Supplemental Material: Dimensional crossover for the beyond-mean-field correction in Bose gases

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BEYOND-MEAN-FIELD CORRECTION

In this paper, we use the field theoretic approach of Hugenholtz and Pines [1] to conveniently include the beyond-mean-field corrections for the weakly interacting Bose gas. In their paper, Hugenholtz and Pines are able to connect the ground state energy to the Green's function of the excited particles. The Green's function can then be evaluated using the standard procedure in quantum field theory and they obtain in beyond-mean-field order

$$E - \frac{1}{2}\mu N = \frac{1}{4}\sum_{\boldsymbol{k}\neq 0} \frac{2\tilde{\varepsilon}^2 + 3\tilde{\varepsilon} ng}{\sqrt{\tilde{\varepsilon}^2 + 2\tilde{\varepsilon} ng}} - 2\tilde{\varepsilon} - ng. \tag{1}$$

Here, $\tilde{\varepsilon} = \hbar^2 \boldsymbol{k}^2/2m$ is the single particle excitation spectrum, $g = 4\pi\hbar^2 a_s/m$ is the coupling constant, where a_s denotes the scattering length, and $n = N/V$ is the density of the gas. The chemical potential satisfies the thermodynamic relation $\mu = dE/dN$. Hence, Eq. (1) is a linear first order differential equation for the ground state energy E. We now rewrite the differential equation in terms of the crossover parameter κ and arrive at

$$E - \frac{\kappa}{2}\frac{dE}{d\kappa} = \frac{\lambda}{2} E_0 \sum\!\!\!\!\!\!\int \frac{2\varepsilon^2 + 3\varepsilon\kappa}{\sqrt{\varepsilon^2 + 2\varepsilon\kappa}} - 2\varepsilon - \kappa,$$

where $\varepsilon = \tilde{\varepsilon}/(4\pi\hbar^2/ml_\perp^2) = \pi(u^2+v^2+w^2)/2$ is the single particle excitation spectrum in dimensionless units and we have introduced the energy scale $E_0 = \frac{2\pi\hbar^2}{m}\frac{V}{l_\perp^4 a_s}$. We want to emphasize that this differential equation determines all terms except the mean-field terms of order κ^2. Hence, the divergencies from Bogoliubov theory appearing by a renormalization of the mean-field term are prevented. The general solution of this differential equation takes the from

$$\frac{E}{E_0} = \kappa^2\left[1 + \lambda\left(A(\kappa^*) - \int_{\kappa^*}^{\kappa} d\kappa'\, h(\kappa')\right)\right] \qquad \text{with} \qquad h(\kappa) = \frac{1}{\kappa^3}\sum\!\!\!\!\!\!\int \left[\frac{2\varepsilon^2 + 3\kappa\varepsilon}{\sqrt{\varepsilon^2+2\kappa\varepsilon}} - 2\varepsilon - \kappa\right].$$

The constant $A(\kappa^*)$ determines the initial condition of the differential equation and has to be chosen in order to reproduce the correct mean-field term proportional to κ^2, while κ^* denotes an arbitrary value. The proper determination of $A(\kappa^*)$ is given by the condition to reproduce the correct ground state energy in Eq. (1) of the main text in the three-dimensional regime with $\kappa \gg 1$, which leads to

$$A(\kappa^*) = \int_{\kappa^*}^{\infty} \left[h(\kappa') - h_{3\mathrm{D}}(\kappa')\right] - \int_0^{\kappa^*} d\kappa' h_{3\mathrm{D}}(\kappa'). \tag{2}$$

This relation holds for all values of κ^* so we choose $\kappa^* = \kappa$ and we arrive at $E/E_0 = \kappa^2(1 + \lambda A(\kappa))$. For large values of κ the first term of Eq. (2) vanishes and the second exactly provides the correct three-dimensional LHY correction.

EFFECTIVE SCATTERING LENGTH

We calculate the effective scattering lengths by making use of a two-channel model. In the open channel, we describe the two particles by the wave function $\psi(\boldsymbol{x},\boldsymbol{y})$, whereas the closed channel is described by a single molecular state $\phi(\boldsymbol{z})$. The two channels are described by

$$\left[E - H_0^a - H_0^b\right]\psi(\boldsymbol{x},\boldsymbol{y}) = \bar{g}\int d\boldsymbol{z}\alpha_\Lambda(\boldsymbol{r})\phi(\boldsymbol{z})\delta(\boldsymbol{z}-\boldsymbol{R}) \quad \text{and} \quad \left[E - \nu_0 - H_0^M\right]\phi(\boldsymbol{z}) = \bar{g}\int d\boldsymbol{x}d\boldsymbol{y}\alpha_\Lambda(\boldsymbol{r})\psi(\boldsymbol{x},\boldsymbol{y})\delta(\boldsymbol{z}-\boldsymbol{R}), \tag{3}$$

where $\boldsymbol{r}$ and $\boldsymbol{R}$ are the relative and center-of-mass coordinate of the two particles, while $\alpha_\Lambda(\boldsymbol{r})$ describes the coupling between the two channels. Here, we introduce a cut-off Λ, such that $\alpha_\Lambda(\boldsymbol{r}) \to \delta(\boldsymbol{r})$ for $\Lambda \to 0$, e.g., $\alpha_\Lambda(\boldsymbol{r}) = e^{-\boldsymbol{r}^2/2\Lambda^2}/(2\pi\Lambda^2)^{3/2}$. Hence, we obtain a pseudopotential in the open channel for $\Lambda \to 0$ but a regular potential for $\Lambda \neq 0$. This regularization procedure of the pseudopotential is a suitable method to avoid divergencies throughout the calculation of the scattering amplitudes. The single particle Hamiltonians are given by $H_0^\sigma = -\frac{\hbar^2}{2m_\sigma}\Delta$ with $m_M = m_a + m_b$. In the following, we restrict ourselves to scattering processes with particles of equal mass $m_a = m_b = m$. To connect the three-dimensional scattering length a_s to the effective lower-dimensional scattering lengths, we first need to solve the coupled Eq. (3) in the three-dimensional case. We transform both equations to the center-of-mass frame. In the center-of-mass frame, the molecular wave function is simply a constant ϕ_c and we obtain

$$\left[\frac{\hbar^2\boldsymbol{k}^2}{m} + \frac{\hbar^2}{m}\Delta\right]\psi(\boldsymbol{r}) = \overline{g}\phi_c\alpha_\Lambda(\boldsymbol{r}) \qquad \text{and} \qquad \left[\frac{\hbar^2\boldsymbol{k}^2}{m} - \nu_0\right]\phi_c = \overline{g}\int d\boldsymbol{r}\psi(\boldsymbol{r})\alpha_\Lambda(\boldsymbol{r}). \tag{4}$$

For the remaining Eq. (4) we choose the ansatz

$$\psi_{\boldsymbol{k}}(\boldsymbol{r}) = e^{\mathrm{i}\boldsymbol{k}\boldsymbol{r}} + \beta\int d\boldsymbol{r}'\alpha_\Lambda(\boldsymbol{r}')G_{\boldsymbol{k}}(\boldsymbol{r}-\boldsymbol{r}'), \tag{5}$$

where we define the Green's function by $\frac{\hbar^2}{m}\left[\boldsymbol{k}^2 + \Delta\right]G_{\boldsymbol{k}}(\boldsymbol{r}) = \delta(\boldsymbol{r})$. The far-field behavior of our ansatz yields a connection between the factor β and the low-energy scattering amplitude,

$$f(\boldsymbol{k}) = -\frac{m}{4\pi\hbar^2}\beta = -\frac{m}{4\pi\hbar^2}\frac{\overline{g}^2}{\frac{\hbar^2\boldsymbol{k}^2}{m} - \nu_0 - \overline{g}^2\overline{G}(0) + \frac{m\overline{g}^2}{4\pi\hbar^2}\mathrm{i}k} \qquad \text{where} \qquad \overline{G}(0) = \int d\boldsymbol{r}\, d\boldsymbol{r}'\alpha_\Lambda(\boldsymbol{r})\alpha_\Lambda(\boldsymbol{r}')G_0(\boldsymbol{r}-\boldsymbol{r}').$$

Thus, the scattering amplitude simplifies to

$$f(\boldsymbol{k}) = -\frac{1}{-\frac{4\pi\hbar^2}{m}\frac{\nu}{\overline{g}^2} + \mathrm{i}k + \mathcal{O}(k^2)} = -\frac{1}{\frac{1}{a_s} + \mathrm{i}k},$$

where we have introduced the renormalized detuning $\nu = \nu_0 + g^2\overline{G}(0)$. Hence, we have found the connection between the scattering length and the parameters of the two-channel model $a_s = -\frac{m}{4\pi\hbar^2}\frac{\overline{g}^2}{\nu}$.

Quasi one-dimensional scattering

With the discussed procedure, we can now treat confined systems analogously and connect the effective scattering length a_{1D} to the three-dimensional scattering length a_s. We will begin with a quasi one-dimensional geometry. We impose periodic boundary conditions and the allowed wave vectors are of the form $k_z \in \mathbb{R}$ in the elongated direction and $k_x = \frac{2\pi}{l_\perp}v$, $k_y = \frac{2\pi}{l_\perp}w$ with $v, w \in \mathbb{Z}$ in the transversal directions. As we are interested in the quasi one-dimensional case, the particles occupy the transverse ground state ($v = w = 0$) and the kinetic energy along the z-direction is much smaller than the level spacing, $k_z \ll 2\pi/l_\perp$. With the restrictions on the wave vectors in mind, the coupled Eq. (4) remain valid and so does our ansatz Eq. (5). Again, it is possible to find a connection between the quasi one-dimensional scattering amplitude and the factor β_{1D} using the behavior of the wave function at large interparticle separations,

$$f_{\mathrm{1D}}(k_z) = \frac{\mathrm{i}}{2}\frac{m}{\hbar^2 l_\perp^2}\beta_{\mathrm{1D}} = \frac{\mathrm{i}}{2}\frac{m}{\hbar^2 l_\perp^2}\frac{\overline{g}^2}{\frac{\hbar^2\boldsymbol{k}^2}{m} - \nu_0 - \overline{g}^2\overline{G}_{\mathrm{1D}}(k_z)},$$

where

$$\begin{aligned}\overline{G}_{\mathrm{1D}}(k_z) &= \int d\boldsymbol{r}\, d\boldsymbol{r}'\alpha_\Lambda(\boldsymbol{r})\alpha_\Lambda(\boldsymbol{r}')G^{\mathrm{1D}}_{k_z}(\boldsymbol{r}-\boldsymbol{r}') = \frac{m}{\hbar^2}\frac{1}{l_\perp^2}\sum_{p_x,p_y}\int\frac{dp_z}{2\pi}\frac{\hat{\alpha}_\Lambda^2(\boldsymbol{p})}{k_z^2 - p^2 + \mathrm{i}\eta} \\ &= -\frac{\mathrm{i}m}{2\hbar^2 l_\perp^2 k_z} + \frac{m}{\hbar^2 l_\perp^2}\sum_{(p_x,p_y)\neq(0,0)}\int\frac{dp_z}{2\pi}\frac{\hat{\alpha}_\Lambda^2(\boldsymbol{p})}{k_z^2 - p^2 + \mathrm{i}\eta} = -\frac{\mathrm{i}m}{2\hbar^2 l_\perp^2 k_z} + \overline{G}'_{\mathrm{1D}}(k_z).\end{aligned}$$

Here, $\hat{\alpha}_\Lambda(\boldsymbol{p})$ denotes the Fourier transformed of $\hat{\alpha}_\Lambda(\boldsymbol{r})$ and we separated the mode $p_x = p_y = 0$ in the last step. Finally, we arrive at an expression for the quasi one-dimensional scattering amplitude

$$f_{\rm 1D}(k_z) = -\frac{1}{1 + \mathrm{i}k_z\left(\frac{2\hbar^2 l_\perp^2}{m}\left(\frac{\nu}{\overline{g}^2} + \overline{G}'_{\rm 1D}(0) - \overline{G}(0)\right)\right) + O(k_z^2)} = -\frac{1}{1 + \mathrm{i}k_z a_{\rm 1D}},$$

where the effective scattering length $a_{\rm 1D}$ is given by

$$a_{\rm 1D} = -\frac{1}{2\pi}\frac{l_\perp^2}{a_s}\left(1 - C_{\rm 1D}\frac{a_s}{l_\perp}\right) \qquad \text{with} \qquad C_{\rm 1D} = \int dvdw \frac{1}{\sqrt{v^2+w^2}} - {\sum_{v,w}}' \frac{1}{\sqrt{v^2+w^2}} \approx 3.899.$$

This expression describes the confinement-induced resonance for a setup with periodic boundary conditions.

Quasi two-dimensional scattering

Next, we calculate the effective two-dimensional scattering length $a_{\rm 2D}$ in the same manner. The allowed wave vectors are of the form $k_y, k_z \in \mathbb{R}$ and $k_x = \frac{2\pi}{l_\perp}w$ with $w \in \mathbb{Z}$. The motion is restricted to the transverse ground state and an analysis of the far-field behavior of the given geometry gives an expression for the scattering amplitude

$$f_{\rm 2D}(k_\rho) = -\frac{m}{4\hbar^2}\sqrt{\frac{2\mathrm{i}}{\pi k_\rho}}\frac{1}{l_\perp}\beta_{\rm 2D} = -\frac{m}{4\hbar^2}\sqrt{\frac{2\mathrm{i}}{\pi k_\rho}}\frac{1}{l_\perp}\frac{\overline{g}^2}{\frac{\hbar^2 k_\rho^2}{m} - \nu_0 - \overline{g}^2\overline{G}_{\rm 2D}(k_\rho)},$$

where we have introduced $k_\rho^2 = k_y^2 + k_z^2$, and

$$\overline{G}_{\rm 2D}(k_\rho) = \frac{m}{\hbar^2}\frac{1}{l_\perp}\sum_{p_x}\int \frac{dp_y dp_z}{(2\pi)^2}\frac{\hat{\alpha}_\Lambda^2(\boldsymbol{p})}{k_\rho^2 - p^2 + \mathrm{i}\eta}.$$

Replacing the bare detuning ν_0 with the physical detuning ν, we arrive at

$$f_{\rm 2D}(k_\rho) = -\sqrt{\frac{2\pi\mathrm{i}}{k_\rho}}\frac{1}{\frac{l_\perp}{a_s} - \frac{4\pi\hbar^2 l_\perp}{m}\left(\overline{G}^{\rm 2D}(k_\rho) - \overline{G}(0)\right)} = -\sqrt{\frac{2\pi\mathrm{i}}{k_\rho}}\frac{1}{\mathrm{i}\pi - \ln\left(k_\rho^2 l_\perp^2 e^{-l_\perp/a_s}\right)} = -\sqrt{\frac{2\pi\mathrm{i}}{k_\rho}}\frac{1}{\mathrm{i}\pi - \ln\left(k_\rho^2 a_{\rm 2D}^2 e^{2\gamma}/4\right)},$$

where $\gamma \approx 0.577$ is the Euler-Mascheroni constant, and we have used the relation

$$\overline{G}^{\rm 2D}(k_\rho) - \overline{G}(0) = \frac{m}{4\pi\hbar^2 l_\perp}\left(-\mathrm{i}\pi + \ln(k_\rho^2 l_\perp^2)\right).$$

Hence, the effective scattering length in the quasi two-dimensional geometry is given by $a_{\rm 2D} = 2l_\perp e^{-\frac{l_\perp}{2a_s}}e^{-\gamma}$.

Harmonic confinement

In this part, we provide a derivation of the effective two-dimensional scattering length $a_{\rm 2D}$ in a setup with harmonic confinement. Again, we resort to the previously studied two-channel model. The introduction of a harmonic trapping potential in x-direction changes the single particle Hamiltonian to $H_0 = -\frac{\hbar^2}{2m}\Delta + \frac{m\omega^2}{2}x^2$ and gives rise to the harmonic oscillator length $l_\perp = \sqrt{\hbar/m\omega_\perp}$. The coupled Schrödinger equations in the center-of-mass frame reduce to

$$\left[\frac{\hbar^2\boldsymbol{k}_\rho^2}{m} + \hbar\omega_\perp\left(n + \frac{1}{2}\right) + \frac{\hbar^2}{m}\Delta - \frac{m\omega_\perp^2}{4}x^2\right]\psi(\boldsymbol{r}) = \overline{g}\phi_c\alpha_\Lambda, \text{ and } \left[\frac{\hbar^2\boldsymbol{k}_\rho^2}{m} + \hbar\omega_\perp\left(n + \frac{1}{2}\right) - \nu_0\right]\phi_c = \overline{g}\int d\boldsymbol{r}\alpha_\Lambda(\boldsymbol{r})\psi(\boldsymbol{r}),$$

where $\boldsymbol{k}_\rho = (k_y, k_z)^T$. We are interested in the quasi two-dimensional regime so the motion is restricted to the transverse ground state and our ansatz for the scattering wave function takes the form

$$\psi(\boldsymbol{r}) = e^{\mathrm{i}\boldsymbol{k}_\rho\boldsymbol{\rho}}\varphi_0(x) + \beta_{\rm 2D}^h\int d\boldsymbol{r}'\alpha_\Lambda(\boldsymbol{r}')G_{k_\rho,0}(\boldsymbol{r},\boldsymbol{r}')$$

with $\boldsymbol{\rho}=(y,z)^T$. In addition, $\varphi_n(x)$ denote the eigenfunctions of the harmonic oscillator. Then, the Green's function is given by

$$G_{k_\rho,0}(\boldsymbol{r},\boldsymbol{r}')=\frac{m}{\hbar^2}\int\frac{d\boldsymbol{p}^2}{(2\pi)^2}\sum_{n=0}^{\infty}\frac{e^{\mathrm{i}\boldsymbol{p}(\boldsymbol{\rho}-\boldsymbol{\rho}')}\varphi_n(x)\varphi_n(x')}{\boldsymbol{k}_\rho^2-\boldsymbol{p}^2-n/l_\perp^2+\mathrm{i}\eta},$$

Again, we find the connection to the quasi two-dimensional scattering amplitude $f^h_{\mathrm{2D}}(\boldsymbol{k}_\rho)$,

$$f^h_{\mathrm{2D}}(\boldsymbol{k}_\rho)=-\frac{m}{4\hbar^2}\sqrt{\frac{2\mathrm{i}}{\pi k_\rho}}\left(\frac{1}{2\pi l_\perp^2}\right)^{1/4}\beta^h_{\mathrm{2D}}=-\frac{m}{4\hbar^2}\sqrt{\frac{2\mathrm{i}}{\pi k_\rho}}\left(\frac{1}{2\pi l_\perp^2}\right)^{1/4}\frac{\overline{g}^2\left(\frac{1}{2\pi l_\perp^2}\right)^{1/4}}{\frac{\hbar^2\boldsymbol{k}_\rho^2}{m}-\nu_0-\overline{g}^2\overline{G}^h_{\mathrm{2D}}(\boldsymbol{k}_\rho)},$$

where

$$\overline{G}^h_{\mathrm{2D}}(\boldsymbol{k}_\rho)=\int d\boldsymbol{r}\,d\boldsymbol{r}'\alpha_\Lambda(\boldsymbol{r})\alpha_\Lambda(\boldsymbol{r}')G_{k_\rho,n=0}(\boldsymbol{r},\boldsymbol{r}').$$

Thus, the quasi two-dimensional scattering amplitude in terms of the physical detuning ν simplifies to

$$f^h_{\mathrm{2D}}(\boldsymbol{k}_\rho)=-\frac{m}{4\hbar^2}\sqrt{\frac{2\mathrm{i}}{\pi k_\rho}}\frac{1}{\sqrt{2\pi l_\perp^2}}\frac{\overline{g}^2}{-\nu-\overline{g}^2\left(\overline{G}^h_{\mathrm{2D}}(\boldsymbol{k}_\rho)-\overline{G}(0)\right)}.$$

For the evaluation of the difference $\overline{G}^h_{\mathrm{2D}}(\boldsymbol{k}_\rho)-\overline{G}(0)$ it is suitable to choose $\alpha_\Lambda(\boldsymbol{r})=e^{-\boldsymbol{\rho}/2\Lambda^2}\delta(x)/2\pi\Lambda^2$ without loss of generality. Then, we can analytically evaluate the summation over the harmonic oscillator modes in $\overline{G}^h_{\mathrm{2D}}(\boldsymbol{k}_\rho)$ and are left with the integration in momentum space,

$$\overline{G}^h_{\mathrm{2D}}(\boldsymbol{k}_\rho)-\overline{G}(0)=\frac{m}{4\pi\hbar^2}\frac{1}{\sqrt{2\pi l_\perp^2}}\left(\overline{C}^h_{\mathrm{2D}}-\mathrm{i}\pi+\ln\left(\boldsymbol{k}_\rho^2l_\perp^2/2\right)\right),$$

where

$$\overline{C}^h_{\mathrm{2D}}=\int_0^\infty du\,2u\left(\frac{\sqrt{\pi}}{u}-\frac{\sqrt{\pi}\Gamma(u^2)}{\Gamma(u^2+\frac{1}{2})}+\frac{1}{u^2(1+u)}\right)\approx 1.938\,.$$

Hence, the scattering amplitude $f^h_{\mathrm{2D}}(\boldsymbol{k}_\rho)$ is

$$f^h_{\mathrm{2D}}(\boldsymbol{k}_\rho)=-\sqrt{\frac{2\pi\mathrm{i}}{k_\rho}}\frac{1}{\mathrm{i}\pi-\ln\left(\boldsymbol{k}_\rho^2l_\perp^2e^{\overline{C}^h_{\mathrm{2D}}-\sqrt{2\pi}l_\perp/a_s}\right)}=-\sqrt{\frac{2\pi\mathrm{i}}{k_\rho}}\frac{1}{\mathrm{i}\pi-\ln\left(\boldsymbol{k}_\rho^2a^h_{\mathrm{2D}}e^{2\gamma}/4\right)},$$

and we found an expression for the scattering length, $a^h_{\mathrm{2D}}=\sqrt{2}l_\perp e^{-\gamma-\overline{C}^h_{\mathrm{2D}}/2}e^{-\sqrt{2\pi}l_\perp/2a_s}$. A closer analysis of $\overline{C}^h_{\mathrm{2D}}$ reveals the connection to the constant B found in [2, 3], $B=2\pi e^{-\overline{C}^h_{\mathrm{2D}}}\approx 0.905$, and we obtain

$$a^h_{\mathrm{2D}}=2l_\perp\sqrt{\frac{\pi}{B}}e^{-\sqrt{2\pi}l_\perp/2a_s-\gamma}.$$

The constant C^h_{2D} of the main text is then given by $C^h_{\mathrm{2D}}=(2\pi)^{3/2}\sqrt{e}/B$. One can obtain the quasi one-dimensional scattering length in a harmonic confinement in the same manner and for completeness we will only give the expression first found by [4] here,

$$a^h_{\mathrm{1D}}=-\frac{l_\perp^2}{a_s}\left(1-\frac{C^h_{\mathrm{1D}}}{\sqrt{2}}\frac{a_s}{l_\perp}\right)\qquad\text{with}\qquad C^h_{\mathrm{1D}}\approx 1.4603\,.$$

3D-1D CROSSOVER

In the following, we will briefly sketch the derivation of the function describing the entire crossover regime in the 3D-1D crossover. We start with

$$\frac{E-E_{\mathrm{3D}}}{E_0}=\kappa^2\lambda\int_\kappa^\infty d\kappa'\,\left[h(\kappa')-h_{\mathrm{3D}}(\kappa')\right]$$

and perform the integration over κ' first. The upper boundary of the integration vanishes and we obtain

$$\frac{E_{\text{1D}}-E_{\text{3D}}}{E_0}=\lambda\int du\left[\sum_{n,m}f(\varepsilon)-\int dvdw\, f(\varepsilon)\right],\qquad\text{where}\qquad f(\varepsilon)=\sqrt{\varepsilon^2+2\varepsilon\kappa}-\varepsilon-\kappa.$$

We proceed by rewriting the integration over u by deforming the integration path into the complex plane. Then, the integral takes the from

$$\int_{-\infty}^{\infty}du\,\sqrt{\varepsilon^2+2\varepsilon\kappa}-\varepsilon-\kappa=-\frac{4\kappa^{3/2}}{\sqrt{\pi}}\int_0^1 dt\,\frac{\sqrt{t(1-t)}}{\sqrt{t+\frac{\pi}{4\kappa}(v^2+w^2)}}.$$

This allows us now to nicely separate the different contributions to the ground state energy,

$$\begin{aligned}
&\int dv\,dw\frac{1}{\sqrt{t+\frac{\pi}{4\kappa}(v^2+w^2)}}-\sum_{v,w}\frac{1}{\sqrt{t+\frac{\pi}{4\kappa}(v^2+w^2)}}\\
&=-\frac{1}{\sqrt{t}}+{\sum_{v,w}}'\frac{1}{\sqrt{\frac{\pi}{4\kappa}(v^2+w^2)}}-{\sum_{v,w}}'\frac{1}{\sqrt{t+\frac{\pi}{4\kappa}(v^2+w^2)}}\\
&\quad+\left.\int dv\,dw\frac{1}{\sqrt{\frac{\pi}{4\kappa}(v^2+w^2)}}-{\sum_{v,w}}'\frac{1}{\sqrt{\frac{\pi}{4\kappa}(v^2+w^2)}}\right\}=\sqrt{\frac{4\kappa}{\pi}}C_{\text{1D}}\\
&\quad+\left.\int dv\,dw\int\frac{1}{\sqrt{t+\frac{\pi}{4\kappa}(v^2+w^2)}}-\int dv\,dw\frac{1}{\sqrt{\frac{\pi}{4\kappa}(v^2+w^2)}}\right\}=-8\sqrt{t}\kappa.
\end{aligned}$$

Hence, we arrive at

$$\frac{E-E_{\text{3D}}}{\lambda E_0}=-\frac{8}{3\sqrt{\pi}}\kappa^{3/2}+C_{\text{1D}}\kappa^2-\frac{128}{15\sqrt{\pi}}\kappa^{5/2}+\frac{8\kappa^2}{\pi}\int_0^1 dt\,\sqrt{t(1-t)}\left[{\sum_{v,w}}'\frac{1}{\sqrt{v^2+w^2}}-{\sum_{v,w}}'\frac{1}{\sqrt{v^2+w^2+\frac{4\kappa t}{\pi}}}\right].$$

As a last step, we want to get rid of the double summation. Therefore, we use $1/\sqrt{A}=\int_0^\infty d\tau\, e^{-\tau A}/\sqrt{\tau}$ to write

$$\begin{aligned}
{\sum_{v,w}}'\left[\frac{1}{\sqrt{v^2+w^2}}-\frac{1}{\sqrt{v^2+w^2+\frac{4\kappa t}{\pi}}}\right]&=\int_0^\infty\frac{d\tau}{\sqrt{\pi\tau}}{\sum_{v,w}}'e^{-\tau(v^2+w^2)}(1-e^{-\tau 4\kappa t/\pi})\\
&=\int_0^\infty\frac{d\tau}{\sqrt{\pi\tau}}\left(\vartheta_3(0,e^{-\tau})^2-1\right)\left(1-e^{-\tau 4\kappa t/\pi}\right),
\end{aligned}$$

where we have made use of the definition of the Jacobi theta function $\vartheta_3(z,q)=\sum_n q^{n^2}\cos(2nz)$. Finally, we perform the integration over t,

$$\int_0^1 dt\sqrt{t(1-t)}\left(1-e^{-4\tau\kappa t/\pi}\right)=\frac{\pi}{8}\left(1-\frac{e^{-\frac{2\tau\kappa}{\pi}}I_1(\frac{2\tau\kappa}{\pi})}{\tau\kappa/\pi}\right),$$

and arrive at the final result

$$\frac{E_{\text{1D}}}{E_0}=\kappa^2(1+\lambda C_{\text{1D}})-\lambda\frac{8}{3\sqrt{\pi}}\kappa^{3/2}+\lambda\kappa^2\int_0^\infty\frac{d\tau}{\sqrt{\pi\tau}}\left[\vartheta_3\left(0,e^{-\tau}\right)^2-1\right]\left[1-\frac{e^{-\frac{2\tau\kappa}{\pi}}I_1\left(\frac{2\tau\kappa}{\pi}\right)}{\tau\kappa/\pi}\right].$$

3D regime with $\kappa\gg 1$

Using the relation $\vartheta_3\left(0,e^{-\pi x}\right)=\vartheta_3\left(0,e^{-\pi/x}\right)/\sqrt{x}$, which is straightforward to prove using the Poisson summation formula, we find a suitable expression to perform the analytic expansion for $\kappa\gg 1$,

$$\kappa^2 \int_0^\infty \frac{d\tau}{\sqrt{\pi\tau}} \left[\vartheta_3\left(0, e^{-\tau}\right)^2 - 1\right] \left[1 - \frac{e^{-\frac{2\tau\kappa}{\pi}} I_1\left(\frac{2\tau\kappa}{\pi}\right)}{\tau\kappa/\pi}\right] = \kappa^2 \int_0^\infty \frac{d\tau}{\sqrt{\pi\tau}} \left[\vartheta_3\left(0, e^{-\tau}\right)^2 - \frac{\pi}{\tau}\right] \left[1 - \frac{e^{-\frac{2\kappa\pi}{\tau}} I_1\left(\frac{2\pi\kappa}{\tau}\right)}{\pi\kappa/\tau}\right]$$

$$= \kappa^2 \int_0^\infty \frac{d\tau}{\sqrt{\pi\tau}} \left[1 - \frac{e^{-\frac{2\kappa\pi}{\tau}} I_1\left(\frac{2\pi\kappa}{\tau}\right)}{\pi\kappa/\tau}\right] \Bigg\} = \frac{128}{15\sqrt{\pi}}\kappa^{5/2} \tag{6}$$

$$+ \kappa^2 \int_0^\infty \frac{d\tau}{\sqrt{\pi\tau}} \left[\vartheta_3\left(0, e^{-\tau}\right)^2 - \frac{\pi}{\tau} - 1\right] \Bigg\} = -\kappa^2 C_{\rm 1D} \tag{7}$$

$$+ \kappa^2 \int_0^\infty \frac{d\tau}{\sqrt{\pi\tau}} \frac{\pi}{\tau} \left[\frac{e^{-\frac{2\kappa\pi}{\tau}} I_1\left(\frac{2\pi\kappa}{\tau}\right)}{\pi\kappa/\tau}\right] \Bigg\} = \frac{8}{3\sqrt{\pi}}\kappa^{3/2} \tag{8}$$

$$- \int_0^\infty \frac{d\tau}{\sqrt{\pi\tau}} \left[\vartheta_3\left(0, e^{-\tau}\right)^2 - 1\right] \left[\frac{e^{-\frac{2\kappa\pi}{\tau}} I_1\left(\frac{2\pi\kappa}{\tau}\right)}{\pi\kappa/\tau}\right] \Bigg\} \approx -\kappa^2 \int_0^\infty \frac{d\tau}{\sqrt{\pi\tau}} \left[\vartheta_3\left(0, e^{-\tau}\right)^2 - 1\right] \sqrt{\frac{2}{\pi}} \left(\frac{\tau}{2\kappa\pi}\right)^{3/2} = -\sqrt{\kappa}\frac{A_{\rm 1D}}{2\pi^{5/2}}$$

Inserting into Eq. (9) of the main text, we see that Eq. (6) provides the correct 3D results, while Eq. (7) cancels the correction to the mean-field shift, Eq. (8) cancels the 1D beyond-mean-field correction, while the last term can now be expanded in the small parameter τ/κ, which provides the leading correction for large $\kappa \gg 1$ with $A_{\rm 1D} = \sum'_{v,w} \left(v^2 + w^2\right)^{-2} \approx 6.0268$.

HARMONIC CONFINEMENT

Bogoliubov theory in the 1D geometry with harmonic confinement

In the following, we study a system, where the mean-field energy is canceled by a second type of interaction, and the ground state remains always in the lowest harmonic mode of the transverse confinement within mean-field theory. Then, we can derive the Bogoliubov excitation spectrum in analogy to the situation with periodic boundary conditions and perform a perturbation expansion for small values of $\kappa = n_{\rm 1D} a_s$. We express the bosonic field operator in eigenstates of the non-interacting theory,

$$\psi(\boldsymbol{r}) = \frac{1}{\sqrt{L}} \sum_{v,w} \int \frac{dk}{2\pi} e^{\mathrm{i}kx} \varphi_v(y)\varphi_w(z) b_{k,vw} \qquad \text{with} \qquad \varphi_w(z) = \frac{1}{\sqrt{2^w w!\, l_\perp \sqrt{\pi}}} H_w(z/l_\perp) e^{-z^2/2l_\perp^2}$$

the eigenfunctions of the harmonic oscillator and the oscillator length $l_\perp = \sqrt{\hbar/m\omega_\perp}$. The quantum many-body Hamiltonian takes the form

$$H = \sum_k \sum_{v,w} \tilde{\epsilon}_{k,vw} b^\dagger_{k,vw} b_{k,vw} + \frac{1}{2L} \sum_{\substack{k,k',\\ q}} \sum_{\substack{i,i',w,w',\\ j,j',v,v'}} V^{ij,i'j'}_{vw,v'w'}(q) b^\dagger_{k,vw} b^\dagger_{k',v'w'} b_{k'-q,i'j'} b_{k+q,vw}$$

with the excitation spectrum $\tilde{\epsilon}_{k,vw} = \frac{\hbar^2 k^2}{2m} + \hbar\omega_\perp(v+w)$ and the interaction potential

$$V^{ij,i'j'}_{vw,v'w'}(q) = \int dx\, dy\, dy'\, dz\, dz'\, e^{iqx} \varphi_v(y)\varphi_{v'}(y')\varphi_w(z)\varphi_{w'}(z')\varphi_{i'}(y')\varphi_i(y)\varphi_j(z)\varphi_{j'}(z') V(x,y,y',z,z').$$

In order to achieve the cancellation of the mean-field energy, we add an attractive interaction with a very long range. The combined interaction potential is suitably chosen as

$$V(x,y,y',z,z') = g\delta(y-y')\delta(z-z') \left(\delta(x) - \frac{1}{\sqrt{\pi r_0^2}} e^{-x^2/r_0^2}\right).$$

The first term is the contact interaction with $g = 4\pi\hbar^2 a_s/m$, whereas the attractive second part will not contribute to the beyond-mean-field corrections due to its long-range character. The mean-field part, however, is strongly influenced by the additional interaction, as $\mu = V^{00,00}_{00,00}(0) = 0$. Following the standard approach by Bogoliubov, we replace the lowest state $b_{0,00}$ by a macroscopic occupation $\sqrt{N_0}$. Then, we express the condensate fraction by $N_0 = N - \sum'_{k,vw} b^\dagger_{k,vw} b_{k,vw}$ and obtain in leading order in N

$$H = \sum_k \sum_{vw} \tilde{\epsilon}_{k,vw} b^\dagger_{k,vw} b_{k,vw} + \frac{n_{\rm 1D}}{2} \sum'_{\substack{k,v,w,\\ v',w'}} \left[2V_{vw,v'w'} b^\dagger_{k,vw} b_{k,v'w'} + V_{vw,v'w'} b^\dagger_{k,vw} b^\dagger_{-k,v'w'} + V_{vw,v'w'} b_{k,vw} b_{-k,v'w'} \right],$$

where we have introduced $V_{vw,v'w'} = g \int dy\, dz\, \varphi_v(y)\varphi_{v'}(y)\varphi_0(y)^2 \varphi_w(z)\varphi_{w'}(z)\varphi_0(z)^2$ and the primed sum indicates the absence of the condensate mode. The determination of the Bogoliubov excitation spectrum is most conveniently achieved using the following approach: We start with the Heisenberg equation

$$\mathrm{i}\hbar \dot{b}_{k,vw} = [b_{k,vw}, H] = \tilde{\epsilon}_{k,vw} b_{k,vw} + n_{\rm 1D} \sum_{v',w'} V_{vw,v'w'} (b_{k,v'w'} + b_{-k,v'w'}).$$

Then, the second time derivation simplifies to

$$(\mathrm{i}\hbar)^2 \ddot{b}_{k,vw} = \tilde{\epsilon}^2_{k,vw} b_{k,vw} + \tilde{\epsilon}_{k,vw} n_{\rm 1D} \sum_{v'w'} V_{vw,v'w'} \left(b_{k,v'w'} + b^\dagger_{-k,v'w'} \right) + n_{\rm 1D} \sum_{v'w'} V_{vw,v'w'} \left(\tilde{\epsilon}_{k,v'w'} b_{k,v'w'} - \tilde{\epsilon}_{k,v'w'} b^\dagger_{-k,v'w'} \right). \tag{9}$$

Adding and subtracting the adjoint of Eq. (9) and making use of the Bogoliubov transformation $b_{k,vw} = \sum_{\alpha,\beta} u^{\alpha\beta}_{k,vw} a_{k,\alpha\beta} + v^{\alpha\beta}_{k,vw} a^\dagger_{-k,\alpha\beta}$, we obtain two equations for the excitation spectrum $\tilde{E}_{k,\alpha\beta}$,

$$\tilde{E}^2_{k,\alpha\beta} f^{+,\alpha\beta}_{k,vw} = \tilde{\epsilon}^2_{k,vw} f^{+,\alpha\beta}_{k,vw} + 2\tilde{\epsilon}_{k,vw} n_{\rm 1D} \sum_{v'w'} V_{vw,v'w'} f^{+,\alpha\beta}_{k,v'w'} \qquad \text{and}$$
$$\tilde{E}^2_{k,\alpha\beta} f^{-,\alpha\beta}_{k,vw} = \tilde{\epsilon}^2_{k,vw} f^{-,\alpha\beta}_{k,vw} + 2 n_{\rm 1D} \sum_{v'w'} \tilde{\epsilon}_{k,v'w'} V_{v'w',vw} f^{-,\alpha\beta}_{k,v'w'}, \tag{10}$$

with $f^{\pm,\alpha\beta}_{k,vw} = u^{\alpha\beta}_{k,vw} \pm v^{\alpha\beta}_{k,vw}$. Both equations are connected by the relation $f^{+,\alpha\beta}_{k,vw} = \frac{\tilde{\epsilon}_{k,vw}}{E_{k,\alpha\beta}} f^{-,\alpha\beta}_{k,vw}$. As the Bogoliubov transformation has to be canonical, the Bogoliubov functions satisfy $\delta_{\alpha,\gamma}\delta_{\beta,\delta} = \sum_{vw} f^{+,\alpha\beta}_{k,vw} f^{-,\gamma\delta}_{k,vw}$, which determines the normalization of the Bogoliubov functions $f^{\pm,\alpha\beta}_{k,vw}$. The solution of Eq. (10) provides the excitation spectrum $\tilde{E}_{k,\alpha\beta}$, the Bogoliubov function $f^{\pm,\alpha\beta}_{k,vw}$ and hence the amplitude $v^{\alpha\beta}_{k,vw}$. This allows us to determine the beyond-mean-field correction by the approach of Hugenholtz and Pines,

$$E^h_{\rm 1D} - \frac{1}{2}\mu N = \frac{L}{2} \int \frac{dk}{2\pi} \sum_{\substack{v,\alpha\\ w,\beta}} \left[\tilde{\epsilon}_{k,vw} - \tilde{E}_{k,\alpha\beta} \right] |v^{\alpha\beta}_{k,vw}|^2. \tag{11}$$

In general, Eq.(10) has to be solved numerically. In the quasi-one-dimensional regime $\kappa = n_{\rm 1D} a_s \ll 1$, however, we can perform a perturbation expansion for $E_{k,\alpha\beta}$ and $f^{\pm,\alpha\beta}_{k,vw}$ in κ to obtain the leading contributions. A consistent expansion in κ turns out to be challenging and we will describe the procedure in detail. We introduce the dimensionless single particle excitation spectrum $\epsilon_{u,vw} = \tilde{\epsilon}_{k,vw}/(4\pi\hbar^2/ml^2_\perp) = \pi(u^2 + (v+w)/(2\pi^2))/2$ and the dimensionless Bogoliubov spectrum $E_{u,\alpha\beta} = \tilde{E}_{k,\alpha\beta}/(4\pi\hbar^2/ml^2_\perp)$, where $u = kl_\perp/(2\pi)$. In what follows, we will discuss the right-hand side of Eq. (11) and separate the different contributions,

$$\frac{L}{2} \int \frac{dk}{2\pi} \sum_{\substack{v,\alpha\\ w,\beta}} \left[\tilde{\epsilon}_{k,vw} - \tilde{E}_{k,\alpha\beta} \right] |v^{\alpha\beta}_{k,vw}|^2 = 2\pi E^h_0 \lambda \Bigg(\int du\, [\epsilon_{u,00} - E_{u,00}] \, |v^{00}_{u,00}|^2 + \int du \sum'_{v,w} [\epsilon_{u,vw} - \epsilon_{u,00}] \, |v^{00}_{u,vw}|^2 \tag{12}$$
$$+ \int du \sum'_{\alpha,\beta} [\epsilon_{u,00} - E_{u,\alpha\beta}] \, |v^{\alpha\beta}_{u,00}|^2 + \int du \sum'_{v,w} \sum'_{\alpha,\beta} [\epsilon_{u,vw} - E_{u,\alpha\beta}] \, |v^{\alpha\beta}_{u,vw}|^2 \Bigg),$$

where we have introduced the energy scale $E^h_0 = \hbar\omega_\perp L/a_s$ and $\lambda = a_s/l_\perp$. For the first term we determine the Bogoliubov spectrum within second order perturbation theory,

$$E^2_{u,00} = \epsilon^2_{u,00} + 2\epsilon_{u,00}\kappa\eta_{00}\eta_{00} + 4\kappa^2 \sum'_{v,w} \frac{\epsilon_{u,00}\epsilon_{u,vw}\eta^2_{v0}\eta^2_{w0}}{\epsilon^2_{u,00} - \epsilon^2_{u,vw}},$$

where we have introduced the overlap of the harmonic oscillator wave functions

$$\eta_{vw} = l_\perp \int dz\, \varphi_v(z)\varphi_w(z)\varphi_0(z)^2 = \begin{cases} \frac{1}{\sqrt{2\pi}} \frac{(-1)^{(v-w)/2}}{2^{v+w}} \frac{(v+w)!}{(\frac{v+w}{2})!} \frac{1}{\sqrt{v!w!}} & v+w = \text{even} \\ 0 & v+w = \text{odd}. \end{cases}$$

The momentum dependence of the third term can be neglected, and we can evaluate the remaining sum over the harmonic oscillator modes,

$$\sum_{v,w}{}' \frac{\eta_{v0}^2\eta_{w0}^2}{v+w} = \frac{\ln(4/3)}{8\pi^2}. \tag{13}$$

Thus, the Bogoliubov spectrum reads

$$E_{u,00}^2 = \epsilon_{u,00}^2 + 2\epsilon_{u,00}\left(\frac{\kappa}{2\pi} - \frac{\ln(4/3)}{\pi}\kappa^2\right).$$

For the Bogoliubov function $f_{u,00}^{+,00}$ on the other hand, it is sufficient only to include the lowest order $(f_{u,00}^{+,00})^2 = \frac{\epsilon_{u,00}}{E_{u,00}}$. Any higher order will only provide contribution of order $\kappa^{7/2}$ or higher. We obtain the amplitude

$$|v_{u,00}^{00}|^2 = \frac{1}{4}\left(1 - \frac{E_{u,00}}{\epsilon_{u,00}}\right)^2 (f_{u,00}^{+,00})^2 = \frac{1}{4}\left(1 - \frac{E_{u,00}}{\epsilon_{u,00}}\right)^2 \frac{\epsilon_{u,00}}{E_{u,00}}$$

and perform the momentum integration, which yields

$$2\pi\lambda \int du\, [\epsilon_{u,00} - E_{u,00}]\, |v_{u,00}^{00}|^2 = \lambda\left(-\frac{\sqrt{2}}{3\pi}\kappa^{3/2} + \frac{\sqrt{2}\ln(4/3)}{\pi}\kappa^{5/2}\right).$$

The first term will provide the one-dimensional beyond-mean-field correction, whereas the second is part of the leading correction. In contrast to the situation with periodic boundary conditions, the leading correction to the one-dimensional behavior is not of order κ^3, but the coupling of the condensate mode to higher harmonic oscillator modes leads to a three-dimensional behavior of the form $\kappa^{5/2}$. Terms of order κ^3 appear in our calculations as well but they are not the dominant correction anymore. In the following we will see, that a consistent expansion up to κ^3 would require to calculate the Bogoliubov functions within second order perturbation theory, which makes a consistent expansion very cumbersome.

We will now continue with the second term of Eq. (12). As $(v,w) \neq (\alpha,\beta) = (0,0)$, we need to calculate the perturbative correction to the Bogoliubov function $f_{u,vw}^{\pm,00}$. In our calculation we determine $f_{u,vw}^{+,00}$ within first order, take care of the normalization in the relevant order and obtain

$$|v_{u,vw}^{00}|^2 = \frac{1}{4}\left(1 - \frac{E_{u,00}}{\epsilon_{u,vw}}\right)^2 (f_{u,vw}^{+,00})^2 = \kappa^2 \frac{\epsilon_{u,00}}{E_{u,00}}\left(1 - \frac{E_{u,00}}{\epsilon_{u,vw}}\right)^2 \left(\frac{\epsilon_{u,vw}\eta_{v0}\eta_{w0}}{\epsilon_{u,00}^2 - \epsilon_{u,vw}^2}\right)^2.$$

Although we calculated the Bogoliubov spectrum $E_{u,00}$ up to second order, it is sufficient to use only the lowest order order, as $v_{u,vw}^{00}$ itself contains orders of κ^2 and higher. Again, we can perform the momentum integration and expand the result in orders of κ,

$$2\pi\lambda \int du \sum_{v,w}{}' [\epsilon_{u,vw} - \epsilon_{u,00}]\, |v_{u,vw}^{00}|^2 = \lambda \sum_{vw}{}' \eta_{v0}^2\eta_{w0}^2 \left(\frac{2\pi^2}{\sqrt{v+w}}\kappa^2 - \frac{16\sqrt{2}\pi}{v+w}\kappa^{5/2} + O(\kappa^3)\right). \tag{14}$$

The first term of order κ^2 carries the κ-dependence of the amplitude $|v_{u,vw}^{00}|^2$. Hence, evaluating the Bogoliubov functions up to second order immediately yields additional contributions of order κ^3. In the following we will waive to include those. The second term is another contribution to the energy of order $\kappa^{5/2}$ with the same double sum as in Eq. (13).

The procedure for the third term of Eq. (12) is very similar to the previous one. It is sufficient to calculate the Bogoliubov spectrum $E_{u,\alpha\beta}$ and the amplitudes $v_{u,00}^{\alpha\beta}$ within first order,

$$E_{u,\alpha\beta}^2 = \epsilon_{u,\alpha\beta}^2 + 2\kappa\epsilon_{u,\alpha\beta}\eta_{\alpha\alpha}\eta_{\beta\beta} \qquad \text{and} \qquad |v_{u,00}^{\alpha\beta}|^2 = \kappa^2 \frac{\epsilon_{u,\alpha\beta}}{E_{u,\alpha\beta}}\left(1 - \frac{E_{u,\alpha\beta}}{\epsilon_{u,00}}\right)^2 \left(\frac{\epsilon_{u,00}\eta_{\alpha0}\eta_{\beta0}}{\epsilon_{u,\alpha\beta}^2 - \epsilon_{u,00}^2}\right)^2.$$

The expansion for small values of $\kappa \ll 1$ and the momentum integration yields another contribution of order κ^2,

$$2\pi\lambda \int du \sum_{\alpha,\beta}{}' \left[\epsilon_{u,00} - E_{u,\alpha\beta}\right] |v^{\alpha\beta}_{u,00}|^2 = -\lambda \sum_{\alpha,\beta}{}' 2\pi^2 \frac{\eta^2_{\alpha 0}\eta^2_{\beta 0}}{\sqrt{\alpha+\beta}} \kappa^2 + O(\kappa^3),$$

which exactly cancels the κ^2 contribution of Eq. (14).

For the last term in Eq. (12), even the lowest order in the Bogoliubov spectrum $E_{u,\alpha\beta}$ and the amplitudes

$$|v^{\alpha\beta}_{u,vw}|^2 = \frac{1}{4}\left(1 - \frac{E_{u,\alpha\beta}}{\epsilon_{u,vw}}\right)^2 \frac{\epsilon_{u,vw}}{E_{u,\alpha\beta}} \delta_{v\alpha}\delta_{w\beta} \tag{15}$$

immediately leads to contributions of order κ^3.

In conclusion, we arrive at a differential equation for $\kappa \ll 1$,

$$E^h_{\rm 1D} - \frac{1}{2}\mu N = E^h_{\rm 1D} - \frac{\kappa}{2}\frac{dE^h_{\rm 1D}}{d\kappa} = \lambda E^h_0 \left(-\frac{\sqrt{2}}{3\pi}\kappa^{3/2} - \frac{\sqrt{2}\ln(4/3)}{\pi}\kappa^{5/2} + O(\kappa^3)\right) \tag{16}$$

and its solution reads

$$\frac{E^h_{\rm 1D}}{E^h_0} = \lambda \frac{C^h_{\rm 1D}}{\sqrt{2}}\kappa^2 - \lambda\frac{4\sqrt{2}}{3\pi}\kappa^{3/2} + \lambda\frac{4\sqrt{2}\ln(4/3)}{\pi}\kappa^{5/2} + O(\kappa^3). \tag{17}$$

Note that the first term of order κ^2 is not determined by the differential equation but enters as a constraint, as the crossover has to include the leading correction to the confinement-induced resonance, which became evident in the study for periodic boundary conditions. For a harmonic confinement, we confirm this by a full numerical evaluation of the ground state energy in the crossover form 3D to 1D. This requires the full numerical evaluation of the Bogoliubov excitation spectrum $E_{u,\alpha\beta}$ and the determination of the factors $|v^{\alpha\beta}_{u,vw}|^2$. Inserting the result in Eq. (11) and performing the summation and integration numerically allows us to solve the differential equation and obtain the ground state energy $E^h_{\rm 1D}$ in the full crossover. We can fix the integration constant at large densities ($\kappa \gg 1$) by requiring the correct mean-field term from the local-density approximation. Then, we can derive the behavior in the one-dimensional regime $\kappa \ll 1$. From our numerical calculations, we obtain the leading corrections in Eq. (14) in the main text, and recover the analytical expressions for $C^h_{\rm 1D}$ with an error of 1%. Remarkably, from the 3D result in local-density approximation, we recover from the crossover the expected LHY correction, as well as the correction stemming from the regularization of the 1D scattering length due to the transverse confinement.

Finally, we want to comment on the situation in the 2D geometry with harmonic confinement. The analysis of the ground state energy in the two-dimensional regime is carried out analogously to the 1D scenario. We find, that the leading correction to the beyond-mean-field energy is of order $\kappa^3\ln(\kappa)$. This is problematic as a consistent expansion up to order κ^3 would require to determine the Bogoliubov functions $f^{\pm,\alpha}_{\mathbf{k},w}$ within second order perturbation theory, as we have already seen in the one-dimensional case.

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