excitations ($\varepsilon_\nu \approx T$) near the trap center, for $T \gg \mu$ we obtain

$$\langle[\delta\hat{\phi}(\mathbf{R})]^2\rangle \approx (mT/\pi\hbar^2 n_{0m}) \ln(R/\lambda_T). \tag{3}$$

We also find that Eq. (3) holds at any T for a homogeneous gas of density n_{0m}, where at $T \ll \mu$ it reproduces the well-known result (see [2]). In a trapped gas for $T \ll \mu$, due to the contribution of low-energy excitations, Eq. (3) acquires a numerical coefficient ranging from 1 at $R \ll R_{TF}$ to approximately 3 at $R \approx R_{TF}$.

The character of the Bose-condensed state is determined by the phase fluctuations at $R \sim R_{TF} \sim (N/n_{0m})^{1/2}$. For a large number of particles, assuming that $\ln N \gg \ln(n_{0m}\lambda^2)$, from Eq. (3) we find

$$\langle[\delta\phi(R_{TF})]^2\rangle \approx (T/T_c)(mg/4\pi\hbar^2)^{1/2} \ln N. \tag{4}$$

In quasi-2D trapped alkali gases one can expect a value $\sim 10^{-2}$ or larger for the small parameter $mg/2\pi\hbar^2$, and the number of trapped atoms N ranging from 10^4 to 10^6. For $T \gtrsim \mu$ the quantity $n_{0m}\lambda_T^2 \approx (\pi\hbar^2/mg)(\mu/T)$ and $\ln N$ is always significantly larger than $\ln(n_{0m}\lambda_T^2)$.

Then, from Eq. (4) we identify two BEC regimes. At temperatures well below T_c the phase fluctuations are small, and there is a true condensate. For intermediate temperatures $T < T_c$ the phase fluctuations are large and, as the density fluctuations are suppressed, one has a quasicondensate (condensate with fluctuating phase).

The characteristic radius of the phase fluctuations $R_\phi \approx \lambda_T \exp(\pi\hbar^2 n_{0m}/mT)$, following from Eq. (3) under the condition $\langle[\delta\hat{\phi}(\mathbf{R})]^2\rangle \sim 1$, greatly exceeds the healing length. Therefore, the quasicondensate has the same Thomas-Fermi density profile as the true condensate. Correlation properties at distances smaller than R_ϕ and, in particular, local density correlators are also the same. Hence, one expects the same reduction of inelastic decay rates as in 3D condensates [6]. However, the phase coherence properties of a quasicondensate are drastically different. For example, in the MIT-type [17] of experiment on interference of two independently prepared quasicondensates the interference fringes will be essentially smeared out.

We now calculate the mean-field interparticle interaction in a quasi-2D Bose-condensed gas, relying on the binary approximation. The coupling constant g is influenced by the trapping field in the direction z of the tight confinement. For a harmonic tight confinement, the motion of two atoms interacting with each other via the potential $V(r)$ can be still separated into their relative and center-of-mass motion. The former is governed by $V(r)$ together with the potential $V_H(z) = m\omega_0^2 z^2/4$ originating from the tight harmonic confinement. Then, similarly to the 3D case (see, e.g., [18]), to zero order in perturbation theory the coupling constant is equal to the vertex of interparticle interaction in vacuum at zero momenta and frequency $E = 2\mu$. For low $E > 0$ this vertex coincides with the amplitude of scattering at energy E and, hence, is given by [19]

$$g = f(E) = \int d\mathbf{r}\psi(\mathbf{r})V(r)\psi_f^*(\mathbf{r})\,. \tag{5}$$

The wave function of the relative motion of a pair of atoms, $\psi(\mathbf{r})$, satisfies the Schrödinger equation

$$[-(\hbar^2/m)\Delta + V(\mathbf{r}) + V_H(z) - \hbar\omega_0/2]\psi(\mathbf{r}) = E\psi(\mathbf{r})\,. \tag{6}$$

The wave function of the free x, y motion $\psi_f(\mathbf{r}) = \varphi_0(z)\exp(i\mathbf{q}\cdot\boldsymbol{\rho})$, with $\varphi_0(z)$ being the ground state wave function for the potential $V_H(z)$, $\boldsymbol{\rho} = \{x, y\}$, and $q = (2mE/\hbar^2)^{1/2}$. As the vertex of interaction is an analytical function of E, the coupling constant for $\mu < 0$ is obtained by analytical continuation of $f(E)$ to $E < 0$.

The possibility to omit higher orders in perturbation theory requires the above criterion $(mg/2\pi\hbar^2) \ll 1$.

The solution of the quasi-2D scattering problem from Eq. (6) contains two distance scales: the extension of $\psi(\mathbf{r})$ in the z direction, $l = (\hbar/m\omega_0)^{1/2}$, and the characteristic radius R_e of the potential $V(r)$. In alkalis it ranges from 20 Å for Li to 100 Å for Cs. At low energies ($qR_e \ll 1$) the amplitude $f(E)$ is determined by the scattering of the s wave for the motion in the x, y plane.

We first consider the limiting case $l \gg R_e$. Then the relative motion of atoms in the region of interatomic interaction is not influenced by the tight confinement, and $\psi(\mathbf{r})$ in Eq. (5) differs only by a normalization coefficient from the 3D wave function at zero energy:

$$\psi(r) = \eta\varphi_0(0)\psi_{3\mathrm{D}}(r)\,. \tag{7}$$

At $r \gg R_e$ we have $\psi_{3\mathrm{D}} = 1 - a/r$. Hence, for $R_e \ll r \ll l$, Eq. (7) takes the form $\psi = \psi_{\mathrm{as}}(r) = \eta\varphi_0(0)(1 - a/r)$. This expression serves as a boundary condition at $r \to 0$ for the solution of Eq. (6) with $V(r) = 0$ ($r \gg R_e$). The latter can be expressed through the Green function $G(\mathbf{r}, \mathbf{r}')$ of this equation:

$$\psi(\mathbf{r}) = \varphi_0(z)\exp(i\mathbf{q}\cdot\boldsymbol{\rho}) + AG(\mathbf{r}, 0)\,. \tag{8}$$

The coefficients A and η are obtained by matching the solution (8) at $r \to 0$ with $\psi_{\mathrm{as}}(r)$.

Similarly to the case of a purely 1D harmonic oscillator (see, e.g., [20]), we have

$$G(\mathbf{r}, 0) = \frac{1}{l}\int_0^\infty dt\,\frac{\exp[i(z^2\cot t/4l^2 - q^2l^2t - t/2 + \rho^2/4tl^2)]}{t\sqrt{(4\pi i)^3\sin t}}\,.$$

Under the condition $ql \ll 1$ ($\mu \ll \hbar\omega_0$) at $r \ll l$ we obtain

$$G \approx \frac{1}{4\pi r} + \frac{1}{2(2\pi)^{3/2}l}\left[\ln\!\left(\frac{1}{\pi q^2 l^2}\right) + i\pi\right]. \tag{9}$$

Omitting the imaginary part of G (9) and comparing Eq. (8) with ψ_{as}, we immediately find

$$\eta = -\frac{A}{4\pi a\varphi_0(0)} = \left[1 + \frac{a}{\sqrt{2\pi}\,l}\ln\!\left(\frac{1}{\pi q^2 l^2}\right)\right]^{-1}. \tag{10}$$

In Eq. (5) one can put $\psi_f = \varphi_0(0) = (1/2\pi l^2)^{1/4}$. Then, using the well-known result $\int d\mathbf{r}\psi_{3\mathrm{D}}(r)V(r) = 4\pi\hbar^2 a/m$, Eqs. (5), (7), and (10) lead to the coupling constant

$$g = \frac{2\sqrt{2\pi}\,\hbar^2}{m}\,\frac{1}{l/a + (1/\sqrt{2\pi})\ln(1/\pi q^2 l^2)}\,. \tag{11}$$

For $\mu < 0$, analytical continuation of Eqs. (9) and (11) to $E = \hbar^2 q^2/m < 0$ leads to the replacement $E \to |E| = 2|\mu|$ in the definition of q.

The coupling constant in quasi-2D depends on $q = (2m|\mu|/\hbar^2)^{1/2}$ and, hence, on the condensate density. In the limit $l \gg |a|$ the logarithmic term in Eq. (11) is not important, and g becomes density independent. In this case the quasi-2D gas can be treated as a 3D condensate with the density profile $\propto \exp(-z^2/l^2)$ in the z direction.

As follows from Eq. (11), for repulsive mean-field interaction in 3D ($a > 0$) the interaction in quasi-2D is also repulsive. For $a < 0$ the dependence of g (and the scattering amplitude f) on l has a resonance character (cf. Fig. 1): The coupling constant changes sign from negative (attraction) at very large l to positive for $l < l_* = (|a|/\sqrt{2\pi})\ln(1/\pi q^2 l^2)$. The resonance originates from the appearance of a bound state with zero energy for a pair of atoms in the quasi-2D geometry at $l \approx l_*$. This should describe the case of Cs, where $a \lesssim -600$ Å [14,21] and the condition $l \gg R_e$ assumed in Eq. (11) is satisfied at $l < l_*$. Near the resonance point l_* the quantity $(m|g|/2\pi\hbar^2)$ becomes large, which violates the perturbation theory for a Bose-condensed gas and makes Eqs. (5) and (11) invalid.

For $l \lesssim R_e$ (except for very small l) we used directly Eqs. (5) and (6) and calculated numerically the coupling constant g for Li, Na, Rb, and Cs. The potential $V(r)$ was modeled by the Van der Waals tail, with a hard core at a distance $R_0 \ll R_e$ selected to support many bound states and reproduce the scattering length a. The numerical results differ slightly from the predictions of Eq. (11). For Rb and Cs both are presented in Fig. 1.

The nature of the $g(l)$ dependence in quasi-2D can be understood just relying on the values of g in the purely 2D and 3D cases. In 2D at low energies the mean-field interaction is always repulsive. This striking difference from the

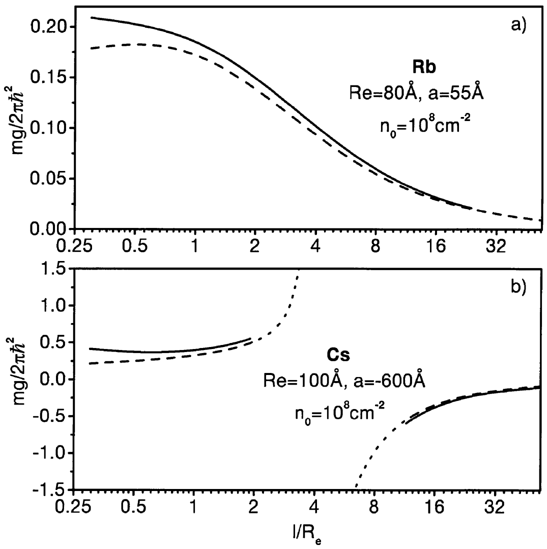

FIG. 1. The parameter $mg/2\pi\hbar^2$ versus l/R_e at fixed n_0 for Rb (a) and Cs (b). Solid curves correspond to the numerical results, dashed curves to Eq. (11). The dotted curve in (b) shows the result of Eq. (11) in the region where $m|g|/2\pi\hbar^2 \sim 1$.

3D case can be found from the solution of the 2D scattering problem in [19] and originates from the 2D kinematics: At distances, where $R_e \ll \rho \ll q^{-1}$ ($q \to 0$), the solution of the Schrödinger equation for the (free) relative motion in a pair of atoms reads $\psi \propto \ln(\rho/d)/\ln(1/qd)$ ($d > 0$). We always have $|\psi|^2$ increasing with ρ, unless we touch resonances corresponding to the presence of a bound state with zero energy ($d \to \infty$). This means that it is favorable for particles to be at larger ρ, i.e., they repel each other.

In quasi-2D for very large l the sign of the interparticle interaction is the same as in 3D. With decreasing l, the 2D features in the relative motion of atoms become pronounced, which is described by the logarithmic term in Eq. (11). Hence, for $a > 0$ the interaction remains repulsive, whereas for $a < 0$ the attraction turns to repulsion.

The obtained results are promising for tunable BEC in quasi-2D gases, based on variations of the tight confinement and, hence, l. However, as in the MIT studies of tunable 3D BEC by using Feshbach resonances [22], an "underwater stone" concerns inelastic losses: Variation of l can change the rates of inelastic processes. For optically trapped atoms in the lowest Zeeman state the most important decay process is three-body recombination.

This process occurs at interparticle distances $r \lesssim \max\{R_e, |a|\}$ [23,24]. We will restrict ourselves to the case where $l \gtrsim |a|$ and is also significantly larger than R_e. Then the character of recombination collisions remains three-dimensional, and one can treat them in a similar way as in a 3D gas with the density profile $(n_0/\sqrt{\pi}\, l)\exp(-z^2/l^2)$.

However, the normalization coefficient of the wave function in the incoming channel will be influenced by the tight confinement. Relying on the Jastrow approximation, we write this wave function as a product of the three wave functions $\psi(\mathbf{r}_{ik})$, each of them being a solution of the binary collision problem Eq. (6). In our limiting case the solution is given by Eq. (7) divided by $\varphi_0(0)$ to reconstruct the density profile in the z direction. The outgoing wave function remains the same as in 3D, since one has a molecule and an atom with very large kinetic energies.

Thus, in the Jastrow approach we have an additional factor η^3 for the amplitude and η^6 for the probability of recombination in a quasi-2D gas compared to the 3D case. Averaging over the density profile in the z direction, we can relate the quasi-2D rate constant α to the rate constant in 3D (see [24] for a table of α_{3D} in alkalis):

$$\alpha = (\eta^6/\pi l^2)\alpha_{3D} . \qquad (12)$$

As η is given by Eq. (10), for $a > 0$ the dependence $\alpha(l)$ is smooth. For $a < 0$ the rate constant α peaks at $l \approx l_*$ and decreases as $l^4/(l_* - l)^6$ at smaller l. This indicates a possibility to reduce recombination losses while maintaining a repulsive mean-field interaction ($g > 0$). For Cs already at $l \approx 200$ Å ($l_* \approx 500$ Å) we have $\alpha \sim 10^{-17}$ cm^4/s, and at densities 10^8 cm^{-2} the lifetime $\tau > 1$ s.

The predicted possibility to modify the mean-field interaction and reduce inelastic losses by varying the frequency of the tight confinement opens new handles on tunable BEC in quasi-2D gases. These experiments can be combined with measurements of nontrivial phase coherence properties of condensates with fluctuating phase.

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