

Supplemental material: Three-body-interacting bosons in free space

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THREE-BODY SCHRÖDINGER EQUATION

In this section we derive the three-body Schrödinger equation, discuss its solution in the adiabatic hyperspherical approximation, and comment on the 2D kinematic problem and on the case of finite g_2 and E .

The three sets of rescaled Jacobi coordinates $\mathbf{\Pi}_i = \{\mathbf{x}_i, \mathbf{y}_i\}$, $i = 1, 2, 3$, are defined by

$$\begin{aligned}\mathbf{x}_i &= (2\mathbf{r}_i - \mathbf{r}_j - \mathbf{r}_k)/\sqrt{3}, \\ \mathbf{y}_i &= \mathbf{r}_k - \mathbf{r}_j,\end{aligned}\quad (\text{S1})$$

where $\{i, j, k\}$ are cyclic permutations of $\{1, 2, 3\}$. One can switch from one set to another by using the formulas

$$\begin{aligned}\mathbf{x}_1 &= -\mathbf{x}_2/2 + \sqrt{3}\mathbf{y}_2/2, \\ \mathbf{y}_1 &= -\sqrt{3}\mathbf{x}_2/2 - \mathbf{y}_2/2,\end{aligned}$$

and

$$\begin{aligned}\mathbf{x}_1 &= -\mathbf{x}_3/2 - \sqrt{3}\mathbf{y}_3/2, \\ \mathbf{y}_1 &= \sqrt{3}\mathbf{x}_3/2 - \mathbf{y}_3/2.\end{aligned}$$

We look for the three-body wavefunction in the form

$$\begin{aligned}|true_3\rangle &= (|\uparrow\rangle|\uparrow\rangle|\uparrow\rangle + |\downarrow\rangle|\downarrow\rangle|\downarrow\rangle)\phi_0(\mathbf{\Pi}_1) \\ &+ (|\uparrow\rangle|\downarrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle|\uparrow\rangle)\phi_1(\mathbf{\Pi}_1) \\ &+ (|\downarrow\rangle|\uparrow\rangle|\downarrow\rangle + |\uparrow\rangle|\downarrow\rangle|\uparrow\rangle)\phi_2(\mathbf{\Pi}_1) \\ &+ (|\downarrow\rangle|\downarrow\rangle|\uparrow\rangle + |\uparrow\rangle|\uparrow\rangle|\downarrow\rangle)\phi_3(\mathbf{\Pi}_1),\end{aligned}$$

where the bosonic symmetry requires

$$\phi_0(\mathbf{\Pi}_1) = \phi_0(\mathbf{\Pi}_2) = \phi_0(\mathbf{\Pi}_3) \quad (\text{S2})$$

and

$$\phi_i(\mathbf{\Pi}_1) = \phi_1(\mathbf{\Pi}_i), \quad i = 2, 3. \quad (\text{S3})$$

The Schrödinger equation for ϕ_0 then reads

$$[-\nabla_{\mathbf{\Pi}_1}^2 + 4t - E + \sum_{j=1}^3 V_{\uparrow\uparrow}(y_j)]\phi_0 - t \sum_{j=0}^3 \phi_j = 0 \quad (\text{S4})$$

and for $i \neq 0$

$$[-\nabla_{\mathbf{\Pi}_1}^2 + 4t - E + V_{\uparrow\uparrow}(y_i) + \sum_{j \neq i} V_{\uparrow\downarrow}(y_j)]\phi_i - t \sum_{j=0}^3 \phi_j = 0. \quad (\text{S5})$$

For zero total angular momentum the configurational space of the problem is three-dimensional: the three-body wavefunction depends only on the hyperradius

$\Pi = |\mathbf{\Pi}|$ and two (hyper)angles θ and ϕ defined by $x_1 = \Pi \cos(\theta/2)$, $y_1 = \Pi \sin(\theta/2)$, and ϕ is the angle between \mathbf{x}_1 and \mathbf{y}_1 . Then the kinetic energy operator reads

$$-\nabla_{\mathbf{\Pi}_1}^2 = -\frac{\partial^2}{\partial \Pi^2} - \frac{3}{\Pi} \frac{\partial}{\partial \Pi} - \frac{4}{\Pi^2} \left(\frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right). \quad (\text{S6})$$

Note that the hyperangular part is nothing else than the usual angular kinetic energy operator in 3D where θ and ϕ are polar and azimuthal angles, respectively. The fundamental hyperangular domain is $1/12$ of the θ - ϕ sphere defined by the inequalities $0 < \phi < \pi/2$ and $\theta < \arcsin(1/\sqrt{1 + \cos^2 \phi})$. The boundary conditions for the wavefunction on the borders of the domain can be derived from Eqs. (S2-S3).

The set of four equations (S4-S5) is solved numerically by using the adiabatic hyperspherical approach. Namely, we first diagonalize the hyperspherical part, i.e., we fix Π and in the kinetic energy operator (S6) we keep only the second line. We choose a grid which has many points at small θ to allow for a better accuracy at large Π . Then, we solve the coupled channel hyperradial problem keeping typically 40 hyperspherical channels. However, we find that for our parameters 10 channels is sufficient for convergence.

The calculated $|true_3\rangle$ is then substituted into

$$g_3 = \langle free_3 | \hat{V} | true_3 \rangle - 3 \langle free_2 | \hat{V} | true_2 \rangle. \quad (\text{S7})$$

In principle, at $t = t_c$ the two-body contribution vanishes. However, let us mention an important technical detail. The brackets $\langle \dots \rangle$ in the three-body term in Eq. (S7) imply integration over the whole three-body configurational space. In practice, we deal with integrals of the type $\int_{\Pi < \Pi_{\max}} d^4 \Pi$ which, in fact, do not converge at $\Pi_{\max} \rightarrow \infty$ since the missing contribution from large hyperradii remains important for any Π_{\max} because of the $1/y_i^3$ tails of the dipole-dipole potentials. The problem can, in principle, be cured analytically. However, a simpler solution is to treat the three- and two-body terms in Eq. (S7) on equal footing, i.e., when calculating the two-body contribution we use the same region of integration $\Pi < \Pi_{\max}$ by introducing a third dummy particle, not interacting with the first two. In fact, as an additional consistency check we solve this three-body problem (with a dummy particle) by using the same machinery developed for the three-body case although with different symmetry requirements for the wavefunction. We can then double check the accuracy of the three-body numerical scheme

since the two-body wavefunction is known very precisely from an ordinary “two-body” calculation.

Finally, let us discuss in more detail the case $t \neq t_c$, i.e., finite g_2 . For the sake of simplicity and readability we use the language of coupling constants, i.e., coefficients in front of powers of density in the low-density expansion of the energy functional. In 3D these coupling constants are the zero energy limits of the corresponding well-behaving vertex functions. Unfortunately, the formal zero energy limits of low-dimensional vertex functions are not physically reasonable. A significant part of the main text is sacrificed for sorting out this issue in the 2D case (the 1D case is discussed below in this Supplemental Material). For any finite two-body interaction the two-body wavefunction vanishes at finite interparticle distances in the limit $E \rightarrow 0$ because of the normalization condition. Therefore, strictly speaking, in the limit $E \rightarrow 0$ three interacting particles never approach each other to distances where they can feel the three-body potential. However, if E is larger than $\propto \exp(-4\pi/|g_2|)$, the normalization is no longer a problem, the particles do approach each other, and the three-body vertex function saturates to a finite value and then only weakly depends on E . We calculate the three-body wavefunction numerically for various values of t and E and see no dramatic changes in its shape as one crosses the point $g_2 = 0$ (apart from expected changes related to the finite collision momentum). The results are consistent with the fact that the three-body effective interaction acts at distances (hyperradii) $\sim 1/\sqrt{t}$ whereas the system feels the two-body interaction only at distances $\propto \exp(2\pi/|g_2|)$. Therefore, one can explicitly introduce g_3 as a “constant” in the region defined by the inequalities $|g_2| \ll 1$ and $t \exp(-4\pi/|g_2|) \ll E \ll t$. Note that this region includes the point $g_2 = E = 0$.

THREE-BODY PROBLEM IN THE ZERO-RANGE APPROXIMATION. 2D CASE

In this section we derive the Skorniakov and Ter-Martirosian (STM) equations for the 2D four-channel three-body problem in the zero-range approximation (see Ref. [1] for a general introduction to this method). We then solve the STM equations and obtain the three-body coupling constant $g_{3,zr}$ by using a systematic expansion

with respect to the small parameter $\xi = 1/\ln \sqrt{E_{\uparrow\uparrow}/E_{\uparrow\downarrow}}$.

In the zero-range approximation the interaction potentials $V_{\uparrow\uparrow}$ and $V_{\uparrow\downarrow}$ in Eqs. (S4-S5) are substituted by boundary conditions on the wavefunctions ϕ_i . Namely, we have for $i = 1, 2, 3$

$$\phi_0|_{y_i \rightarrow 0} \propto \ln(\sqrt{E_{\uparrow\uparrow}} y_i e^\gamma / 2), \quad (\text{S8})$$

$$\phi_i|_{y_i \rightarrow 0} \propto \ln(\sqrt{E_{\uparrow\uparrow}} y_i e^\gamma / 2), \quad (\text{S9})$$

$$\phi_i|_{y_{j \neq i} \rightarrow 0} \propto \ln(\sqrt{E_{\uparrow\downarrow}} y_j e^\gamma / 2). \quad (\text{S10})$$

The proportionality symbols are understood in the sense that, for example, Eq. (S8) fixes the ratio $A_2/A_1 = \ln(\sqrt{E_{\uparrow\uparrow}} e^\gamma / 2)$ in the small- y_1 expansion $\phi_0 \approx A_1 \ln(y_1) + A_2$.

By introducing a new set of wavefunctions $\{\psi_i\}$

$$\begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \quad (\text{S11})$$

we write Eqs. (S4-S5) in the form

$$(-\nabla_{\mathbf{\Pi}_1}^2 - E)\psi_0 = \sum_{j=1}^3 f_0(\mathbf{x}_j)\delta(\mathbf{y}_j) \quad (\text{S12})$$

and, for $i \neq 0$,

$$(-\nabla_{\mathbf{\Pi}_1}^2 + 4t - E)\psi_i = f_1(\mathbf{x}_i)\delta(\mathbf{y}_i) + \sum_{j \neq i} f_2(\mathbf{x}_j)\delta(\mathbf{y}_j). \quad (\text{S13})$$

The functions f_0 , f_1 , and f_2 are defined by

$$f_i(\mathbf{x}_1) = -2\pi \lim_{y_1 \rightarrow 0} \left[y_1 \frac{\partial \psi_i(\mathbf{x}_1, \mathbf{y}_1)}{\partial y_1} \right], \quad i = 0, 1, 2. \quad (\text{S14})$$

One can check that the right hand sides in Eqs. (S12-S13) together with the definition (S14) allow for the appearance of log-type singularities of ψ at small y_i consistent with the symmetry conditions (S2-S3) (the sets $\{\psi_i\}$ and $\{\phi_i\}$ satisfy the same symmetry requirements).

We now invert Eqs. (S12-S13) and express ψ_i as functionals of f_i :

$$\psi_0(\mathbf{\Pi}_1) = \psi_{00}(\mathbf{\Pi}_1) + \sum_{j=1}^3 \int G_E[\sqrt{(\mathbf{x}_j - \mathbf{x})^2 + y_j^2}] f_0(\mathbf{x}) d^2x, \quad (\text{S15})$$

$$\psi_i(\mathbf{\Pi}_1) = \int G_{E-4t}[\sqrt{(\mathbf{x}_i - \mathbf{x})^2 + y_i^2}] f_1(\mathbf{x}) d^2x + \sum_{j \neq i} \int G_{E-4t}[\sqrt{(\mathbf{x}_j - \mathbf{x})^2 + y_j^2}] f_2(\mathbf{x}) d^2x, \quad (\text{S16})$$

where ψ_{00} is a general (properly symmetrized) solution of the homogeneous Eq. (S12). It plays the role of the incoming (free) wave. The Green function G_E is the solution of $(-\nabla_{\mathbf{II}}^2 - E)G_E(\mathbf{II}) = \delta(\mathbf{II})$ corresponding to the outgoing wave (for $E > 0$). We do not have free terms in Eq. (S16) since we assume $E < 4t$. In fact, in the four-dimensional \mathbf{II} -space the wavefunctions $\psi_{i \neq 0}$ are localized in the areas defined by $|y_j| \lesssim 1/\sqrt{4t - E}$, i.e., they correspond to the virtually excited “bound” pairs (see discussion in the main text). Therefore, the outgoing scattered wave is given solely by the sum in the right hand side of Eq. (S15).

In the case $E = 0$ and $t = t_c = \sqrt{E_{\uparrow\uparrow}E_{\uparrow\downarrow}}/4$ the two-body scattering is absent, the three-body problem is equivalent to the 4D scattering by a finite-range potential, and $\psi_{00}(\mathbf{II}_1) \equiv \psi_{00} = \text{const.}$ The wavefunction ψ_0 at large hyperradii reads

$$\psi_0 = \psi_{00} + 3G_0(\mathbf{II}_1) \int f_0(\mathbf{x})d^2x = \psi_{00} + 3 \frac{\int f_0(\mathbf{x})d^2x}{4\pi^2\Pi_1^2} \quad (\text{S17})$$

and the three-body coupling constant equals

$$g_3 = -(9/4) \int f_0(\mathbf{x})d^2x/\psi_{00}. \quad (\text{S18})$$

Equation (S18) can be derived from Eq. (S17) by switching from the potential to kinetic energy operators in the definition $g_3 = \langle free_3 | \hat{V} | true_3 \rangle$ and then using the Gauss-Ostrogradsky theorem which allows one to express g_3 in the form of an integral of $\nabla_{\mathbf{II}_1} \psi_0$ over a four-dimensional sphere with infinitely large radius where we substitute the asymptote (S17). One should also take into account an additional factor 3/4 which comes from the Jacobian determinant of the coordinate transformation (S1). Alternatively, one can simply note that the asymptote (S17) is the zero energy eigenstate of the four-dimensional problem of scattering by the pseudopotential U_3 defined by

$$U_3\psi = g_3\delta(\sqrt{3}\mathbf{x}_1/2)\delta(\mathbf{y}_1)\text{Reg}\psi, \quad (\text{S19})$$

where $\text{Reg}\psi = \partial[\Pi_1^2\psi(\mathbf{II}_1)]/\partial\Pi_1^2|_{\Pi_1 \rightarrow 0}$.

The function f_0 is calculated in the following manner. We first obtain the functions ϕ_i by inverting Eq. (S11),

$$\begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & 3 & -1 & -1 \\ 1 & -1 & 3 & -1 \\ 1 & -1 & -1 & 3 \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad (\text{S20})$$

and substituting Eqs. (S15-S16) into the right hand side of Eq. (S20). Coupled integral equations for f_i (STM equations) are then derived by applying the boundary conditions (S8-S10). As we explain in the main text and in the end of the previous section of this Supplemental Material, if $t \neq t_c$, one has to work with finite energies in

order to avoid the 2D kinematic issue. Here for simplicity we present the final equations for the particular case $t = t_c = \sqrt{E_{\uparrow\uparrow}E_{\uparrow\downarrow}}/4$ where we can safely set $E = 0$. It is also convenient to switch from $f_i(\mathbf{x})$ to the Fourier transformed functions $\tilde{f}_i(\mathbf{p}) = \int f_i(\mathbf{x}) \exp(-i\sqrt{4t_c}\mathbf{p}\mathbf{x})d^2x$, where we measure the momentum in units of $\sqrt{4t_c}$. Then the STM equations read

$$\tilde{f}_0 = \xi[\hat{L}_{-1}\tilde{f}_1 - \ln(1+p^2)(2\tilde{f}_2 - \tilde{f}_1)], \quad (\text{S21})$$

$$\tilde{f}_1 = \xi[-\hat{L}_{-1}\tilde{f}_2 + \ln(1+p^2)\tilde{f}_1], \quad (\text{S22})$$

$$2\tilde{f}_2 - \tilde{f}_1 = \xi(\hat{L}_0 - 2\ln p)\tilde{f}_0 + \xi(4\pi^3\psi_{00}/t_c)\delta(\mathbf{p}), \quad (\text{S23})$$

where the integral operator \hat{L}_ϵ is defined by

$$\hat{L}_\epsilon\tilde{f}(\mathbf{p}) = \frac{2}{\pi} \int \frac{\tilde{f}(\mathbf{k})d^2k}{p^2 + k^2 + \mathbf{p}\mathbf{k} - 3\epsilon/4} \quad (\text{S24})$$

and $\xi = 1/\ln\sqrt{E_{\uparrow\uparrow}/E_{\uparrow\downarrow}}$. In the dipolar case this parameter is small as required for the validity of the zero-range approximation. Accordingly, we have arranged different terms in Eqs. (S21-S23) for the following iterative procedure. Starting from the zeroth order $\tilde{f}_0^{(0)} = \tilde{f}_1^{(0)} = \tilde{f}_2^{(0)} = 0$ the next order approximation is obtained by substituting the previous one into the right hand side of Eqs. (S21-S23). In particular, the first order approximation is $\tilde{f}_0^{(1)} = \tilde{f}_1^{(1)} = 0$, $\tilde{f}_2^{(1)} = \xi(2\pi^3\psi_{00}/t_c)\delta(\mathbf{p})$, the second - $\tilde{f}_0^{(2)} = 0$, $\tilde{f}_1^{(2)} = 2\tilde{f}_2^{(2)} = -\xi^2(4\pi^2\psi_{00}/t_c)/(p^2 + 3/4)$, etc. By continuing this procedure up to the fourth iteration we obtain

$$\tilde{f}_0(0) = -(32\pi^2\xi^3\psi_{00}/3t_c)(1 - 3\ln(4/3)\xi + \dots), \quad (\text{S25})$$

from which we get $g_{3,zr}$ claimed in the paper.

Note that in contrast to the initial bilayer problem, the zero-range version of the purely repulsive potential $V_{\uparrow\uparrow}$ supports a bound state. In fact, this leads to two two-body bound states in our multi-channel model. One of them is in the spinor channel ($|\uparrow\rangle|\uparrow\rangle - |\downarrow\rangle|\downarrow\rangle$) which, on the two-body level, is decoupled from $(|+\rangle)^2$ and $(|-\rangle)^2$ and has been completely neglected because of the $\uparrow\downarrow$ symmetry of the Hamiltonian. Its binding energy is exactly $E_{\uparrow\uparrow}$. The second one corresponds to the pole of the zero-range scattering amplitude

$$f^{(2D)}(q) = \frac{2\pi}{\ln \frac{\kappa}{q} - \ln \frac{\kappa^2}{E_{\uparrow\downarrow}} \ln \frac{\kappa^2}{E_{\uparrow\uparrow}} / \ln \frac{\kappa^4}{E_{\uparrow\downarrow}E_{\uparrow\uparrow}} + i\frac{\pi}{2}}. \quad (\text{S26})$$

Its energy satisfies the relation

$$\ln(\epsilon/\sqrt{E_{\uparrow\downarrow}E_{\uparrow\uparrow}}) \ln(1 + \epsilon/\sqrt{E_{\uparrow\downarrow}E_{\uparrow\uparrow}}) = (\ln\sqrt{E_{\uparrow\uparrow}/E_{\uparrow\downarrow}})^2, \quad (\text{S27})$$

which for small ξ also gives $\epsilon \approx E_{\uparrow\uparrow}$. Formally, in the zero-range approximation both of these states can be populated by three-body recombination. The phenomenon is characterized by the appearance of poles

of functions f_i at momenta $p_0 \approx \exp(1/2\xi)$ and, consequently, by an imaginary contribution to $\tilde{f}_0(0)$ and g_3 . One can show that these imaginary parts are $\propto \exp(-1/2\xi)$ and, accordingly, never appear in the Taylor expansion (S25) [2]. In the ‘‘atomic’’ case these two states can also be spurious. For example, for positive and small $a_{\uparrow\uparrow} \ll l_0$ the actual two-body bound states are three-dimensional and their energies have nothing to do with the quantity $E_{\uparrow\uparrow} = B/(\pi l_0^2) \exp(\sqrt{2\pi}l_0/a_{\uparrow\uparrow})$, which is just a characteristic of the 2D two-body scattering at low collision energy. In this case the three-body recombination has the usual 3D scaling $\propto a_{\uparrow\uparrow}^4$ and is small.

3D CASE

In this section we generalize our zero-range theory to the 3D case.

The two-body wavefunction in the zero-range approximation reads

$$\Psi = (|+\rangle)^2 \left[\frac{\sin(qr)}{qr} + f^{(3D)}(q) \frac{e^{iqr}}{r} \right] + C(|-\rangle)^2 \frac{e^{-\kappa r}}{r}. \quad (\text{S28})$$

Consequently the functions $\phi_{\uparrow\uparrow}$ and $\phi_{\uparrow\downarrow}$ equal

$$\phi_{\uparrow\uparrow/\uparrow\downarrow}(r) = \frac{\sin(qr)}{qr} + f^{(3D)}(q) \frac{e^{iqr}}{r} \pm C \frac{e^{-\kappa r}}{r}. \quad (\text{S29})$$

The scattering amplitude

$$f^{(3D)}(q) = - \left[\frac{2 - (a_{\uparrow\uparrow} + a_{\uparrow\downarrow})\kappa}{a_{\uparrow\uparrow} + a_{\uparrow\downarrow} - 2a_{\uparrow\uparrow}a_{\uparrow\downarrow}\kappa} + iq \right]^{-1} \quad (\text{S30})$$

and coefficient

$$C = \frac{a_{\uparrow\downarrow} - a_{\uparrow\uparrow}}{2 - (a_{\uparrow\uparrow} + a_{\uparrow\downarrow})\kappa + iq(a_{\uparrow\uparrow} + a_{\uparrow\downarrow} - 2a_{\uparrow\uparrow}a_{\uparrow\downarrow}\kappa)} \quad (\text{S31})$$

are obtained by applying the zero-range boundary conditions

$$\phi_{\sigma\sigma'}|_{r \rightarrow 0} \propto 1 - a_{\sigma\sigma'}/r \quad (\text{S32})$$

to Eq. (S29).

The three-body approach of the previous sections of this Supplemental Material is modified as follows. The coordinates \mathbf{r} , \mathbf{x} , and \mathbf{y} are now three-dimensional and $\mathbf{\Pi}$ is six-dimensional. The dimension of the integrals should be changed accordingly. The boundary conditions (S8-S9) in the 3D case become

$$\phi_0|_{y_i \rightarrow 0} \propto 1 - a_{\uparrow\uparrow}/y_i, \quad (\text{S33})$$

$$\phi_i|_{y_i \rightarrow 0} \propto 1 - a_{\uparrow\uparrow}/y_i, \quad (\text{S34})$$

$$\phi_i|_{y_j \neq i \rightarrow 0} \propto 1 - a_{\uparrow\downarrow}/y_j, \quad (\text{S35})$$

and the functions f_i are defined by

$$f_i(\mathbf{x}_1) = 4\pi \lim_{y_1 \rightarrow 0} y_1 \psi_i(\mathbf{x}_1, \mathbf{y}_1), \quad i = 0, 1, 2. \quad (\text{S36})$$

An important simplification of the 3D case compared to the low-D ones is that Eq. (S7) is meaningful in the limit $E \rightarrow 0$ and the three-body interaction is clearly distinguished from the two-body one even when the latter is finite. In 3D we have

$$g_3 = -(9\sqrt{3}/8) \int [f_0(\mathbf{x}) - \lim_{\mathbf{x} \rightarrow \infty} f_0(\mathbf{x})] d^3x / \psi_{00}, \quad (\text{S37})$$

where the subtracted part proportional to $\lim_{\mathbf{x} \rightarrow \infty} f_0(\mathbf{x})$ is the (tripled) two-body contribution.

The STM equations are derived in the same manner as in the 2D case. Now we write them for $E = 0$ and for arbitrary t not necessarily equal to t_c , but we still measure momentum in units of $\sqrt{4t_c} = (1/a_{\uparrow\uparrow} + 1/a_{\uparrow\downarrow})/2$,

$$\tilde{f}_0 = \xi[\hat{L}_{-t/t_c} \tilde{f}_1 + (1 - \sqrt{t/t_c + p^2})(2\tilde{f}_2 - \tilde{f}_1)], \quad (\text{S38})$$

$$\tilde{f}_1 = \xi[-\hat{L}_{-t/t_c} \tilde{f}_2 - (1 - \sqrt{t/t_c + p^2})\tilde{f}_1], \quad (\text{S39})$$

$$2\tilde{f}_2 - \tilde{f}_1 = \xi(\hat{L}_0 + 1 - p)\tilde{f}_0 + \xi(2\pi^4 \psi_{00}/t_c^2)\delta(\mathbf{p}), \quad (\text{S40})$$

where $\xi = (a_{\uparrow\downarrow} + a_{\uparrow\uparrow})/(a_{\uparrow\downarrow} - a_{\uparrow\uparrow})$ and

$$\hat{L}_\epsilon \tilde{f}(\mathbf{p}) = \frac{2}{\pi^2 \sqrt{3}} \int \frac{\tilde{f}(\mathbf{k}) d^3k}{p^2 + k^2 + \mathbf{p}\mathbf{k} - 3\epsilon/4}. \quad (\text{S41})$$

For small ξ we can perform the same iterative procedure as in the 2D case up to the third iteration arriving at the leading order term

$$g_3 \approx 3\pi^2 \xi^3 / \sqrt{t_c^3 t}, \quad (\text{S42})$$

which for $t = t_c$ gives the expression presented in the main text. In contrast to the 2D case we can not continue the expansion further because momenta of order $p \sim 1/\xi$ become important starting from the next iteration. This renders the problem (S38-S40) non-perturbative: the terms proportional to $\xi^4/t_c^2 \propto a^4$ depend on the short-range Efimov physics, three-body parameters in the $\uparrow\uparrow\uparrow$ and $\uparrow\uparrow\downarrow$ three-body channels, etc. In particular, the imaginary part of g_3 is of order $\xi^4/t_c^2 \propto a^4$ and is related to the three-body recombination to weakly and deeply bound dimer states.

Finally, we note that by tuning $a_{\uparrow\downarrow}$, $a_{\uparrow\uparrow}$ and t one can have *any* effective scattering length

$$a_{\text{eff}} = \frac{a_{\uparrow\uparrow} + a_{\uparrow\downarrow} - 2a_{\uparrow\uparrow}a_{\uparrow\downarrow}\sqrt{4t}}{2 - (a_{\uparrow\uparrow} + a_{\uparrow\downarrow})\sqrt{4t}}. \quad (\text{S43})$$

In particular, $a_{\text{eff}} = \infty$ is obtained for $\sqrt{t} = 1/(a_{\uparrow\downarrow} + a_{\uparrow\uparrow})$. An interesting question then concerns the effective three-body parameter which governs the Efimov physics of the unitary Bose gas obtained in this manner. Since it depends on a number of parameters ($a_{\uparrow\downarrow}$, $a_{\uparrow\uparrow}$, and the three-body parameters in the $\uparrow\uparrow\uparrow$ and $\uparrow\uparrow\downarrow$ channels), it is likely that one can, in principle, control it. However, more important is whether one can significantly reduce its imaginary part, i.e., minimize the three-body relaxation rate. Unfortunately, we can not give an affirmative

answer. We have tried to set $a_{\uparrow\uparrow}$ to zero, thus eliminating also the three-body parameter in the $\uparrow\uparrow\uparrow$ channel. Then $a_{\uparrow\downarrow} = 1/\sqrt{t}$ sets the length scale: on longer length scales the system behaves as a usual spinless unitary Bose gas with the scaling factor ≈ 22.7 , on shorter length scales we deal with the Efimov physics in the $\uparrow\uparrow\downarrow$ channel, which is characterized by a quite different scaling factor ≈ 1986.1 . Clearly, by changing $a_{\uparrow\downarrow}$ (and \sqrt{t} accordingly) we can change the effective three-body parameter. However, we find that the effective inelasticity parameter can not be much reduced compared to the one for the $\uparrow\uparrow\downarrow$ channel.

1D CASE

Let us now discuss the 1D case in more detail. In the zero-range approximation the scattering happens only in the even (s -wave) channel. The two-body wavefunction reads

$$\Psi = (|+\rangle)^2 [\cos(qr) + f^{(1D)}(q)e^{iq|r|}] + C(|-\rangle)^2 e^{-\kappa|r|} \quad (\text{S44})$$

and the functions $\phi_{\uparrow\uparrow}$ and $\phi_{\uparrow\downarrow}$ equal

$$\phi_{\uparrow\uparrow/\uparrow\downarrow}(r) = \cos(qr) + f^{(1D)}(q)e^{iq|r|} \pm Ce^{-\kappa|r|}. \quad (\text{S45})$$

The scattering amplitude

$$f^{(1D)}(q) = - \left[1 + iq \frac{a_{1,\uparrow\uparrow} + a_{1,\uparrow\downarrow} - 2a_{1,\uparrow\uparrow}a_{1,\uparrow\downarrow}\kappa}{2 - (a_{1,\uparrow\uparrow} + a_{1,\uparrow\downarrow})\kappa} \right]^{-1} \quad (\text{S46})$$

and coefficient

$$C = \frac{iq(a_{\uparrow\uparrow} - a_{\uparrow\downarrow})}{2 - (a_{\uparrow\uparrow} + a_{\uparrow\downarrow})\kappa + iq(a_{\uparrow\uparrow} + a_{\uparrow\downarrow} - 2a_{\uparrow\uparrow}a_{\uparrow\downarrow}\kappa)} \quad (\text{S47})$$

are obtained by applying the boundary conditions

$$\left. \frac{\partial \phi_{\sigma\sigma'}}{\partial r} \right|_{r \rightarrow 0^+} - \left. \frac{\partial \phi_{\sigma\sigma'}}{\partial r} \right|_{r \rightarrow 0^-} = - \frac{2}{a_{1,\sigma\sigma'}} \phi_{\sigma\sigma'}(0) \quad (\text{S48})$$

to Eq. (S45). The scattering amplitude (S46) corresponds to the two-body scattering by the pseudopotential $g_2(q)\delta(r)$, which depends on the scattering momentum q ,

$$g_2(q) = -2 \frac{2 - (a_{1,\uparrow\uparrow} + a_{1,\uparrow\downarrow})\sqrt{4t - q^2}}{a_{1,\uparrow\uparrow} + a_{1,\uparrow\downarrow} - 2a_{1,\uparrow\uparrow}a_{1,\uparrow\downarrow}\sqrt{4t - q^2}}. \quad (\text{S49})$$

The approximation $g_2(q) = g_2(0) = g_2$ can be made for $q \ll \sqrt{t}$ if $(a_{1,\uparrow\uparrow} - a_{1,\uparrow\downarrow})/(a_{1,\uparrow\uparrow} + a_{1,\uparrow\downarrow})$ is not anomalously small. The zero crossing occurs at $t = t_c$, where $\sqrt{t_c} = 1/(a_{1,\uparrow\uparrow} + a_{1,\uparrow\downarrow})$. It is useful to note that at this point a finite momentum can not render the two-body interaction strong since the ratio $g_2(q)|_{t=t_c}/q$, which measures the ‘‘strength’’ of the interaction in 1D, is linear in q and, therefore, remains small for small finite q .

The configurational space $\mathbf{\Pi} = \{x, y\}$ of the 1D three-body problem is two-dimensional. The effective three-body potential acts at distances $\Pi \lesssim 1/\kappa \sim 1/\sqrt{t}$ and

originates from the interaction of the $(|-\rangle)^2$ -component of the wavefunction (S44) with the third particle in state $|+\rangle$. The effect of this finite range three-body interaction depends on the value of g_2 and the energy. For example, if $g_2 = 0$, the three-body scattering is equivalent to the 2D s -wave scattering by a finite range potential. The corresponding vertex function should scale as [3]

$$\Gamma_3(E) \approx 2\sqrt{3}\pi/[\ln(\epsilon_3/E) + i\pi] \quad (\text{S50})$$

for small E . When $g_2 = \infty$ (fermionized or Tonks gas), the scattering is equivalent to the 2D g -wave scattering (angular momentum $l = 3$) because the three-body wavefunction has six nodes on a circle drawn around the origin in the two-dimensional $\mathbf{\Pi}$ -space. In this case the threshold law for the scattering amplitude is $\propto E^3$. Clearly, for finite g_2 the kinematic regime of scattering is determined by the ratio g_2/\sqrt{E} . For *any* finite g_2 the system will fermionize at energies $E \lesssim g_2^2$. Nevertheless, we can neglect two-body interactions for $\Pi < 1/|g_2|$ and, if this radius is larger than the range of the three-body interaction $1/\sqrt{t}$, the quantity ϵ_3 is meaningful. It is a continuous function of $t - t_c$ close to the zero crossing and can be used, for example, to construct a local three-body pseudopotential. Moreover, it characterizes the actual three-body interaction as opposed to the kinematic effects that happen at distances $\Pi \sim 1/|g_2|, 1/\sqrt{E}$ where the structure of the three-body wavefunction changes solely due to the two-body interactions.

We thus calculate ϵ_3 in the case $g_2 = 0$. If one insists on introducing the three-body coupling constant g_3 , this can be done in the same manner as we do in the main text when defining g_2 in the 2D case. Namely,

$$\Gamma_3(E) \approx 2\sqrt{3}\pi/[\ln(4t/E) + 2\sqrt{3}\pi/g_3 + i\pi], \quad (\text{S51})$$

where g_3 is defined as $g_3 = 2\sqrt{3}\pi/\ln(\epsilon_3/4t)$ and is assumed to be small. Then in the first order Born approximation one can use the pseudopotential

$$U_3 = (2/\sqrt{3})g_3\delta(\mathbf{\Pi}) = g_3\delta(\sqrt{3}x/2)\delta(y). \quad (\text{S52})$$

For example, the chemical potential of a weakly purely three-body-interacting 1D (quasi)condensate can be estimated as $\mu \approx g_3 n^2/2$, although it would be more appropriate to use the self-consistent equation $\mu \approx \Gamma_3(3\mu)n^2/2$ in analogy with the weakly two-body-interacting 2D Bose gas.

The 1D STM equations are derived in the same manner as in the 2D and 3D cases. The boundary conditions (S8-S9) in the 1D case read

$$\left. \frac{\partial \phi_0}{\partial y_i} \right|_{y_i \rightarrow 0^+} - \left. \frac{\partial \phi_0}{\partial y_i} \right|_{y_i \rightarrow 0^-} = - \frac{2}{a_{1,\uparrow\uparrow}} \phi_0|_{y_i=0}, \quad (\text{S53})$$

$$\left. \frac{\partial \phi_i}{\partial y_i} \right|_{y_i \rightarrow 0^+} - \left. \frac{\partial \phi_i}{\partial y_i} \right|_{y_i \rightarrow 0^-} = - \frac{2}{a_{1,\uparrow\uparrow}} \phi_i|_{y_i=0}, \quad (\text{S54})$$

$$\left. \frac{\partial \phi_i}{\partial y_j} \right|_{y_j \rightarrow 0^+} - \left. \frac{\partial \phi_i}{\partial y_j} \right|_{y_j \rightarrow 0^-} = - \frac{2}{a_{1,\uparrow\downarrow}} \phi_i|_{y_j=0}, \quad j \neq i, \quad (\text{S55})$$

and the functions f_i are defined by

$$f_i(\mathbf{x}_1) = -\frac{\partial \phi_i}{\partial y_1} \Big|_{y_1 \rightarrow 0^+} + \frac{\partial \phi_i}{\partial y_1} \Big|_{y_1 \rightarrow 0^-}, \quad i = 0, 1, 2. \quad (\text{S56})$$

In the low energy limit we can set $E = 0$ in Eq. (S16). However, in Eq. (S15) we have to keep E small but finite since the Green function in this case scales as $G_E(\Pi) \approx -\ln(\sqrt{-E}e^\gamma \Pi/2)/(2\pi)$. Then, (for $g_2 = 0$) the large- Π asymptote of the three-body wavefunction reads

$$\psi_0(\Pi) = \psi_{00} - \frac{3}{2\pi} \ln \frac{\sqrt{-E}e^\gamma \Pi}{2} \int f_0(\mathbf{x}) dx. \quad (\text{S57})$$

On the other hand ψ_0 should be proportional to $\ln(\sqrt{\epsilon_3}e^\gamma \Pi/2)$ consistent with Eq. (S50). We thus find

$$\epsilon_3 = -E \exp \left[-\frac{4\pi\psi_{00}}{3 \int f_0(\mathbf{x}) dx} \right]. \quad (\text{S58})$$

We will now show that in the limit $E \rightarrow 0$ the dependence of f_0 on E is such that ϵ_3 is energy independent.

The 1D STM equations read

$$\tilde{f}_0 = \xi[\hat{L}_{-1}\tilde{f}_1 + (1/\sqrt{1+p^2} - 1)(2\tilde{f}_2 - \tilde{f}_1)], \quad (\text{S59})$$

$$\tilde{f}_1 = \xi[-\hat{L}_{-1}\tilde{f}_2 - (1/\sqrt{1+p^2} - 1)\tilde{f}_1], \quad (\text{S60})$$

$$2\tilde{f}_2 - \tilde{f}_1 = \xi(\hat{L}_\epsilon + 1/\sqrt{p^2 - \epsilon} - 1)\tilde{f}_0 + 4\pi\xi\psi_{00}\delta(p), \quad (\text{S61})$$

where $\xi = (a_{1,\uparrow\uparrow} + a_{1,\uparrow\downarrow})/(a_{1,\uparrow\downarrow} - a_{1,\uparrow\uparrow})$, $\tilde{f}_i(p) = \int f_i(x) \exp(-i\sqrt{4t_c}px) dx$, and we measure momenta in

units of $\sqrt{4t_c}$. In Eq. (S61) $\epsilon = E/4t_c \ll 1$ and we have set E to zero in Eqs. (S59) and (S60). The integral operator in 1D is defined by

$$\hat{L}_\epsilon \tilde{f}(p) = \frac{\sqrt{3}}{\pi} \int \frac{\tilde{f}(k) dk}{p^2 + k^2 + pk - 3\epsilon/4}. \quad (\text{S62})$$

The ϵ dependence in Eqs. (S59-S61) can be singled out by using the equations

$$\hat{L}_\epsilon 1 = 2/\sqrt{p^2 - \epsilon} \quad (\text{S63})$$

and

$$\lim_{\epsilon \rightarrow 0} \hat{L}_{-1} \frac{1}{\sqrt{p^2 - \epsilon}} = F(p) - \frac{\sqrt{3}}{\pi} \frac{\ln(-\epsilon)}{p^2 + 3/4}, \quad (\text{S64})$$

where

$$F(p) = \frac{\sqrt{3} \ln(4p^2 + 3)}{\pi} + \frac{2p \operatorname{arccot}(\sqrt{3}\sqrt{p^2 + 1}/p)}{\pi(p^2 + 3/4)\sqrt{p^2 + 1}}. \quad (\text{S65})$$

We then define a new function $\tilde{h}_2(p)$ by

$$\tilde{f}_2(p) = \tilde{h}_2(p) + \frac{\xi}{2} \left[\frac{3\tilde{f}_0(0)}{\sqrt{p^2 - \epsilon}} + 4\pi\psi_{00}\delta(p) \right], \quad (\text{S66})$$

substitute Eq. (S66) into Eqs. (S59-S61), and take the limit $\epsilon \rightarrow 0$ where it exists. The result is

$$\tilde{f}_0(p) = \xi \{ \hat{L}_{-1}\tilde{f}_1(p) + (1/\sqrt{1+p^2} - 1)[2\tilde{h}_2(p) - \tilde{f}_1(p)] \} + 3\xi^2(1/\sqrt{p^2 + 1} - 1)|p|^{-1}\tilde{f}_0(0), \quad (\text{S67})$$

$$\tilde{f}_1(p) = \xi[-\hat{L}_{-1}\tilde{h}_2(p) - (1/\sqrt{1+p^2} - 1)\tilde{f}_1(p)] - (3/2)\xi^2 F(p)\tilde{f}_0(0) - 2\sqrt{3}\xi^2 \Lambda/(p^2 + 3/4), \quad (\text{S68})$$

$$2\tilde{h}_2(p) - \tilde{f}_1(p) = \xi(\hat{L}_0 + 1/|p|)[\tilde{f}_0(p) - \tilde{f}_0(0)] - \xi\tilde{f}_0(p), \quad (\text{S69})$$

where $\Lambda = \psi_{00} - (3/4\pi)\tilde{f}_0(0) \ln(-\epsilon)$ and we can set it to 1. Then, Eqs. (S67-S69) do not contain ϵ , and, therefore, \tilde{f}_0 , \tilde{f}_1 , and \tilde{h}_2 do not depend on energy. The ϵ dependence is transferred to ψ_{00} which now equals $\psi_{00} = 1 + (3/4\pi)\tilde{f}_0(0) \ln(-\epsilon)$. Equation (S57) then becomes explicitly energy independent, $\psi_0(\Pi) \approx 1 - (3/2\pi)\tilde{f}_0(0) \ln(\sqrt{4t_c}e^\gamma \Pi/2)$, and Eq. (S58) reduces to $\epsilon_3 = 4t_c \exp[-4\pi/3\tilde{f}_0(0)]$.

Equations (S67-S69) can be solved iteratively for small ξ as we do for higher dimensions. In fact, in the purely 1D case there are no two-body bound states for $|\xi| < 1$ and the three-body interaction is absolutely conservative

(at least in the purely 1D model). The leading terms are

$$\tilde{f}_0(0) = -(8\xi^3/\sqrt{3})[1 + (2/3 - 2\sqrt{3}/\pi)\xi + (3\sqrt{3} - 5)\xi^2 + \dots], \quad (\text{S70})$$

which means that the three-body interaction is weakly repulsive (attractive) for small positive (negative) ξ . The interaction gradually increases with $|\xi|$ and ϵ_3 becomes of order t_c when $|\xi| \sim 1$. We have not looked at finite energies, but from the structure of the zero energy wavefunction it is clear that there are no trimer states for $\xi > 0$ and the interaction can be considered as a soft core repulsion. In contrast, for $\xi < 0$ there is a node of ψ_0 at large Π and there is a trimer state, the binding energy of which for small $|\xi|$ equals ϵ_3 .

The point $\xi = 1$ on the repulsive side is equivalent to

$a_{1,\uparrow\uparrow} = 0$, i.e., particles of the same type are impenetrable. This gives a strong three-body repulsion. The repulsion can be increased even further by making ξ somewhat larger than 1 (small positive $a_{1,\uparrow\uparrow}$) and thus entering the so-called super-Tonks regime. However, similarly to the 2D and 3D cases, there are two two-body bound states of small size $\approx a_{1,\uparrow\uparrow}$, three-body recombination to which is possible. For $\xi < -1$ ($a_{1,\uparrow\downarrow} > 0$), there is one such bound state. The rate of recombination to these states can be found numerically from Eq. (S67-S69) but we leave this task for future studies.

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- [2] In fact, for a given ξ the iterative procedure diverges and the series (S25) is asymptotic, which is also consistent with the scaling of the correction $\propto \exp(-1/2\xi)$. We find that in the region of our interest ($r_* > 0.5$) the next two iterations do not improve the expansion (S25).
- [3] Equation (S50) differs from the usual two-body 2D vertex function by the factor $\sqrt{3}/2$ originating from the Jacobian of the transformation (S1) or, equivalently, from the fact that the three-body reduced mass is $2/\sqrt{3}$ times larger than the two-body one.

[1] D. S. Petrov, in *Many-Body Physics with Ultra-Cold Gases: Lecture Notes of the Les Houches Summer Schools*,