Exact Distribution of the Maximal Height of *p* Vicious Walkers

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Using path-integral techniques, we compute exactly the distribution of the maximal height H_p of p nonintersecting Brownian walkers over a unit time interval in one dimension, both for excursions p watermelons with a wall, and bridges p watermelons without a wall, for all integer $p \geq 1$. For large p, we show that $\langle H_p \rangle \sim \sqrt{2p}$ (excursions) whereas $\langle H_p \rangle \sim \sqrt{p}$ (bridges). Our exact results prove that previous numerical experiments only measured the preasymptotic behaviors and not the correct asymptotic ones. In addition, our method establishes a physical connection between vicious walkers and random matrix theory.

Introduction.—Since the pioneering work of de Gennes [1], followed up by Fisher [2], the subject of vicious (nonintersecting) random walkers has attracted a lot of interest among physicists. It has been studied in the context of wetting and melting [2], networks of polymers [3] and fibrous structures [1], persistence properties in nonequilibrium systems [4] and stochastic growth models [5,6]. There also exist connections between the vicious walker problem and the random matrix theory (RMT) [7–9], including for instance Dyson's Brownian motion [10]. These connections to RMT have rekindled recent interest in the vicious walker problem and have led to new interesting questions. However, despite extensive recent mathematical literature on the subject, the connections to RMT have so far been established using mostly combinatorial approaches. Given the nonintersection constraint in the vicious walker problem, it is natural to expect a free Fermion approach to make its connection to RMT physically more explicit. The aim of this Letter is to present such an approach which, in addition, allows us to derive a variety of new exact results in the vicious walker problem.

Physically, one-dimensional vicious walkers play an important role in describing the elementary topological excitations in the $p \times 1$ commensurate adsorbed phases close to the commensurate-incommensurate (C-IC) transition [11]. In the commensurate phase the elementary excitations are pairs of dislocations at a given distance with p nonintersecting domain walls emerging from one and terminating at the other. This is just a "watermelon" pattern configuration of p nonintersecting Brownian bridges [see Fig. 1(b)]. The sizes of such defects and their fluctuations become important near the phase transition. An important quantity that characterises the transverse fluctuations of the defect is the maximal height of the p vicious walkers in a fixed time (here time signifies the fixed longitudinal distance between the pair). Such extreme value questions have recently been studied extensively for a *single* Brownian bridge or an excursion (with certain constraints) in the

context of the maximal height of a fluctuating interface [12,13]. In this Letter, we obtain exactly the distribution of the maximal height for p nonintersecting Brownian bridges and excursions.

Motivated by the geometry of elementary excitations discussed above, we thus focus on "watermelons" pattern configurations [see Figs. 1(a) and 1(b)] where p nonintersecting Brownian walkers $x_1(\tau) < \cdots < x_p(\tau)$, starting at 0 at time $\tau = 0$, arrive at the same position at $\tau = 1$. We consider both "p watermelons with a wall" [Fig. 1(a)], where the walkers stay positive in the time interval [0, 1] and "p-watermelons without wall" [Fig. 1(b)] where the walkers are free to cross the origin in between. Our main focus is on H_p, the maximal height of the top walker in [0, 1], $H_p = \text{Max}_\tau [x_p(\tau)0, \leq \tau \leq 1]$.

In particular, we are interested in the cumulative distribution $F_p(M) = \text{Prob}[H_p \leq M]$ and in the moments $\langle H_p^s \rangle$. For $p = 1$, there exist well known results [14], e.g., $\langle H_1 \rangle = \sqrt{\pi/2}$ for an excursion, or $\langle H_1 \rangle = \sqrt{\pi/8}$ for a bridge. Recently, Bonichon and Mosbah (BM) [15], using an algorithm based on exact enumerative formulas [16], conjectured, from numerical simulations, that for $p > 1$, $\langle H_p \rangle_{\text{num}} \simeq \sqrt{1.67p - 0.06}$ for watermelons with a wall
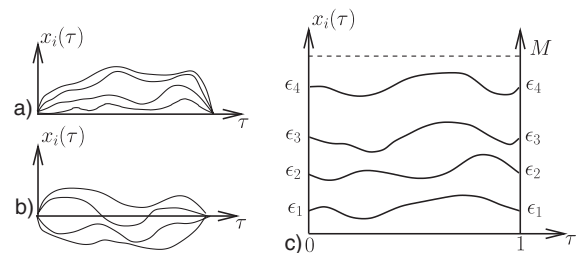

FIG. 1.   (a) Four watermelons with a wall. (b) Four watermelons without a wall. (c) Illustration of the method to compute $F_4(M)$ using path-integral techniques where appropriate cutoffs ϵ_i's have been introduced.

and $\langle H_p \rangle_{\text{num}} \simeq \sqrt{0.82p - 0.46}$ for watermelons without wall. These results stimulated several recent works [17–21] aiming at an analytical derivation of these estimates.

On the other hand, exploiting the recent connection between watermelons and the Airy processes [9,22], setting $\tilde{x}_p(\tau) = x_p(\tau)/\sqrt{\tau(1-\tau)}$, one expects that, in the limit $p \to \infty$, $\tilde{x}_p(\tau) = A\sqrt{p} + p^{-1/6}\xi$ where $A = 2^{3/2}$ (excursions) and $A = 2$ (bridges), where ξ is the Airy$_2$ process [9,22] of a suitably rescaled time parameter. Thus in the large p limit, the top curve approaches a limit shape, $x_p(\tau) \to A\sqrt{p}\sqrt{\tau(1-\tau)}$. Since the maximum of the top curve occurs at the midpoint $\tau = 1/2$, one expects that for $p \gg 1$, $\langle H_p \rangle \sim \langle x_p(\tau = \frac{1}{2}) \rangle \sim \sqrt{2p}$ for excursions and, similarly, $\langle H_p \rangle \sim \sqrt{p}$ for bridges. These exact asymptotic estimates differ considerably from the numerical estimates of BM suggesting that the latter only describe the preasymptotic behavior of $\langle H_p \rangle$. However, it calls for an explanation why this preasymptotic behavior as measured by BM should be about $\sqrt{1.67p}$ and $\sqrt{0.82p}$.

In this Letter, we present a method based on path integrals associated to corresponding free Fermions models to compute exactly $F_p(M)$. Our exact formula is useful for a number of reasons. It provides the exact asymptotic tails of the distribution of H_p which were not known before. For the average height, our formula explains the aforementioned discrepancy between the estimates of BM and the exact asymptotic behaviors of $\langle H_p \rangle$. We show that for moderate values of p (preasymptotic behavior), one obtains $\langle H_p \rangle \propto \pi\sqrt{p/6} = \sqrt{1.644\,93\cdots p}$ for excursions and $\langle H_p \rangle \propto \pi\sqrt{p/12} = \sqrt{0.822\,467\cdots p}$ for bridges, in nice agreement with BM's estimates. Finally, we show how our method allows for a physical derivation of the connection between p-watermelons' configurations and RMT.

Method.—To calculate the cumulative distribution $F_p(M)$, we use a path-integral method which needs to be suitably adapted to this problem. Indeed one notices that the p-watermelons' configurations described above [see, e.g., Figs. 1(a) and 1(b)] are ill defined for systems in continuous space and time. For such Brownian walks, it is well known that if two walkers cross each other once, they will recross each other infinitely many times immediately after the first crossing. Therefore, it is impossible to enforce the constraint $x_i(0) = x_{i+1}(0) = 0$ and simultaneously forcing $x_i(\tau) < x_{i+1}(\tau)$ immediately after. The cleanest way to circumvent this problem is to consider discrete time random walks moving on a discrete one-dimensional lattice (so called Dyck path): this is the method used in Refs. [16,17,19,20]. By taking the diffusion continuum limit, one would then arrive at nonintersecting Brownian motions [23]. This method is however mathematically cumbersome. Alternatively, following Refs. [12,24], we can go around this problem by assuming that the starting and finishing positions of the p walkers are $0 < \epsilon_1 < ... < \epsilon_p$ [see Fig. 1(c)]). Only at the end we take

the limit $\epsilon_i \to 0$ and show that it is well defined. In addition, in order to compute $F_p(M)$, we put an absorbing hard wall at M such that

$$F_p(M) = \lim_{\epsilon_i \to 0} \left[\frac{N(\epsilon, M)}{N(\epsilon, M \to \infty)} \right], \tag{1}$$

where $\epsilon \equiv \epsilon_1, \cdots, \epsilon_p$ and $N(\epsilon, M)$ is the probability that the p Brownian paths starting at $0 < \epsilon_1 < ... < \epsilon_p$ at $\tau = 0$ come back to the same points at $\tau = 1$ without crossing each other and staying within the interval $[\mu, M]$, with $\mu = 0$ for excursions and $\mu \to -\infty$ for bridges. This procedure is depicted in Fig. 1(c).

The probability measure associated to p unconstrained Brownian paths $x_1(\tau), .., x_p(\tau)$ over the time interval $[0, 1]$ is proportional to $\exp[-\frac{1}{2}\sum_{i=1}^{p}\int_0^1 (\frac{dx_i}{d\tau})^2 d\tau]$. Here, we have to incorporate the constraint that they stay in the interval $[\mu, M]$. Therefore one can use path-integral techniques to write $N(\epsilon, M)$ in Eq. (1) as the propagator

$$N(\epsilon, M) = \langle \epsilon | e^{-\hat{H}_M} | \epsilon \rangle, \tag{2}$$

with $\hat{H}_M = \sum_{i=1}^{p} [\frac{-1}{2}\frac{\partial^2}{\partial x_i^2} + V(x_i)]$, where $V(x)$ is a confining potential with $V(x) = 0$ if $x \in [\mu, M]$ and $V(x) = \infty$ outside this interval. Denoting by E the eigenvalues of $\hat{H}_M$ and $|E\rangle$ the corresponding eigenvectors one has

$$N(\epsilon, M) = \sum_E |\Psi_E(\epsilon)|^2 e^{-E}, \tag{3}$$

where we introduced the notation $\langle \mathbf{x}|E \rangle = \Psi_E(\mathbf{x})$. Importantly, to take into account the fact that we are considering here nonintersecting Brownian paths, the many-body wave function $\Psi_E(\mathbf{x}) \equiv \Psi_E(x_1, .., x_p)$ must be Fermionic, i.e., it vanishes if any of the two coordinates are equal. This many-body antisymmetric wave function is thus constructed from the one-body eigenfunctions of $\hat{H}_M$ by forming the associated Slater determinant.

Watermelons with a wall.—In that case $\mu = 0$ and the one-body eigenfunctions are given by $\phi_n(x) = \sqrt{\frac{2}{M}}\sin\frac{n\pi x}{M}$ with discrete eigenvalues $\frac{n^2\pi^2}{2M^2}$, $n \in \mathbb{N}^*$. Therefore one has

$$\Psi_E(\epsilon) = \frac{1}{\sqrt{p!}} \det_{1 \le i,j \le p} \phi_{n_i}(\epsilon_j), \qquad E = \frac{\pi^2}{2M^2}\mathbf{n}^2, \tag{4}$$

where we use the notation $\mathbf{n}^2 = \sum_{i=1}^{p} n_i^2$, $n_i \in \mathbb{N}^*$. From this expression (4), one checks that, in the limit $\epsilon_1 \to 0, \cdots, \epsilon_p \to 0$, powers of ϵ_i's cancel between the numerator and the denominator in Eq. (1), yielding

$$F_p(M) = \frac{A_p}{M^{2p^2+p}} \sum_{n_1,\cdots,n_p} [\Xi(\mathbf{n})]^2 e^{-(\pi^2/2M^2)\mathbf{n}^2},$$

$$\Xi(\mathbf{n}) = \prod_{1 \le j < k \le p} (n_j^2 - n_k^2) \prod_{i=1}^{p} n_i, \tag{5}$$

where A_p, a constant independent of M, is determined by requiring that $\lim_{M\to\infty} F(M) = 1$. It can be evaluated

using a Selberg's integral [25] yielding $A_p = \pi^{2p^2+p}/[2^{p^2-p/2}\prod_{j=0}^{p-1}\Gamma(2+j)\Gamma(\frac{3}{2}+j)]$. For $p=1$, our expression gives back the well known result for a Brownian excursion [26]. For $p=2$, we have checked, using the Poisson summation formula that our expressions in Eq. (5) yield back the result of Ref. [18]. For generic p, the probability distribution function (PDF) $F'_p(M)$ is bell shaped, exhibiting a single mode. At variance with previous studies [18,19], our expression (5) is easily amenable to an asymptotic analysis for small M. Indeed, when $M \to 0$, the leading contribution to the sum in (5) comes from $n_i = i$ and its $p!$ permutations, yielding for $M \to 0$

$$F_p(M) \sim \frac{\alpha_p}{M^{2p^2+p}} e^{-(\pi^2/12M^2)p(p+1)(2p+1)}, \quad (6)$$

where α_p can be explicitly computed, yielding for instance $\alpha_2 = 12\pi^9$. For large M, one can use the Poisson summation formula to obtain $1 - F_p(M) \propto \exp(-2M^2)$.

From the distribution in Eq. (5), one can compute the moments of the distribution $\langle H_p^s \rangle$. For $p \geq 2$, one obtains that $\langle H_p \rangle$ can be expressed in terms of integrals involving the Jacobi theta function $\vartheta(u) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 u}$ and its derivatives, thus recovering, by a simpler physical derivation, the results of Refs. [17–20]. In particular, one has $\langle H_2 \rangle = 1.82262\ldots$ [17]. For moderate values of p, one observes that the main contribution to the average $\langle H_p \rangle = \int_0^\infty M F'_p(M) dM$ comes from relatively small M where $F'_p(M)$ is dominated, as before in Eq. (6), by the terms where $n_i = i$ and its $p!$ permutations. It is easy to see that the PDF, restricted to this first term (6) exhibits a maximum for $M^* \sim \pi\sqrt{p/6}$. Therefore, one expects that $\langle H_p \rangle \sim M^* = \sqrt{1.64493\cdots p}$, in good agreement with the estimates of BM [15]. For larger values of p the average $\langle H_p \rangle$ picks up contributions from larger values of M where $F'_p(M)$ can not be approximated by a single term as in Eq. (6) and therefore the estimate of BM ceases to be correct. Instead, one has the exact asymptotic behavior $\langle H_p \rangle \sim \sqrt{2p}$ for $p \gg 1$, which can be obtained directly from our formula in Eq. (5) [27].

Watermelons without walls.—In the case of Brownian bridges, one can apply the same formalism as above (1)–(3) with $\mu \to -\infty$, i.e. $\hat{H}_M = \sum_{i=1}^p \frac{-1}{2}\frac{\partial^2}{\partial x_i^2}$. In that case, the one-body eigenfunctions are given by $\psi_k(x) = \sqrt{\frac{2}{\pi}}\sin[k(M-x)]$ with a continuous spectrum $E_k = k^2/2$, $k \in \mathbb{R}^+$. Therefore, $\Psi_E(\epsilon)$ entering the expression of $N(\epsilon, M)$ in Eq. (3) is formally given by Eq. (4) where ϕ_{n_i} is replaced by ψ_{k_i} and $E = \frac{k^2}{2}$. One obtains

$$F_p(M) = \frac{B_p}{M^{p^2}}\int_0^\infty dy_1 \cdots \int_0^\infty dy_p\, e^{-(\mathbf{y}^2/2M^2)}\Theta_p(\mathbf{y})^2,$$

$$\Theta_p(\mathbf{y}) = \det_{1 \leq i,j \leq p} y_i^{j-1}\cos\left(y_i + j\frac{\pi}{2}\right), \quad (7)$$

where $B_p = 2^{2p}/[(2\pi)^{p/2}\prod_{j=1}^p \Gamma(j+1)]$. This yields, for

instance, $F_2(M) = 1 - 4M^2 e^{-2M^2} - e^{-4M^2}$. From Eq. (7), one obtains the asymptotic behavior for $M \to 0$ as

$$F_p(M) \propto M^{p^2+p}, \quad (8)$$

whereas for large M one has $1 - F_p(M) \propto \exp(-2M^2)$. As in the case of watermelons with a wall, the PDF $F'_p(M)$ is also bell shaped with a single mode. Notice, however, that the presence of the wall has drastic effects on the small M behavior of $F_p(M)$ [see Eq. (6) and (8)] whereas, as expected, it has less influence for large M.

From $F_p(M)$ in (7), one computes the moments $\langle H_p^s \rangle$, yielding $\langle H_2 \rangle = \frac{1+\sqrt{2}}{4}\sqrt{\pi}$ or $\langle H_3 \rangle = \frac{45+36\sqrt{2}-8\sqrt{6}}{96}\sqrt{\pi}$, recovering (to leading order) recent results obtained by rather involved combinatorial techniques [20].

To make contact with BM's estimates, one first focuses on $p = 2$ and notices that $\Theta_2(y_1, y_2)$ in Eq. (7) exhibits saddles for $y_1 = \pm\pi/2$, $y_2 = \pm\pi$ and for symmetric points obtained by permutations: this is shown in Fig. 2(a). In fact this property can be generalized to higher values of p and one can show that $\Theta_p(\mathbf{y})$ has saddles which are located around $y_1 = \pm\pi/2$, $y_2 = \pm\pi, \cdots, y_p = \pm p\pi/2$ and the points obtained by permutations. Of course $\Theta_p(\mathbf{y})$ develops saddles for higher values of $\mathbf{y}^2$ but their weights are exponentially suppressed in Eq. (7). For moderate values of p, one expects that $\langle H_p \rangle$ is dominated by these saddles $y_1 = \pm\pi/2$, $y_2 = \pm\pi, \cdots, y_p = \pm p\pi/2$. Therefore performing a saddle point calculation, one has $F_p(M) \propto e^{-p^2\chi((M/\sqrt{p}))}$, with $\chi(y) = \log y + \pi^2/(24y^2)$, which has a minimum for $y^* = \pi/\sqrt{12}$. This yields $\langle H_p \rangle \sim \pi\sqrt{p/12} = \sqrt{0.822467\cdots p}$, in good agreement with the estimates of BM [15]. For larger values of p one expects that $\langle H_p \rangle$ picks up contributions from larger values of M where $F_p(M)$ can not be reduced to these first saddles. In Fig. 2(b), one shows a comparison between the exact value of $\langle H_p \rangle^2$ computed from Eq. (7) and the estimate of BM. This clearly shows that the estimate of BM corre-
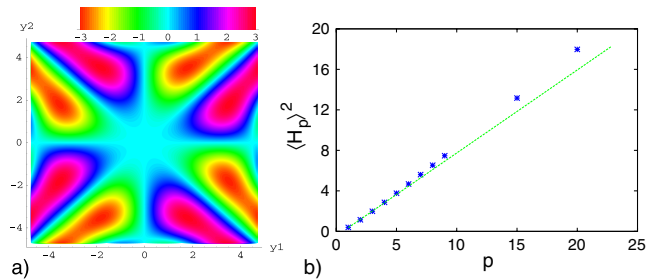

FIG. 2 (color online). (a) Contour plot of $\Theta_2(y_1, y_2) = y_2 \sin y_1 \cos y_2 - y_1 \cos y_1 \sin y_2$ given in (7). It exhibits saddles for $(y_1, y_2) = (\pm\pi/2, \pm\pi)$ and symmetric points obtained by permutations. (b) Plot of $\langle H_p \rangle^2$ as a function of p. The dotted line is the estimate from BM [15]. The quality of this estimate for $p \lesssim 10$ has its origin in the saddles of $\Theta_p(\mathbf{y})$ shown, for $p = 2$, on the left panel. For larger values of p one has instead $\langle H_p \rangle^2 \propto p$.

spond to the preasymptotic behavior. Instead, for large p, one expects here $\langle H_p \rangle \propto \sqrt{p}$.

Extension of the method.—The method presented here can be used to derive many other results. As an interesting example, showing explicitly the connection between watermelons and RMT, we compute the joint probability distribution $P_{\text{joint}}(x_1, \cdots, x_p, \tau)$, first for p bridges. Following the same steps as above, Eq. (1)–(3), and using the Markov property of Brownian paths, one has

$$P_{\text{joint}}(\mathbf{x}, \tau) = \lim_{\epsilon_i \to 0} \frac{\langle \boldsymbol{\epsilon} | e^{-\tau \hat{H}_0} | \mathbf{x} \rangle \langle \mathbf{x} | e^{-(1-\tau)\hat{H}_0} | \boldsymbol{\epsilon} \rangle}{\langle \boldsymbol{\epsilon} | e^{-\hat{H}_0} | \boldsymbol{\epsilon} \rangle} \quad (9)$$

with $\hat{H}_0 = \sum_{i=1}^{p} \frac{-1}{2} \frac{\partial^2}{\partial x_i^2}$. One can show that powers of ϵ_i's cancel between the numerator and the denominator in (9), yielding $P_{\text{joint}}(\mathbf{x}, \tau) \propto Q(\mathbf{x}, \tau) Q(\mathbf{x}, 1 - \tau)$ with

$$Q(\mathbf{x}, \tau) = \int d\mathbf{k} \prod_{i<j} (k_i - k_j) e^{-(\tau \mathbf{k}^2/2)} \det_{1 \le m, n \le p} e^{(ix_m k_n)}, \quad (10)$$

where $\int d\mathbf{k} \equiv \int_{-\infty}^{\infty} dk_1 \cdots \int_{-\infty}^{\infty} dk_p$. After some algebra to evaluate the integrals in Eq. (10) one finally obtains, for p watermelons without wall

$$P_{\text{joint}}(\mathbf{x}, \tau) = Z_p^{-1} \sigma(\tau)^{-p^2} \prod_{i<j} (x_i - x_j)^2 e^{-(\mathbf{x}^2/2\sigma^2(\tau))}, \quad (11)$$

with $\sigma(\tau) = \sqrt{\tau(1-\tau)}$ and Z_p a normalization constant. This expression in Eq. (11) shows that this joint probability is exactly the one of the eigenvalues of the Gaussian unitary ensemble of random matrices (GUE) [7,8,10]. In particular, for $p \gg 1$, defining the rescaled variable $\eta = \sqrt{2} p^{1/6} \left(\frac{x_p(\tau)}{\sqrt{2}\sigma(\tau)} - \sqrt{2p} \right)$, one obtains that the cumulative distribution of η is given by Proba$[\eta \le x] = \mathcal{F}_2(x)$, the Tracy-Widom distribution for $\beta = 2$ [28].

For excursions, a similar calculation shows that

$$P_{\text{joint}}(\mathbf{x}, \tau) = Z_p'^{-1} \sigma(\tau)^{-p(2p+1)} [\Xi(\mathbf{x})]^2 e^{-(\mathbf{x}^2/2\sigma^2(\tau))}, \quad (12)$$

where $\Xi(\mathbf{x})$ is defined in (5) and Z_p' a normalization constant. Hence the joint distribution of $y_i = x_i^2/2\sigma^2(\tau)$ is formally identical to the distribution of the eigenvalues of Wishart matrices [25] with $M - N = \frac{1}{2}$, and $N = p$. In that case, from the results for the largest eigenvalue of Wishart matrices we conclude that for $p \gg 1$, the cumulative distribution of the rescaled variable $\zeta = 2^{2/3} p^{1/6} \left(\frac{x_p(\tau)}{\sqrt{2}\sigma(\tau)} - 2\sqrt{p} \right)$ is again given by $\mathcal{F}_2(x)$ [29].

Conclusion.—To conclude, using methods of many-body physics, where appropriate cutoffs ϵ_i's have been introduced [see Fig. 1(c)], we have obtained exact results for the distribution of the maximal height for p watermelons with a wall (5) and without walls (7), which is physically relevant to describe the geometrical properties of dislocations arising in $p \times 1$ commensurate adsorbed phases close to the C-IC transition. Our expressions ex-

plain the discrepancy between the estimates of BM [15] and the true asymptotic behaviors for the average $\langle H_p \rangle$. Besides, we obtained a quantitative description of the preasymptotic regime actually measured in the numerical experiments of BM. We hope that the path-integral method presented here, which is rather general, and the precise connection to RMT will allow further future studies.

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