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Non-Markovian persistence in the diluted Ising model at criticality

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Abstract. – We investigate global persistence properties for the non-equilibrium critical dynamics of the randomly diluted Ising model. The disorder-averaged persistence probability $\overline{P_c}(t)$ of the global magnetization is found to decay algebraically with an exponent θ_c that we compute analytically in a dimensional expansion in $d = 4 - \epsilon$. Corrections to Markov process are found to occur already at one loop order and θ_c is thus a novel exponent characterizing this disordered critical point. Our result is thoroughly compared with Monte Carlo simulations in $d = 3$, which also include a measurement of the initial slip exponent. Taking carefully into account corrections to scaling, θ_c is found to be universal, independent of the dilution factor p along the critical line at $T_c(p)$, and in good agreement with our one-loop calculation.

Persistence properties, and related first-passage-time problems, have been intensively investigated both theoretically and experimentally during these last few years [1]. Originally introduced in the context of non-equilibrium coarsening dynamics of ferromagnets at zero temperature [2], the persistence exponent θ describes the long time decay $P(t) \sim t^{-\theta}$ of the probability $P(t)$ that a stochastic variable, here the local order parameter, does not cross a threshold value during a time interval t. Interestingly, this exponent θ appeared to be highly non-trivial, even for simple pure systems such as diffusion [3], and therefore it has motivated numerous investigations in various pure models. However, except for few models in one dimension [4, 5], very little is known about this exponent for disordered systems.

Although for a ferromagnet at finite T, one expects that the persistence $P(t)$ associated to the *local* magnetization has an exponential tail, persistence properties received a special attention in the context of critical dynamics of pure ferromagnets at T_c. Indeed, it has been proposed [6] that the probability $P_c(t)$ that the global magnetization M has not changed sign in the time interval t following a quench from a random initial configuration, decays algebraically at large time $P_c(t) \sim t^{-\theta_c}$. In this context, analytical progress is made possible thanks to the property that, in the thermodynamic limit, the global order parameter remains Gaussian at all finite times t. Indeed, for a d-dimensional system of linear size L, $M(t)$ is the sum of L^d random variables which are correlated only over a *finite* correlation length $\xi(t)$. Thus, in the thermodynamic limit $L/\xi(t) \gg 1$, the Central Limit Theorem (CLT) asserts that $M(t)$ is a Gaussian process, for which powerful tools have been developed to compute the persistence properties [1, 7]. Remarkably, under the *additional* assumption that M is a Markovian process, θ_c can be related to the other critical exponents via the scaling relation

$\theta_c = \mu \equiv (\lambda - d + 1 - \eta/2) z^{-1}$, with z and λ the dynamical and autocorrelation [8,9] exponent, respectively, and η the (static) Fisher exponent.

Nevertheless, as argued in ref. [6], M is in general non-Markovian and thus θ_c is a *new* exponent associated to critical dynamics. For the non-conserved critical dynamics of pure O(N) model, corrections to this scaling relation were indeed found to occur at two-loops order [10], in rather good agreement with numerical simulations in dimensions $d = 2,\ 3$ [11]. Global persistence has also been studied in various contexts, *e.g.* for directed percolation where corrections to Markov process were found at one-loop order [12]. However the question whether such a *universal* persistence exponent can also be defined, and computed, in the presence of quenched disorder has not been addressed in detail.

On the other hand, there is currently a wide interest in the slow relaxational dynamics following a quench at a critical point [13]. Although simpler to study than glasses, they display interesting properties such as two-time dynamics (aging) or violation of the Fluctuation Dissipation Theorem, as found in more complex glassy systems [14]. Recently, some progress has been achieved in the characterization of the effects of the disorder on these properties [15–17]. In this letter, taking advantage of these recent studies, we study persistence properties, both analytically using the Renormalization Group (RG) and numerically via Monte Carlo simulations, of the randomly diluted Ising model:

$$H = -\sum_{\langle ij \rangle} \rho_i \rho_j s_i s_j, \tag{1}$$

where s_i are Ising spins on a d-dimensional hypercubic lattice and ρ_i are *quenched* random variables such that $\rho_i = 1$ with probability p and 0 otherwise. For the experimentally relevant case of dimension $d = 3$ [18], for which the specific-heat exponent of the pure model is positive, the disorder is expected, according to Harris criterion [19], to modify the universality class of the transition. For $1 - p \ll 1$, the large-scale properties of (1) at criticality are then described by the a O(1) model with a random mass term, the so-called Random Ising Model (RIM) [20]:

$$H^{\psi}[\varphi] = \int \mathrm{d}^d x \left[\frac{1}{2} (\nabla \varphi)^2 + \frac{1}{2} [r_0 + \psi(x)] \varphi^2 + \frac{g_0}{4!} \varphi^4 \right], \tag{2}$$

where $\varphi \equiv \varphi(x)$, $\psi(x)$ is a Gaussian random variable $\overline{\psi(x)\psi(x')} = \Delta_0 \delta^d(x - x')$ and r_0, the bare mass, is adjusted so that the renormalized one is zero.

We first study the relaxational dynamics of the Randomly diluted Ising Model (2) in dimension $d = 4 - \epsilon$ described by a Langevin equation:

$$\eta \frac{\partial}{\partial t} \varphi(x,t) = -\frac{\delta H^{\psi}[\varphi]}{\delta \varphi(x,t)} + \zeta(x,t), \tag{3}$$

where $\langle \zeta(x,t) \rangle = 0$ and $\langle \zeta(x,t) \zeta(x',t') \rangle = 2\eta T \delta(x - x') \delta(t - t')$ is the thermal noise and η the friction coefficient, set to 1 in the following. At initial time $t_i = 0$, the system is in a random configuration with zero mean magnetization and short-range correlations $[\varphi(x, t = 0)\varphi(y, t = 0)]_i = \tau_0^{-1} \delta^d(x - y)$, where τ_0^{-1} is irrelevant in the RG sense [8] (it will thus be set to zero in the following). Defining the global magnetization $M(t)$ as

$$M(t) = \frac{1}{N_{\mathrm{occ}}} \sum_i \rho_i s_i(t) = \frac{1}{L^d} \int_x \mathrm{d}^d x \varphi(x,t), \tag{4}$$

where N_{occ} is the total number of occupied sites, we are interested in the disorder-averaged probability $\overline{P_c}(t)$ that the magnetization has not changed sign in the time interval t following the quench. To that purpose, we introduce the correlation $\mathcal{C}_{tt_w} = \overline{\langle M(t)M(t_w)\rangle}$ and the linear response $\mathcal{R}_{tt_w}$ to a small external field $f(t_w)$, $\mathcal{R}_{tt_w} = \overline{\delta\langle M(t)\rangle/\delta f(t_w)}$ where $\overline{...}$ and $\langle ...\rangle$ denote, respectively, averages over the disorder and thermal fluctuations. At one-loop order, $\mathcal{C}_{tt_w}$ and $\mathcal{R}_{tt_w}$ take the scaling forms [16] $\mathcal{C}_{tt_w} = A_c t^{\frac{2-\eta}{z}} \Phi_c(t/t_w)$ and $\mathcal{R}_{tt_w} = A_r t^{\frac{2-\eta-z}{z}} \Phi_r(t/t_w)$, where A_c, A_r are non-universal constants and, $\Phi_{c,r}(x)$ —not shown here for clarity— are universal scaling functions such that $\Phi_{c,r}(1) = 1$. In a previous publication [17], we have shown, using the Exact RG for the dynamical effective action [21], that they are given, at this order, by the solution of the following equations:

$$\partial_t \mathcal{R}_{tt_w} + \sigma(t)\mathcal{R}_{tt_w} + \int_{t_w}^{t} \mathrm{d}t_1 \Sigma_{tt_1} \mathcal{R}_{t_1 t_w} = 0 \quad , \qquad \sigma(t) = -\int_{t_i}^{t} \mathrm{d}t_1 \Sigma_{tt_1}, \tag{5}$$

$$\mathcal{C}_{tt_w} = 2T \int_{t_i}^{t_w} \mathrm{d}t_1 \mathcal{R}_{tt_1} \mathcal{R}_{t_w t_1} + \int_{t_i}^{t} \mathrm{d}t_1 \int_{t_i}^{t_w} \mathrm{d}t_2 \mathcal{R}_{tt_1} D_{t_1 t_2} \mathcal{R}_{t_w t_2}, \tag{6}$$

where the self-energy $\Sigma_{t_1 t_2}$ and the noise-disorder kernel $D_{t_1 t_2}$ have been computed in [17]:

$$\Sigma_{t_1 t_2} = -\frac{1}{2}\sqrt{\frac{6\epsilon}{53}} (\gamma(t_1 - t_2))^2 \quad , \qquad D_{t_1 t_2} = \frac{T_c}{2}\sqrt{\frac{6\epsilon}{53}} \left(\gamma(t_1 - t_2) - \gamma(t_1 + t_2)\right), \tag{7}$$

where $\gamma(x) = (x + \Lambda_0^{-2})^{-1}$, Λ_0 being the UV cutoff [22]. These equations (5), (6), suggest to describe the time evolution of the magnetization by an effective Gaussian process $\tilde{M}(t)$:

$$\partial_t \tilde{M}(t) + \sigma(t)\tilde{M}(t) = -\int_0^t \mathrm{d}t_1 \Sigma_{tt_1} \tilde{M}(t_1) + \tilde{\zeta}(t), \tag{8}$$

where $\tilde{\zeta}(t)$ is an effective disorder-induced Gaussian noise with zero mean and correlations $\langle \tilde{\zeta}(t)\tilde{\zeta}(t')\rangle_{\text{eff}} = 2T\delta(t - t') + D_{tt'}$. The idea is then to compute $\overline{P_c}(t)$ as the persistence probability of the process $\tilde{M}(t)$. The rhs of eq. (8) clearly indicates that this process is non-Markovian. However, as $\Sigma_{tt'}$ as well as $D_{tt'}$ are of order $\mathcal{O}(\sqrt{\epsilon})$, one can use the perturbative computation of θ_c around a Markov process initially developed in ref. [7] and further studied in the context of critical dynamics in ref. [10]. In that purpose [1], let us introduce the normalized Gaussian process $m(t) = \tilde{M}(t)/\sqrt{\langle \tilde{M}^2(t)\rangle_{\text{eff}}}$. Let $T = \ln t$, then $m(T)$ is a *stationary* Gaussian process, its persistence properties are then obtained from its autocorrelation function:

$$\langle m(T)m(T_w)\rangle_{\text{eff}} = e^{-\mu(T-T_w)}\mathcal{A}(e^{T-T_w}), \tag{9}$$

$$\mathcal{A}(x) = \left(1 + \frac{1}{4}\sqrt{\frac{6\epsilon}{53}}\left(x \log\frac{x-1}{x+1} - \log\frac{x^2-1}{4x^2}\right)\right) \quad , \qquad \mu = \frac{\lambda - d + 1 - \eta/2}{z}.$$

Under this form (9), one can use the first-order perturbation theory result of ref. [10] to obtain the one-loop estimate:

$$\Delta \equiv \theta_c - \mu = \sqrt{\frac{6\epsilon}{53}}\frac{\sqrt{2}-1}{2} + \mathcal{O}(\epsilon) = 0.06968\ldots \quad \text{in} \quad d = 3, \tag{10}$$

where μ is the value corresponding to a Markov process. The second term in eq. (10) is the first correction due to the non-Markovian nature of the dynamics. Interestingly, this correction is entirely determined by the non-trivial structure of the scaling function $\mathcal{A}(x)$, which is directly related to $\Phi_c(x)$. Notice also that, at variance with the pure O(N) models [6, 10], these corrections in the presence of quenched static disorder already appear at one-loop order.

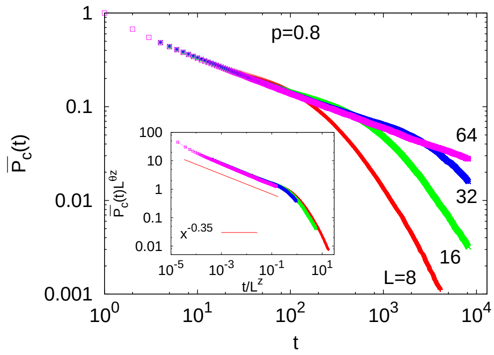

Fig. 1 – Persistence probability $\overline{P_c}(t)$ plotted in the log-log scale for $p = 0.8$ and different L, with $M_0 = 0$. Inset: scaled plot of $L^{\theta_c z}\overline{P_c}(t)$ *vs.* t/L^z for different L with $z = 2.62$ and $\theta = 0.35$.

We now turn to the results from our Monte Carlo simulations of the relaxational dynamics of the randomly diluted Ising model (1) in dimension $d = 3$, which were performed on L^3 cubic lattices with periodic boundary conditions. For $p = 1$, the system is pure and shows ferromagnetic order at $T < T_c(p = 1)$. The critical temperature $T_c(p)$ [23, 24] decreases with p and becomes 0 at the percolation threshold $p = p_c$ (for a $d = 3$ cubic lattice, $p_c \simeq 0.341$). For $p > p_c$, the system is initially prepared in a random initial configuration with zero mean magnetization $M_0 = 0$. Up and down spins are randomly distributed on the occupied sites, mimicking a high-temperature disordered configuration before the quench. At each time step, one site is randomly chosen and the move $s_i \to -s_i$ is accepted or rejected according to Metropolis rule. One time unit corresponds to L^3 such time steps. The exponent θ_c is measured numerically for cubic lattices of linear size $L = 8$, 16, 32 and 64. After a quench to T_c from the initial random configuration each system evolves until the global magnetization first changes sign. $\overline{P}_c(t)$ is then measured as the fraction of surviving systems at each time t, over a number of samples which varies from 2×10^5 for $L = 8$ to 2×10^4 for $L = 64$. Although the RG analysis asserts that the critical exponents are independent of p (at least for $1 - p \ll 1$), the question of universality in disordered systems, and in particular in the present problem [23], is a longstanding issue. Therefore we will compute $\overline{P}_c(t)$ for different values of $p(> p_c) = 0.499, 0.6, 0.65$ and 0.8 along the critical line at $T_c(p)$. In fig. 1 we present the results of $\overline{P}_c(t)$ for $p = 0.8$ and for different lattice sizes. According to standard finite-size scaling [6], one expects the scaling form $\overline{P}_c(t) = t^{-\theta_c} f(t/L^z)$, where z is the dynamical exponent. Keeping the rather well-established value of $z = 2.62(7)$ [17, 25] fixed, θ_c is varied to obtain the best data collapse. The final scaled plot is shown in the inset of fig. 1. We have checked (not shown) that the same analysis can be done for all other values of p studied here. However, the best data collapse is then obtained for a value of θ_c which is p-dependent. This is depicted in fig. 2 (left), where one clearly observes a decrease in slope while the occupation probability p is reduced, which could naively suggest a non-universal value for the exponent $\theta_c(p)$. However, it is now well known that the present model is strongly affected by corrections to scaling [17, 25], characterized by a universal, *i.e.* independent of p, exponent b, which has been estimated in previous works [17, 25] to be $b = 0.23(2)$. Therefore following these previous studies we suggest that the persistence can be written as

$$\overline{P_c}(t) = t^{-\theta_c} f_p(t) \quad , \qquad f_p(t) = A(p)(1 + B(p)t^{-b}), \tag{11}$$

where $A(p)$ and $B(p)$ are fitting parameters. As shown in the inset of fig. 2 (left), the good

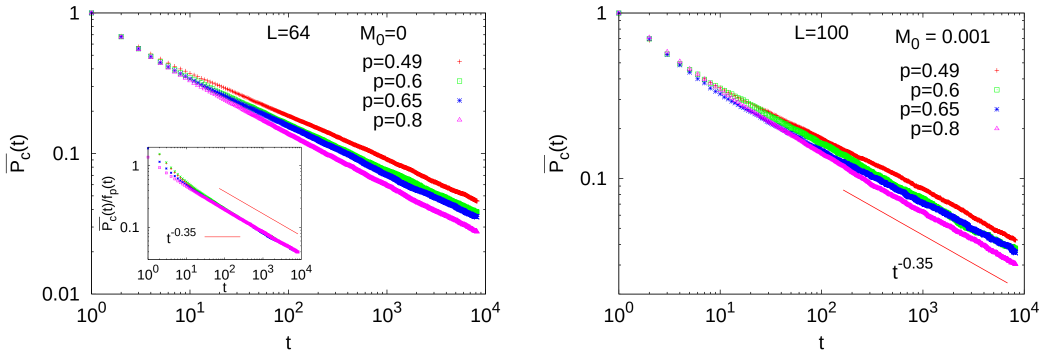

Fig. 2 – Left: persistence probability $\overline{P_c}(t)$ for different $p = 0.49$, 0.6, 0.65 and 0.8 with fixed $L = 64$. The system is initially prepared with $M_0 = 0$. Inset: universality of $\overline{P_c}(t)$ for different p. The function $f_p(t)$ is defined in eq. (11). Right: similar to left, but prepared with a non-zero initial magnetization $M_0 = 0.001$. The linear system size in this case is $L = 100$.

collapse of the quantity $P(t)/f_p(t)$ for different values of p suggest that θ_c is a *universal* exponent. $A(p)$ and $B(p)$ are found to be monotonous decreasing functions of p and such that $B(p = 0.8) \simeq 0$. The value extracted from this collapse is

$$\theta_c = 0.35 \pm 0.01. \tag{12}$$

We have also computed the persistence probability for systems quenched from random configuration with a small initial magnetization $M_0 = 0.001$. The number of up N_{up} and down N_{down} spins is thus: $N_{\text{up}} = (1 + M_0)/2N_{\text{occ}}$ and $N_{\text{down}} = N_{\text{occ}} - N_{\text{up}}$. First we randomly distribute the N_{up} up spins in the occupied sites of the lattice and then fill up the rest with down spins. As noticed previously [11] this protocol allows to reduce the statistical noise and thus to study larger system sizes (this however renders the finite-size scaling analysis more subtle [11]). In fig. 2 (right), we plot the persistence for system size $L = 100$ (the data have been averaged over 2×10^4 ensembles) for $p = 0.499$, 0.6, 0.65 and 0.8. The study of larger system size allows to reduce the corrections to scaling. Indeed, although in the short time scales, the straight lines have slightly different slopes which depends upon p, at later times the slopes varies from 0.36(1) for $p = 0.8$ to 0.35(1) for $p = 0.499$: this confirms the p-independent value of θ_c obtained previously, eq. (12).

In order to compare this numerical value (12) with our one-loop calculation (10), one needs an estimate for μ. Such an estimate is needed not only for the sake of this comparison but also to characterize quantitatively non-Markovian effects. The argument mentioned in the introduction, relying on the CLT, which says that the global magnetization is, in the thermodynamic limit, a Gaussian variable is also valid in the presence of disorder, and we have checked it numerically. Therefore, a finite difference Δ (10) is the signature of a non-Markovian process. A numerical estimate of μ can be done in two different ways [26]. First, using the numerical estimates for $\eta = 0.0374(45)$ from ref. [24], $z = 2.62(7)$ from ref. [25], and $\lambda/z = 1.05(3)$ from ref. [17], one obtains $\mu_{\text{num}} = 0.28(4)$, which gives a first estimate $\Delta_{\text{num}} = 0.07(5)$. Because of the relatively big error bars on λ and z, we propose, alternatively, to express μ in terms of the initial slip exponent , θ' [8], using $\lambda = d - z\theta'$, as $\mu = -\theta' + (1 - \eta/2)/z$. Although θ' has been estimated numerically for pure Ising systems [27, 28], there are no available data for the disordered case: we thus now present a numerical computation of θ'.

To that purpose we study the time evolution of $M(t)$ when the system is quenched from an initial configuration with short-range correlations but a finite, however small, magneti-

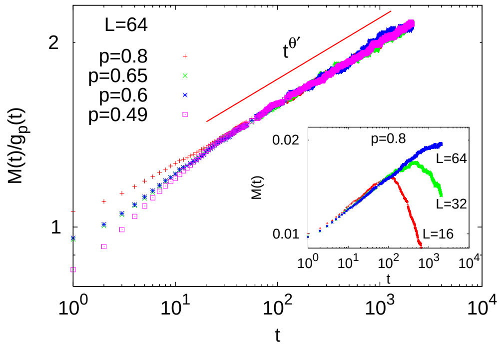

Fig. 3 – Rescaled global magnetization $M(t)/g_p(t)$ as a function of t for $p = 0.49$, 0.6, 0.65 and 0.8 in the log-log scale. The linear system size is $L = 64$, the initial magnetization is $M_0 = 0.01$ and the measured exponent $\theta' = 0.1$. Inset: global magnetization $M(t)$ for $p = 0.8$ and $L = 16, 32$ and 64.

zation M_0. The initial stage of the dynamics is characterized by an increase of the global magnetization, described by a universal power law [8],

$$M(t) \sim M_0 t^{\theta'}. \tag{13}$$

At larger times, $t \gg t_0$, critical fluctuations set in the system and cause the decrease of $M(t)$ to zero as $M(t) \sim t^{-(d-2+\eta)/(2z)}$. In our simulations we use $M_0 = 0.01$ and in all subsequent times we measure $M(t)$ (4) for linear system sizes $L = 8$, 16, 32 and 64. Finally, data are averaged over 8×10^5 samples for $L = 8$ to 10^4 samples for $L = 64$. In the inset of fig. 3, we show a plot of $M(t)$ for $p = 0.8$ and different system sizes. One sees clearly that $M(t)$ is increasing until a time t_0, which is an increasing function of L for the sizes considered here [29] and compatible with the scaling $t_0 \sim L^z$, and then decreases to zero (although the aforementioned scaling for $t \gg t_0$ is not clearly seen here). By computing $M(t)$ for different values of $p = 0.499$, 0.6, 0.65 and 0.8, we observe corrections to scaling which we take into account as $M(t) = t^{\theta'} g_p(t)$ with $g_p(t) = A'(p)[1 + B'(p)t^{-b}]$, $A'(p)$, $B'(p)$ being fitting parameters. As shown in fig. 3, one obtains a reasonably good data collapse of $M(t)/g_p(t)$ *vs.* t for the different values of p. Here also, one finds that $p = 0.8$ is almost unaffected by these corrections to scaling, *i.e.* $B'(0.8) \simeq 0$. After a microscopic time scale, which increases as p is lowered, one observes a universal power law increase (13), from which we get the estimate $\theta' = 0.10(2)$. This value is in good agreement with the two-loops estimate $\theta'_{2\text{ loops}} = 0.0868$ [30] and agrees also quite well with the scaling relation $\lambda = d - z\theta'$ [17]. This gives $\mu_{\text{num}} = 0.27(3)$ and our final numerical estimate

$$\Delta_{\text{num}} = 0.08 \pm 0.04, \tag{14}$$

which is in good agreement with our previous one-loop estimate (10). We also notice that these deviations from a Markov process are slightly larger than for the pure case [11].

In summary, we have shown that the global persistence exponent θ_c carries the signature of non-Markovian effects, and is thus a new critical exponent characterizing this random critical point, that we have computed within a one-loop approximation (10). Our detailed numerical analysis, which relies upon a computation of θ_c and the initial slip exponent θ', supports the existence of these non-Markovian violations (14) and is in good quantitative agreement with our perturbative approach. In addition, θ_c is found to be universal along the critical line. In view of recent progress in the study of aging properties in finite-dimensional glassy phases [31], it would be very interesting to extend the approach presented here to these situations.

* * *

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