Three-Body Interacting Bosons in Free Space

D. S. Petrov
Université Paris-Sud, CNRS, LPTMS, UMR8626, Orsay F-91405, France
(Received 10 December 2013; published 10 March 2014)

We propose a method of controlling two- and three-body interactions in an ultracold Bose gas in any dimension. The method requires us to have two coupled internal single-particle states split in energy such that the upper state is occupied virtually but amply during collisions. By varying system parameters, one can switch off the two-body interaction while maintaining a strong three-body one. The mechanism can be implemented for dipolar bosons in the bilayer configuration with tunneling or in an atomic system by using radio-frequency fields to couple two hyperfine states. One can then aim to observe a purely three-body interacting gas, dilute self-trapped droplets, the paired superfluid phase, Pfaffian state, and other exotic phenomena.

DOI: 10.1103/PhysRevLett.112.103201 PACS numbers: 34.50.-s, 05.30.Jp, 67.85.-d

The Feshbach resonance technique, which allows for tuning the two-body interaction to any value, has been a major breakthrough in the field of quantum gases [1]. Reaching strongly interacting regimes by using this method is proven successful in two-component fermionic mixtures [2] because of the naturally built-in mechanism of suppression of local three-body inelastic processes [3]: the Pauli principle prohibits three fermions to be close to each other as at least two of them are identical. Essentially the same mechanism is responsible for the repulsion between weakly bound molecules in this system, ensuring the mechanical stability. Bosons, having no such protection, are much more fragile. A Bose-Einstein condensate (BEC) collapses in free space even for infinitesimally weak attraction [4] not to mention devastating recombination losses close to resonance [5,6].

A repulsive three-body force can stabilize the system and induce nontrivial many-body effects. A weakly interacting BEC with two-body attraction (coupling constant $g_2 < 0$) and three-body repulsion ($g_3 > 0$) is predicted to be a droplet, the density of which in the absence of external confinement and neglecting the surface tension is flat and equals $n = -3g_2/2g_3$ [7–9]. For a strong (beyond mean-field) two-body attraction the spinless Bose gas can pass from the atomic to paired superfluid phase via an Ising-type transition with peculiar topological properties [10–12]. However, the mechanical stability of the system requires repulsive few-body interactions [13] or other stabilizing mechanisms [14]. Few-body forces are also important for quantum Hall problems: the exact ground state of bosons in the lowest Landau level with a repulsive three-body contact interaction is the Pfaffian state [15] also known as the weak-pairing phase and characterized by non-Abelian excitations [16]. Interestingly, a finite range of the three-body interaction breaks the pairing [17]. On the other hand, there may be a transition from the weak- to strong-pairing Abelian phase [18], presumably driven by varying g_2.

Most proposals for generating effective three-body interactions deal with lattice systems [19–24]. In free space, since three-body effects are significant when $g_3 n$ is of order g_2, staying in the dilute regime requires small g_2 and large g_3. In three dimensions, a resonant three-body force is predicted for large and negative scattering lengths when a three-body Efimov state crosses the three-atom threshold [7,25–27]. This method is associated with strong relaxation losses [28], although nonconservative three-body interactions can also lead to interesting effects [20,29].

This Letter is motivated by the observation that dipolar particles trapped on a single layer and oriented perpendicular to the plane repel each other, whereas in a bilayer configuration there is always a bound state [30–34]. We argue that the bound state emerges from the scattering continuum as one gradually splits the layer into two and reduces the interlayer tunneling amplitude below a critical value $t = t_c$. We show that near this point $g_2 \propto t - t_c$ providing the desired control over the two-body interaction. Next, we find that the three-body interaction near this zero crossing is repulsive and conservative: similar to the fermionic Pauli protection three dipoles are frustrated in the sense that at least two of them are on the same layer and, therefore, experience repulsion. Surprising and counterintuitive is that the effective three-body repulsion strengthens with decreasing the dipole-dipole interaction and interlayer tunneling amplitude. The reason is that the scattering wave function of two dipoles contains a significant contribution of a virtually excited interlayer dimer state of size $\sim 1/\sqrt{t}$. The effective three-body force originates from the interaction of the third particle with this state and becomes stronger with decreasing t. Based on this understanding we propose a general method of controlling few-body interactions applicable for atomic systems in any dimension.

The Hamiltonian of bosonic dipoles in the bilayer geometry with tunneling is written as (cf. [35])

$$H = \int_r \sum_\sigma \Psi^\dagger_{\sigma r}(-\nabla^2_r/2 + t)\Psi_{\sigma r} - t(\Psi^\dagger_{\uparrow r}\Psi_{\downarrow r} + \text{H.c.}) + \frac{1}{2}\int_{r,r'}\sum_{\sigma,\sigma'}\Psi^\dagger_{\sigma r}\Psi^\dagger_{\sigma' r'}V_{\sigma\sigma'}(|\boldsymbol{r}-\boldsymbol{r}'|)\Psi_{\sigma r}\Psi_{\sigma' r'}, \quad (1)$$

where $\Psi^\dagger_{\sigma r}$ is the creation operator of a boson on layer $\sigma(=\uparrow,\downarrow)$ with in-plane coordinate $\boldsymbol{r}$ and we neglect the transverse extension of the wave function within the layers compared to the interlayer distance λ. We adopt the units $\lambda = \hbar = m = 1$. Then, for dipoles oriented perpendicular to the plane the intralayer and interlayer potentials equal $V_{\sigma\sigma}(r) = r_*/r^3$ and $V_{\sigma\sigma'}(r)|_{\sigma\neq\sigma'} = r_*(r^2-2)/(r^2+1)^{5/2}$, respectively, and r_* is the characteristic length scale of the dipole-dipole potential.

The one-body spectrum of (1) consists of two branches with dispersions $\varepsilon_+(k) = k^2/2$ and $\varepsilon_-(k) = 2t + k^2/2$. The corresponding eigenfunctions are $\phi_{\pm,\mathbf{k}} = |\pm\rangle\exp(i\mathbf{k}\cdot\boldsymbol{r})$ with spinor parts $|\pm\rangle = (|\uparrow\rangle \pm |\downarrow\rangle)/\sqrt{2}$. We assume that the temperature and typical interaction scales are lower than t so that the upper branch is excited only virtually during collisions. Moreover, due to the $\uparrow$-$\downarrow$ symmetry, the Hamiltonian (1) couples the lowest two-body spinor configuration $(|+\rangle)^2$ only to the highest one, $(|-\rangle)^2$. The gap is then effectively $4t$. Writing the wave function of the relative motion in the form $(|\uparrow\rangle|\uparrow\rangle + |\downarrow\rangle|\downarrow\rangle)\phi_{\uparrow\uparrow}(\boldsymbol{r}) + (|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle)\phi_{\uparrow\downarrow}(\boldsymbol{r})$, we obtain the two-channel Schrödinger equation

$$\left[-\nabla^2_r - E + \begin{pmatrix} V_{\uparrow\uparrow}(r) + 2t & -2t \\ -2t & V_{\uparrow\downarrow}(r) + 2t \end{pmatrix}\right]\begin{pmatrix}\phi_{\uparrow\uparrow} \\ \phi_{\uparrow\downarrow}\end{pmatrix} = 0. \quad (2)$$

The bound state in the potential $V_{\uparrow\downarrow}$ [30–34,36] is a true eigenstate of our model in the limit $t \to 0$. Its wave function contains only the $\phi_{\uparrow\downarrow}$ part; i.e., the two dipoles are localized on different layers. The binding survives small t since the cost of this localization is of order the (small) tunneling energy. However, for $t > t_c$ the localization becomes too expensive and the bound state crosses the two-particle threshold. Solid line in Fig. 1 shows t_c as a function of r_* obtained numerically from Eq. (2).

The low energy and small momentum properties of this rather unusual two-dimensional scattering problem with long-range interactions are described in terms of the vertex function $\Gamma(E,\mathbf{k},\mathbf{k}')$, where E is the total energy in the center of mass reference frame and $\mathbf{k}$ and $\mathbf{k}'$ are the incoming and outgoing relative momenta, respectively. For sufficiently weakly bound or quasibound state, i.e., when t is close to t_c, we can repeat arguments of Ref. [34] and write

$$\Gamma(E,\mathbf{k},\mathbf{k}') \approx \frac{4\pi}{\ln(4t/E) + 4\pi/g_2 + i\pi} - 2\pi r_*|\mathbf{k}-\mathbf{k}'|, \quad (3)$$

which, in our case, is valid for $E \sim k^2 \sim k'^2 \ll t$. The second term in Eq. (3) accounts for the long-range dipole-dipole interaction tail and the first one is the usual single-pole expression for the scattering amplitude, which effectively integrates out the short-range radial motion and σ degrees of freedom of Eq. (2). "Short-range" in our case means distances smaller than $1/\sqrt{t}$ (see below). For small g_2 one can neglect the logarithmic and imaginary terms in the denominator of Eq. (3) arriving at $\Gamma \approx g_2 - 2\pi r_*|\mathbf{k}-\mathbf{k}'|$. The name coupling constant is thus attached to g_2 with the reservation that the neglected logarithm can become important for exponentially small energies $E \sim t\exp(-4\pi/|g_2|)$, in particular, when looking for poles of the scattering amplitude: as g_2 approaches zero from below, there is an exponentially weakly bound state with the binding energy $\varepsilon_0 = 4t\exp(4\pi/g_2)$.

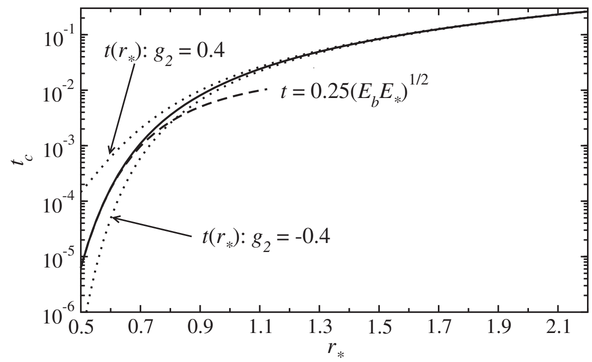

FIG. 1. Critical tunneling amplitude t_c vs. r_* (solid line). For $t < t_c$ the two-body interaction is attractive and supports a bound state. The dotted lines enclose the region $-0.4 < g_2 < 0.4$. The dashed line is the prediction of the zero-range theory $t_c \approx \sqrt{E_{\uparrow\downarrow}E_{\uparrow\uparrow}}/4$ valid for small r_*.

Close to the crossing point g_2 is proportional to $t - t_c$ with a positive coefficient: for $t < t_c$ the two-body interaction is attractive (supports a bound state) and vice versa. To see how sensitive g_2 is to variations of t for different r_*, in Fig. 1 we show the region $-0.4 < g_2 < 0.4$ enclosed by dotted lines. Making g_2 small at large r_* requires a more subtle tuning because of a potential barrier which separates the scattering continuum from the bound state localized in this case at small r. The barrier is given by $V_{\uparrow\uparrow}(r)$ and $V_{\uparrow\downarrow}(r)$, which are both $\propto r_*$ and positive for $r > \sqrt{2}$. We should note that our results become qualitative for $t \sim 1$ ($r_* > 2$ or so) when the thickness of the layers comes into play. One then has to consider a full three-dimensional Hamiltonian instead of the idealized two-dimensional model (1).

For $r_* \ll 1$ we solve the problem analytically by substituting for $V_{\sigma\sigma'}$ zero-range (ZR) pseudopotentials with proper low-energy scattering properties [31]. In terms of Bessel functions the two-body wave function reads

$$\Psi = (|+\rangle)^2[J_0(qr) - if_{\mathrm{zr}}(q)H_0(qr)/4] + C(|-\rangle)^2K_0(\kappa r), \quad (4)$$

where q is the collision momentum and $\kappa = \sqrt{4t - q^2}$. The scattering amplitude

$$f_{\mathrm{zr}}(q) = \frac{2\pi}{\ln\frac{\kappa}{q} - \ln\frac{\kappa^2}{E_{\uparrow\downarrow}}\ln\frac{\kappa^2}{E_{\uparrow\uparrow}}/\ln\frac{\kappa^4}{E_{\uparrow\downarrow}E_{\uparrow\uparrow}} + i\frac{\pi}{2}} \tag{5}$$

and coefficient $C = (2\pi)^{-1} f_{\mathrm{zr}}(q)\ln\frac{E_{\uparrow\downarrow}}{E_{\uparrow\uparrow}}/\ln\frac{\kappa^4}{E_{\uparrow\uparrow}E_{\uparrow\downarrow}}$ are determined from the ZR boundary conditions: $\phi_{\uparrow\uparrow}(r) \propto \ln(\sqrt{E_{\uparrow\uparrow}}re^{\gamma}/2)$ and $\phi_{\uparrow\downarrow}(r) \propto \ln(\sqrt{E_{\uparrow\downarrow}}re^{\gamma}/2)$, where $\gamma \approx 0.5772$ is the Euler constant, $E_{\uparrow\uparrow} = 4\exp(-6\gamma)r_*^{-2}$ [37], and $E_{\uparrow\downarrow} \approx 4\exp(-8/r_*^2)$ [33,34,38]. $f_{\mathrm{zr}}(0)$ vanishes for exponentially small $t = t_c = \sqrt{E_{\uparrow\downarrow}E_{\uparrow\uparrow}}/4 \propto \exp(-4/r_*^2)$. We see that the ZR approximation is justified since typical length scales, $\sim 1/\sqrt{t_c}$, are exponentially large compared to the range of $V_{\sigma\sigma'}$. At $r_* \approx 0.7$ the ZR result for t_c (dashed line in Fig. 1) deviates from the exact one by only about 10%. The ZR approximation also predicts the dependence $g_2 \approx 16\pi(\ln E_{\uparrow\uparrow}/E_{\uparrow\downarrow})^{-2}(t-t_c)/t_c$, valid for $(t-t_c)/t_c \ll 1$ and established by comparing Eq. (5) with the on-shell version of Eq. (3).

We find that for $r_* \lesssim 0.7$ and $q < 2\sqrt{t}$ the exact s-wave scattering amplitude is well approximated by $f(q) \approx f_{\mathrm{zr}}(q) - 8r_*q$, where the last term is the s-wave component of $-2\pi r_*|\mathbf{q}-\mathbf{q}'|$ [see Eq. (3)] taken on the mass shell, $q = q'$. Figure 2 shows $-4\tan\delta(q)$ (solid lines) and $-4\tan\delta_{\mathrm{zr}}(q) - 8r_*q$ (dashed lines) as functions of q for $r_* = 0.65$ and five values of t close to t_c. Here we introduce the s-wave scattering phase shift δ and benefit from the relation $f = -4/(\cot\delta - i) \approx -4\tan\delta$ valid for small $\tan\delta$. We also present $f(q) \approx g_2 - 8r_*q$ (dotted lines), which is the s-wave on-shell version of the approximation $\Gamma \approx g_2 - 2\pi r_*|\mathbf{q}-\mathbf{q}'|$. The inset in Fig. 2 shows the case $r_* = 1.3$ and we omit the ZR curves which are quite far off for this value of r_*.

Let us now discuss the three-body problem. In three dimensions, g_3 can be defined as the interaction energy shift of three condensed bosons in a unit volume with subtracted two-body contributions or, equivalently, in terms of the on-shell vertex functions at zero momenta,

$$g_3 = \langle \mathrm{free}_3|\hat{V}|\mathrm{true}_3\rangle - 3\langle \mathrm{free}_2|\hat{V}|\mathrm{true}_2\rangle, \tag{6}$$

where $\hat{V}$ is the interaction term in Eq. (1), $|\mathrm{free}_n\rangle = (|+\rangle)^n$ is the ground state of n noninteracting bosons, and $|\mathrm{true}_n\rangle$ is the true zero energy n-body eigenstate of Eq. (1), normalized per unit volume. In two dimensions, due to the two-dimensional kinematics [39], the vertex functions should be considered at finite E. In our case the region where Eq. (6) remains approximately constant (thus defining g_3) is given by the inequalities $|g_2| \ll 1$ and $t\exp(-4\pi/|g_2|) \ll E \ll t$. In Fig. 3 (solid line) we show g_3 calculated at the point $g_2 = E = 0$ which belongs to this region. We see that g_3 is always repulsive and rather large. It reaches $g_{3,\mathrm{min}} \approx 1530$ at $r_* = 0.94$. In order to compute $|\mathrm{true}_3\rangle$ we solve the three-body version of Eq. (2) [40] by using the adiabatic hyperspherical approach [41,42].

The surprising enhancement of g_3 for small r_* can be understood from the ZR analysis. In the case $q = 0$ and $f_{\mathrm{zr}}(0) = 0$ the $(|+\rangle)^2$-term in Eq. (4) stands for two noninteracting bosons on a unit surface. The $(|-\rangle)^2$-term describes a "bound" pair of particles. Although $C = -2/\ln\sqrt{E_{\uparrow\uparrow}/E_{\uparrow\downarrow}}$ is small, the pair has an exponentially large size $\sim 1/\sqrt{t_c}$. The effective three-body force originates from the interaction of the third boson (in state $|+\rangle$) with either of these two particles, the collision energy being $\sim t_c$. Due to the bosonic symmetry the dominant s-wave interaction between $|+\rangle$- and $|-\rangle$-particles is given by the repulsive $V_{\uparrow\uparrow}$, which, at these energies, can be substituted by the corresponding vertex part $\approx 4\pi/\ln(E_{\uparrow\uparrow}/t_c) \approx 4\pi/\ln\sqrt{E_{\uparrow\uparrow}/E_{\uparrow\downarrow}}$. Overall, we obtain $g_3 \propto t_c^{-1}(\ln\sqrt{E_{\uparrow\uparrow}/E_{\uparrow\downarrow}})^{-3}$, in which the factor $1/t_c$ provides the main (exponential) dependence on r_*. In a more rigorous perturbation theory [40] the two leading terms in the expansion of g_3 in the small parameter $\xi = 1/\ln\sqrt{E_{\uparrow\uparrow}/E_{\uparrow\downarrow}}$ read $g_{3,\mathrm{zr}} = (24\pi^2/t_c)[\xi^3 - 3\ln(4/3)\xi^4 + \ldots]$ (dashed line in Fig. 3).

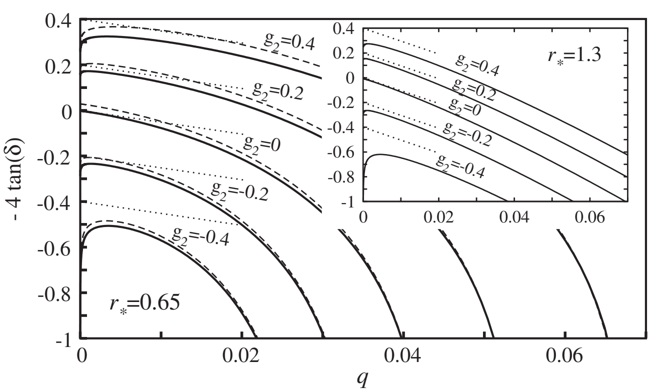

FIG. 2. The function $-4\tan\delta$ vs. the collision momentum q (solid lines) for $r_* = 0.65$ for five values of g_2: $g_2 = 0.4$ ($t = 2.4t_c$), $g_2 = 0.2$ ($t = 1.5t_c$), $g_2 = 0$ ($t = t_c = 5\times10^{-4}$), $g_2 = -0.2$ ($t = 0.65t_c$), and $g_2 = -0.4$ ($t = 0.42t_c$). We also plot the approximations $-4\tan\delta \approx -4\tan\delta_{\mathrm{zr}} - 8r_*q$ (dashed lines) and $-4\tan\delta \approx g_2 - 8r_*q$ (dotted lines). Inset shows the case $r_* = 1.3$ for the same set of g_2, the tunneling amplitudes are $t = 1.045t_c$, $t = 1.023t_c$, $t = t_c = 0.05$, $t = 0.977t_c$, and $t = 0.953t_c$.

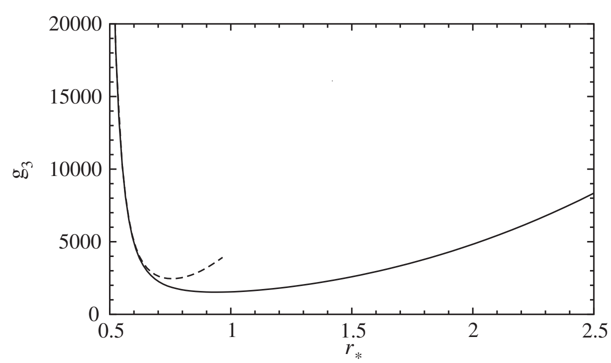

FIG. 3. The three-body coupling constant g_3 (solid) vs. r_* for $t = t_c(r_*)$, i.e., for vanishing effective two-body interaction. The dashed line shows the result of the zero-range approximation (see text).

That g_3 stays positive for all r_* is a manifestation of the three-body frustration: at least two particles are on the same layer and, therefore, experience the repulsive potential $V_{\uparrow\uparrow}$. We claim that this mechanism protects the system against collapse and is responsible for a microscopic suppression of the local three-body correlation function, which clearly means suppressed three-body losses. It has been predicted [43,44] and observed [45] that inelastic processes can be suppressed by tightly confining dipoles to the two-dimensional geometry. Here we argue that the bilayer case with tunneling provides control over the interparticle interaction while preserving this suppression mechanism.

For RbK molecules with the interlayer distance $\lambda = 532$ nm the energy unit $\hbar^2/m\lambda^2$ corresponds to 280 Hz or 13.4 nK, which is rather small. However, for lighter molecules and shorter lattice periods [46] one can gain an order of magnitude or more. It is also useful to keep in mind that similarly to spinor condensates the temperature requirement is rather loose [47], more important is that the BEC chemical potential is smaller than $\hbar^2/m\lambda^2$, which is quite realistic.

The Hamiltonian (1) in the rotating wave approximation also describes an atomic gas in which two hyperfine states ($\uparrow$ and $\downarrow$) are coupled by a resonant microwave or radio frequency field with the Rabi frequency $2t$. Here we can have significantly larger t than in the bilayer dipolar case [48]. However, in order to see three-body effects one has to control the three scattering lengths $a_{\uparrow\uparrow}$, $a_{\downarrow\downarrow}$, and $a_{\uparrow\downarrow} = a_{\downarrow\uparrow}$. The condition $a_{\uparrow\uparrow} = a_{\downarrow\downarrow}$ is not necessary, but for simplicity let us assume that it holds.

An advantage of the "atomic" method is that the ZR approximation is essentially exact. In three dimensions we obtain [40]

$$f^{(3\mathrm{D})}(q) = -\left[\frac{2 - (a_{\uparrow\uparrow} + a_{\uparrow\downarrow})\kappa}{a_{\uparrow\uparrow} + a_{\uparrow\downarrow} - 2a_{\uparrow\uparrow}a_{\uparrow\downarrow}\kappa} + iq\right]^{-1}. \quad (7)$$

The effective scattering length vanishes for $\sqrt{t_c} = (1/a_{\uparrow\uparrow} + 1/a_{\uparrow\downarrow})/4$. The three-body coupling constant can be calculated analytically for small $\xi = (a_{\uparrow\downarrow} + a_{\uparrow\uparrow})/(a_{\uparrow\downarrow} - a_{\uparrow\uparrow})$. Exactly at t_c the leading term reads $g_3 \approx 3\pi^2\xi^3/t_c^2 \approx 48\pi^2 a_{\uparrow\uparrow}^4/\xi$ [40]. This is by $1/\xi \gg 1$ larger than the usual three-body scaling $\propto a^4$ which appears only in the next order and, in particular, contains an imaginary part. Since it gives an estimate for the three-body recombination rate, the ratio of elastic to inelastic three-body interaction is $\propto 1/\xi$. Note, that in order to have a strong elastic three-body repulsion the scattering lengths are not required to be large. However, they should be very close in absolute values and have opposite signs: $a_{\uparrow\uparrow} > 0$ and $a_{\uparrow\downarrow} = -a_{\uparrow\uparrow}(1 + 2\xi)$. Lysebo and Veseth [49] predict rich possibilities for tuning interactions in between various hyperfine states of ^{39}K. In particular, states $F = 1$, $m_F = 0$ and $F = 1$, $m_F = -1$ in the magnetic field region from 50 to 60 Gauss can potentially give the desired effect, but one has to generalize the above theory to the case $a_{\uparrow\uparrow} \neq a_{\downarrow\downarrow}$.

In the two-dimensional case we can use the ZR formulas presented for the bilayer geometry with $E_{\sigma\sigma'} = B/(\pi l_0^2)\exp(\sqrt{2\pi}l_0/a_{\sigma\sigma'})$ [50], where l_0 is the transverse oscillator length and $B \approx 0.9$. Here the restriction on possible values of $a_{\sigma\sigma'}$ is softer than in three dimensions. It originates from the requirement to stay in the two-dimensional regime, $t_c \ll 1/l_0^2$, and reads $\exp[\sqrt{\pi/2}l_0(1/a_{\uparrow\uparrow} + 1/a_{\uparrow\downarrow})] \ll 1$.

The one-dimensional case can be realized by strongly confining atoms in two radial directions (or, for dipoles, in the bitube geometry). Introducing the one-dimensional scattering lengths $a_{1,\sigma\sigma'} = -l_0^2/a_{\sigma\sigma'}(1 - \mathcal{C}' a_{\sigma\sigma'}/l_0)$, where $\mathcal{C}' \approx 1.0326$ [51], from Eq. (2) we obtain [40]

$$f^{(1\mathrm{D})}(q) = -\left[1 + iq\frac{a_{1,\uparrow\uparrow} + a_{1,\uparrow\downarrow} - 2a_{1,\uparrow\uparrow}a_{1,\uparrow\downarrow}\kappa}{2 - (a_{1,\uparrow\uparrow} + a_{1,\uparrow\downarrow})\kappa}\right]^{-1}. \quad (8)$$

The two-body coupling constant vanishes for $\sqrt{t_c} = 1/(a_{1,\uparrow\uparrow} + a_{1,\uparrow\downarrow})$. At this point the three-body problem is equivalent to a two-dimensional one-body scattering by a potential with the range $\sim 1/\sqrt{t_c}$. The three-body interaction is conservative and repulsive for $a_{1,\uparrow\uparrow} < 0$ reaching its maximum at $a_{1,\uparrow\uparrow} = 0$ where the corresponding two-dimensional scattering length is of order $1/\sqrt{t_c}$. On the side $a_{1,\uparrow\uparrow} > 0$ the interaction is stronger but is not conservative due to the recombination to dimer states of size $\sim a_{1,\uparrow\uparrow}$ [40].

We now raise a few questions for future studies. Our intuition built on two-body interacting BECs may have to be reconsidered when three-body forces become important; fragmented condensation, rotonization, or (super)solidity of a three-body interacting gas with or without dipolar tails can not be *a priori* ruled out. Starting from the dilute droplet for $g_2 < 0$ and $g_3 > 0$ [7] and further increasing the two-body attraction, will the system just shrink or eventually form a paired state? What is the interaction between pairs? In two dimensions, when g_2 is small there are two tetramers of exponentially large size [52], the local three-body interaction being a small perturbation. On the other hand, in the bilayer case with $r_* \gtrsim 1$ and $t = 0$ the dimers are rather deeply bound and it is reasonable to assume that they repel each other. If and how the two tetramer states go into the dimer-dimer continuum with increasing $|g_2|$ remains an interesting few-body problem [53]. Note that with the presented method we can also independently control the range of the three-body effective interaction, which can be important, in particular, for quantum Hall problems [17]. Finally, the bilayer setup can be considered as a limiting case of a multilayer one in which the chemical potentials of two layers are significantly lowered. One can consider N layers with generally different chemical potentials and thus try to engineer quite exotic effective few-body interactions (for example, $g_2 = g_3 = 0$ and finite g_4). However, for $N \geq 3$ one should also be aware of inelastic three-body processes.

We thank N. Cooper, O. Dulieu, T. Jolicoeur, G. Shlyapnikov, and M. Zaccanti for fruitful discussions and acknowledge support by the IFRAF Institute.

[1] C. Chin, R. Grimm, P. S. Julienne, and E. Tiesinga, Rev. Mod. Phys. **82**, 1225 (2010).

[2] *BCS-BEC Crossover and the Unitary Fermi Gas*, Lecture Notes in Physics, edited by W. Zwerger (Springer, Berlin, 2011), Vol. 836.

[3] D. S. Petrov, C. Salomon, and G. V. Shlyapnikov, Phys. Rev. Lett. **93**, 090404 (2004); Phys. Rev. A **71**, 012708 (2005).

[4] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).

[5] S. Inouye, M. R. Andrews, J. Stenger, H.-J. Miesner, D. M. Stamper-Kurn, and W. Ketterle, Nature (London) **392**, 151 (1998); J. Stenger, S. Inouye, M. R. Andrews, H.-J. Miesner, D. M. Stamper-Kurn, and W. Ketterle, Phys. Rev. Lett. **82**, 2422 (1999).

[6] J. L. Roberts, N. R. Claussen, S. L. Cornish, and C. E. Wieman, Phys. Rev. Lett. **85**, 728 (2000).

[7] A. Bulgac, Phys. Rev. Lett. **89**, 050402 (2002).

[8] Note the difference with the two-dimensional droplet discussed by H.-W. Hammer and D. T. Son, Phys. Rev. Lett. **93**, 250408 (2004) and with the one-dimensional bright soliton studied by V. E. Zakharov and A. B. Shabat, Sov. Phys. JETP **34**, 62 (1972). They exist when $g_3 = 0$, but their sizes exponentially decrease with the particle number, i.e., in the thermodynamic limit they essentially collapse.

[9] For a discussion of trapped BECs with cubic-quintic nonlinearities, see A. Gammal, T. Federico, L. Tomio, and P. Chomaz, J. Phys. B **33**, 4053 (2000) and references therein.

[10] L. Radzihovsky, J. Park, and P. B. Weichman, Phys. Rev. Lett. **92**, 160402 (2004); L. Radzihovsky, P. B. Weichman, and J. I. Park, Ann. Phys. (Amsterdam) **323**, 2376 (2008).

[11] M. W. J. Romans, R. A. Duine, S. Sachdev, and H. T. C. Stoof, Phys. Rev. Lett. **93**, 020405 (2004).

[12] Y. -W. Lee and Y. -L. Lee, Phys. Rev. B **70**, 224506 (2004).

[13] S. Basu and E. J. Mueller, Phys. Rev. A **78**, 053603 (2008).

[14] P. Nozieres and D. Saint James, J. Phys. (Paris) **43**, 1133 (1982).

[15] M. Greiter, X.-G. Wen, and F. Wilczek, Phys. Rev. Lett. **66**, 3205 (1991).

[16] G. Moore and N. Read, Nucl. Phys. **B360**, 362 (1991).

[17] A. Wójs, C. Tőke, and J. K. Jain, Phys. Rev. Lett. **105**, 196801 (2010).

[18] N. Read and D. Green, Phys. Rev. B **61**, 10267 (2000).

[19] H. P. Büchler, A. Micheli, and P. Zoller, Nat. Phys. **3**, 726 (2007).

[20] A. J. Daley, J. M. Taylor, S. Diehl, M. Baranov, and P. Zoller, Phys. Rev. Lett. **102**, 040402 (2009).

[21] L. Mazza, M. Rizzi, M. Lewenstein, and J. I. Cirac, Phys. Rev. A **82**, 043629 (2010).

[22] K. W. Mahmud and E. Tiesinga, Phys. Rev. A **88**, 023602 (2013).

[23] M. Hafezi, P. Adhikari, and J. M. Taylor, arXiv:1308.0225.

[24] A. J. Daley, and J. Simon, arXiv:1311.1783.

[25] V. Efimov, Sov. J. Nucl. Phys. **29**, 546 (1979).

[26] E. Braaten, H.-W. Hammer, and T. Mehen, Phys. Rev. Lett. **88**, 040401 (2002).

[27] Near a Feshbach resonance a three-body force can originate from the interaction of an atom with a closed-channel molecule, but this interaction is short ranged and can become resonant only accidentally. Nevertheless, the importance of this effect for the bosonic pairing problem has been pointed out in Ref. [13] and for quantum Hall problems in N. R. Cooper, Phys. Rev. Lett. **92**, 220405 (2004).

[28] F. Ferlaino, A. Zenesini, M. Berninger, B. Huang, H.-C. Nägerl, and R. Grimm, Few-Body Syst. **51**, 113 (2011).

[29] M. Roncaglia, M. Rizzi, and J. I. Cirac, Phys. Rev. Lett. **104**, 096803 (2010).

[30] B. Simon, Ann. Phys. (N.Y.) **97**, 279 (1976).

[31] S.-M. Shih and D.-W. Wang, Phys. Rev. A **79**, 065603 (2009).

[32] J. R. Armstrong, N. T. Zinner, D. V. Fedorov, and A. S. Jensen, Europhys. Lett. **91**, 16001 (2010).

[33] M. Klawunn, A. Pikovski, and L. Santos, Phys. Rev. A **82**, 044701 (2010).

[34] M. A. Baranov, A. Micheli, S. Ronen, and P. Zoller, Phys. Rev. A **83**, 043602 (2011).

[35] D.-W. Wang, Phys. Rev. Lett. **98**, 060403 (2007).

[36] We restrict ourselves to the case of a single bound state.

[37] C. Ticknor, Phys. Rev. A **80**, 052702 (2009).

[38] In our calculations we use numerically exact $E_{\uparrow\downarrow}(r_*)$.

[39] The wave functions $|\text{true}_2\rangle$ and $|\text{true}_3\rangle$ vanish in the formal zero energy limit for any finite distance between two particles [check, for example, Eq. (4) in the limit $q \to 0$]. Equation (6) then gives zero similarly to Eq. (3) at $E = 0$.

[40] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.112.103201 for more information on the three-body Schrödinger equation, on its solution in the dipolar bilayer case, and on the zero-range approach in all dimensions.

[41] C. D. Lin, Phys. Rep. **257**, 1 (1995).

[42] E. Nielsen, D. V. Fedorov, A. S. Jensen, and E. Garrido, Phys. Rep. **347**, 373 (2001).

[43] G. Quéméner and J. L. Bohn, Phys. Rev. A **81**, 060701 (2010); **83**, 012705 (2011).

[44] A. Micheli, Z. Idziaszek, G. Pupillo, M. A. Baranov, P. Zoller, and P. S. Julienne, Phys. Rev. Lett. **105**, 073202 (2010).

[45] M. H. G. de Miranda, A. Chotia, B. Neyenhuis, D. Wang, G. Quéméner, S. Ospelkaus, J. L. Bohn, J. Ye, and D. S. Jin, Nat. Phys. **7**, 502 (2011).

[46] For more details on optical control of molecules, see R. Vexiau, Ph. D. thesis, University Paris-Sud, 2012, available at http://hal.archives-ouvertes.fr/tel-00783399/.

[47] D. M. Stamper-Kurn and M. Ueda, Rev. Mod. Phys. **85**, 1191 (2013).

[48] M. Zaccanti (private communication).

[49] M. Lysebo and L. Veseth, Phys. Rev. A **81**, 032702 (2010).

[50] D. S. Petrov and G. V. Shlyapnikov, Phys. Rev. A **64**, 012706 (2001).

[51] M. Olshanii, Phys. Rev. Lett. **81**, 938 (1998).

[52] L. Platter, H.-W. Hammer, U.-G. Meißner, Few-Body Syst. **35**, 169 (2004).

[53] Contrary to the conclusion of A. G. Volosniev, D. V. Fedorov, A. S. Jensen, and N. T. Zinner, Phys. Rev. A **85**, 023609 (2012), we believe that trimers and tetramers of identical dipoles in bilayers without tunneling do exist for small r_*.