SUPPLEMENTAL MATERIAL: GENERALIZED EFIMOV EFFECT IN ONE DIMENSION

Square well regularization and resonance

Consider two particles with the reduced mass μ interacting in one dimension via the regularized inverse square potenital

$$V(r) = -\frac{\alpha}{2\mu r^2}, \quad r > r_0, \qquad V(r) = -D_0 = -\frac{\gamma}{2\mu r_0^2}, \quad 0 < r \leq r_0, \tag{S1}$$

where $r_0 > 0$, $\gamma > 0$. We will also assume $V(-r) = V(r)$, which implies that a wave function should be either even or odd under the reflection $r \to -r$. It should be noted that the regularized potential necessarily supports a parity-even ground state with energy $\epsilon_{\text{GS}} \sim -1/(\mu r_0^2)$. This is in agreement with a general theorem that any attractive well supports at least one bound state in one dimension. Here, however, are interested in the zero energy solution.

For $r > r_0$ a general solution of the relative Schrödinger equation at zero energy is given by

$$\psi(r) = A_+ \Big(\frac{r}{r_0}\Big)^{1/2+\sqrt{1/4-\alpha}} + A_- \Big(\frac{r}{r_0}\Big)^{1/2-\sqrt{1/4-\alpha}}. \tag{S2}$$

On the other hand, for $r \leq r_0$ the solution at zero energy is

$$\psi(r) = C \sin\Big(\sqrt{\gamma}\frac{r}{r_0}\Big) \quad \text{reflection odd}, \qquad \psi(r) = C \cos\Big(\sqrt{\gamma}\frac{r}{r_0}\Big) \quad \text{reflection even}. \tag{S3}$$

By matching the logarithmic derivative at $r = r_0$ one finds for the reflection odd case

$$X \equiv \frac{A_+}{A_-} = \frac{\sqrt{1/4-\alpha} - 1/2 + \sqrt{\gamma}\cot\sqrt{\gamma}}{\sqrt{1/4-\alpha} + 1/2 - \sqrt{\gamma}\cot\sqrt{\gamma}}. \tag{S4}$$

For the reflection even case one must replace $\cot\sqrt{\gamma} \to -\tan\sqrt{\gamma}$ in Eq. (S4). For a generic value of γ, the plus-branch is dominant for $r \gg r_0$ since $X \neq 0$. Note however that one can fine tune the regularized potential such that the wave function follows the minus-branch for $r \gg r_0$. This happens for $X = 0$ resulting in the resonance condition

$$\sqrt{1/4-\alpha} - 1/2 + \sqrt{\gamma}\cot\sqrt{\gamma} = 0 \quad \text{reflection odd}, \qquad \sqrt{1/4-\alpha} - 1/2 - \sqrt{\gamma}\tan\sqrt{\gamma} = 0 \quad \text{reflection even}. \tag{S5}$$

By increasing the depth D_0 of the short-range potential, the total potential supports more and more bound states. Incidentally, the resonance condition signals the appearance of an additional bound state emerging from the zero-energy threshold.

Logarithmic grid and Langer substitution

A numerical solution of the Schrödinger equation

$$[\partial_\theta^2 - v(\theta)]\psi = \kappa^2\psi \tag{S6}$$

with a potential possessing an inverse square singularity requires a grid which is dense near the singularity allowing to resolve the behavior of the wave function close to it. This can be accomplished by the Langer substitution

$$\theta(x) = \exp(x), \qquad \psi(\theta) = \exp\big(\frac{x}{2}\big)y(x) \tag{S7}$$

which transforms Eq. (S6) into

$$\partial_x^2 y(x) + g(x)y(x) = 0 \tag{S8}$$

with

$$g(x) = -\exp(2x)\Big\{\kappa^2 + v[\exp(x)]\Big\} - \frac{1}{4}. \tag{S9}$$

This equation is now solved by the Numerov method on the equidistant x-grid with a lower cutoff $x_{\text{min}} < 0$.

Analytic calculation near $\alpha = 1/8$

Slightly above $\alpha = 1/8$ for the plus-branch the critical mass ratio is large ($\Delta \to \pi/2$) and the angular problem can be attacked analytically. Here we sketch this calculation. First, it is convenient to redefine the angle to be $\phi = \Delta - \theta$. The angular Schrödinger equation (6) to be solved in sector I takes the form

$$\Big[-\partial_\phi^2 - \frac{\alpha}{\sin^2\phi} - \frac{\alpha}{\sin^2(\phi+2\epsilon)}\Big]\psi = 0, \tag{S10}$$

where $\epsilon = \pi/2 - \Delta \ll 1$.

In the regime $\phi \gg \epsilon$ this equation simplifies to

$$\Big(-\partial_\phi^2 - \frac{2\alpha}{\sin^2\phi}\Big)\psi = 0 \tag{S11}$$

with a general solution

$$\psi = (1-\cos\phi)^{\frac{1}{4}} \times \Big[L_1 P_{-1/2}^{\sqrt{1-8\alpha}/2}(\cos\phi) + L_2 Q_{-1/2}^{\sqrt{1-8\alpha}/2}(\cos\phi)\Big], \tag{S12}$$

where $P_n^m(x)$ and $Q_n^m(x)$ are associated Legendre functions. The quantum statistics condition (7) imposed at $\phi = \Delta$ fixes the ratio L_1/L_2.

In the regime $\phi \ll 1$ one can expand the denominators in Eq. (S10) and it transforms to

$$\Big[-\partial_\phi^2 - \frac{\alpha}{\phi^2} - \frac{\alpha}{(\phi+2\epsilon)^2}\Big]\psi = 0 \tag{S13}$$

with a general solution

$$\psi = \epsilon\sqrt{x(2+x)} \times \Big[S_1 P^{\sqrt{1-4\alpha}}_{(\sqrt{1-8\alpha}-1)/2}(1+x) + S_2 Q^{\sqrt{1-4\alpha}}_{(\sqrt{1-8\alpha}-1)/2}(1+x)\Big], \tag{S14}$$

where $x = \phi/\epsilon$. The ratio S_1/S_2 is now determined by applying the plus-branch boundary condition $\psi \sim x^{1/2+\sqrt{1/4-\alpha}}$ at $x = 0^+$ [equivalent to Eq. (8)].

In the intermediate regime $x \gg 1$ and $\phi \ll 1$ the wave functions (S12) and (S14) must now be matched. In this region they are given by

$$\psi = B_+\phi^{\frac{1}{2}(1+\sqrt{1-8\alpha})} + B_-\phi^{\frac{1}{2}(1-\sqrt{1-8\alpha})}, \tag{S15}$$

where $B_+/B_- = \exp(i\zeta)$ with a real phase ζ. The phases extracted from the solutions (S12) and (S14) can differ only by the angle $2\pi n$, where n is an integer. The nodeless wave function ψ is found for $n = 1$. The matching of the phases ζ in the intermediate region with $n = 1$ gives rise to Eq. (9).

Analytic calculation near $\alpha = 0$

Here we show how the mass ratio for the minus-branch can be determined analytically close to the non-interacting point $\alpha = 0$. In this case identical bosons and fermions must be treated differently.

In the case of identical bosons the critical mass ratio vanishes at $\alpha = 0$. We can thus start from the analytical solution of the angular equation (6) at $\Delta = \pi/4$

$$\psi \propto \sin^{1/2-\sqrt{1/4-\alpha}}(\theta + \frac{\pi}{4}) \sin^{1/2-\sqrt{1/4-\alpha}}(\theta - \frac{\pi}{4}) \tag{S16}$$

with the positive eigenvalue

$$-s^2 = 2(1 - 2\sqrt{1/4-\alpha} - 2\alpha) = 4\alpha^2 + O(\alpha^3). \tag{S17}$$

Let us now take into account a small deviation of Δ from $\pi/4$ perturbatively. In order to do this we redefine the angle $\theta = 4\Delta\tilde{\theta}/\pi$ such that the configurational space is the fixed interval $\tilde{\theta} \in (0, \pi/4)$ independent of Δ. After this redefinition, Eq. (6) becomes

$$\Big\{-\partial^2_{\tilde{\theta}} - \frac{\alpha\rho^2}{\sin^2[\rho(\frac{\pi}{4}+\tilde{\theta})]} - \frac{\alpha\rho^2}{\sin^2[\rho(\frac{\pi}{4}-\tilde{\theta})]}\Big\}\psi = -s^2\rho^2\psi, \tag{S18}$$

where $\rho = 4\Delta/\pi > 1$. The linear shift of $-s^2(\rho)$ with respect to $\rho - 1 \ll 1$ is now determined by the first-order perturbation formula

$$\partial(-s^2\rho^2)/\partial\rho = \langle\psi|v_1|\psi\rangle/\langle\psi|\psi\rangle, \tag{S19}$$

where v_1 is the first derivative of the potential in Eq. (S18) with respect to ρ at $\rho = 1$ and ψ is given by Eq. (S16). We have explicitly

$$v_1(\theta) = -\alpha\sec^3(2\theta) \times \big[-3\pi + 8\cos(2\theta) + \pi\cos(4\theta) + 16\theta\sin(2\theta)\big]. \tag{S20}$$

In fact, the integrals on the right hand side of Eq. (S19) can be calculated for a constant ψ, i.e., setting $\alpha = 0$ in Eq. (S16). Then, combining the result with Eq. (S17) we finally obtain

$$-s^2 \approx 4\alpha^2 - 2\alpha(\rho - 1), \tag{S21}$$

from which Eq. (10) follows immediately.

For identical fermions the mass ratio diverges at $\alpha = 0$. Hence one can follow the procedure described in the previous subsection with the wave function (S14) now satisfying the minus-branch boundary condition $\psi \sim x^{1/2-\sqrt{1/4-\alpha}}$ at $x = 0^+$. As a result, one finds Eq. (11).

Born-Oppenheimer approximation for minus-branch

Here we study the minus branch in the regime $M \gg m$, where the Born-Oppenheimer (BO) approximation can be used and the emergence of the Efimov effect can be intuitively understood. Within this approximation the three-body wave function is factorized

$$\Psi(R, r) = \Phi(R)\psi(r; R), \tag{S22}$$

where the function $\psi(r; R)$ corresponds to a bound state of the light particle in the combined potential of two heavy centers located at $R_1 = 0$ and $R_2 = -R$. This wave function satisfies

$$\underbrace{-\frac{1}{2m}\Big[\partial_r^2 + \frac{\alpha}{r^2} + \frac{\alpha}{(r+R)^2}\Big]}_{H_{light}}\psi = \epsilon\psi. \tag{S23}$$

Note that the distance R is the only characteristic length scale in Eq. (S23). Therefore, if a bound state exists at some R, it also exists at any R, and its energy equals $\epsilon_R = -\sigma/mR^2$, where σ is a positive dimensionless number which depends on α and on the choice of the light-heavy boundary condition.

Next, the Born-Oppenheimer approximation assumes that the light particle adiabatically follows displacements of the heavy ones for which ϵ_R acts as the effective interaction potential. Thus, the Schrödinger equation for the relative motion of the heavy particles reads

$$\Big[-\frac{1}{M}\partial_R^2 + \epsilon_{\rm GS}(R)\Big]\Phi = E\Phi. \tag{S24}$$

The energy spectrum E determines the energy spectrum of the three-body system. The spectrum of Eq. (S24) is Efimovian, $E_n \sim e^{-\frac{2\pi n}{s}+\theta}$, where $s \approx \sqrt{\sigma M/m}$. For simplicity, we neglected in Eq. (S24) the diagonal correction to the potential $V_{\rm diag}(R) = \langle\psi|\partial_R^2\psi\rangle/M$ which is subleading and does not modify the argument above.

Now we demonstrate that Eq. (S23) actually supports at least one bound state for the minus-branch condition for any $0 < \alpha < 1/4$. To this end, we consider the minus branch variational wave-function which equals

$$\psi_\kappa \propto r^{1/2-\sqrt{1/4-\alpha}}\exp(-\kappa r) \tag{S25}$$

for $r > 0$ and vanishes for $r < 0$. We find that for any $0 < \alpha < 1/4$ and for a sufficiently small variational parameter κ

$$\langle\psi_\kappa|H_{light}|\psi_\kappa\rangle < 0. \tag{S26}$$

This can be seen by evaluating analytically the variational energy (S26) and Taylor expanding it around $\kappa = 0$. We thus arrive at the conclusion that for the minus-branch boundary condition the BO approximation demonstrates that the energy spectrum of our three-body system is Efimovian for *any* $0 < \alpha < 1/4$ in the limit $M \gg m$.