## Current fluctuations in non-equilibrium diffusive systems: an additivity principle

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We formulate a simple additivity principle allowing to calculate the whole distribution of current fluctuations through a large one dimensional system in contact with two reservoirs at unequal densities from the knowledge of its first two cumulants. This distribution (which in general is non-Gaussian) satisfies the Gallavotti-Cohen symmetry and generalizes the one predicted recently for the symmetric simple exclusion process. The additivity principle can be used to study more complex diffusive networks including loops.

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Understanding the fluctuations of the steady state current through a system in contact with two (or more) heat or particle reservoirs is one of the simplest and most fundamental problems of non-equilibrium physics [1, 2, 3]. For quantum particles such as (weakly interacting) electrons which satisfy the Pauli principle, the whole distribution (the full counting statistics) of the number of particles transferred between the two reservoirs during a long time interval is known [4] and it can be calculated by a number of theoretical approaches [3, 5, 6, 7], ranging from the theory of random matrices [4, 8] to the Boltzmann-Langevin semiclassical description [9].

For systems of purely classical interacting particles [1, 2] in contact with two reservoirs the theory is, to our knowledge, less developed. However, for a number of stochastic models of classical interacting particles [10, 11, 12, 13], the cumulants of the current fluctuations were found to coincide with those previously known of non-interacting quantum particles. It is of course an important issue to know under what condition a classical particle system could present the same distribution of current as in the quantum case.

For most theoretical approaches developped in the quantum or in the classical description, the calculation of the cumulants becomes harder and harder as the degree of the cumulants increases. The goal of the present letter is to show that for classical stochastic models, if one postulates a simple additivity principle for the current fluctuations, the whole distribution of current fluctuations can be calculated from the knowledge of the first two cumulants of the current.

We consider here a one dimensional diffusive open system of length N (with N large) in contact, at its two ends, with two reservoirs of particles at densities  $\rho_a$  and  $\rho_b$ . In the bulk, the system evolves under some conservative stochastic dynamics and, at the boundaries, particles are created or annihilated to match the densities of the reservoirs.

Let  $Q_t$  be the integrated current up to time t, i.e. the number of particles which went through the system during time t. For large N, we shall see that the whole distribution of the fluctuations of  $Q_t$  depends only on two macroscopic parameters  $D(\rho)$  and  $\sigma(\rho)$  defined as follows: Suppose that for  $\rho_a = \rho + \Delta \rho$  and  $\rho_b = \rho$  with  $\Delta \rho$  small, we know that in the steady state Fick's law holds

$$\frac{\langle Q_t \rangle}{t} = \frac{1}{N} D(\rho) \ \Delta \rho \,. \tag{1}$$

Suppose that for  $\rho_a = \rho_b = \rho$  (in which case  $\langle Q_t \rangle = 0$ ), we also know that for large t the variance is

$$\frac{\langle Q_t^2 \rangle}{t} = \frac{1}{N} \sigma(\rho) \,. \tag{2}$$

The main result of the present paper is that, using a simple additivity principle (10,11), we can predict *all* the cumulants of  $Q_t$  for arbitrary  $\rho_a$  and  $\rho_b$ . If we define the integrals  $I_n$  by

$$I_n = \int_{\rho_b}^{\rho_a} D(\rho) \ \sigma(\rho)^{n-1} \ d\rho \,,$$

the first cumulants of  $Q_t$  are given by

$$\frac{\langle Q_t \rangle}{t} = \frac{1}{N} I_1, \quad \frac{\langle Q_t^2 \rangle - \langle Q_t \rangle^2}{t} = \frac{1}{N} \frac{I_2}{I_1} \tag{3}$$

$$\frac{\langle Q_t^3 \rangle_c}{t} = \frac{1}{N} \frac{3(I_3 I_1 - I_2^2)}{I_1^3} \tag{4}$$

$$\frac{\langle Q_t^4 \rangle_c}{t} = \frac{1}{N} \frac{3(5I_4I_1^2 - 14I_1I_2I_3 + 9I_2^3)}{I_1^5} \tag{5}$$

The case  $\rho_a = \rho_b$  can be obtained by letting  $\rho_a$  tend to  $\rho_b$  in the previous expressions.

More generally, all the higher cumulants can be obtained from the knowledge of  $\mu_N$  which characterizes the large t growth of the generating function of  $Q_t$ 

$$\mu_N(\lambda, \rho_a, \rho_b) = \lim_{t \to \infty} t^{-1} \ln \langle e^{\lambda Q_t} \rangle \,. \tag{6}$$

We are going to show that, for large N,  $\mu_N$  takes the following parametric form

$$\mu_N(\lambda, \rho_a, \rho_b) = -\frac{K}{N} \left[ \int_{\rho_b}^{\rho_a} \frac{D(\rho) \, d\rho}{\sqrt{1 + 2K\sigma(\rho)}} \right]^2 + o\left(\frac{1}{N}\right),\tag{7}$$

where  $K = K(\lambda, \rho_a, \rho_b)$  is the solution of

$$\lambda = \int_{\rho_b}^{\rho_a} d\rho \frac{D(\rho)}{\sigma(\rho)} \left[ \frac{1}{\sqrt{1 + 2K\sigma(\rho)}} - 1 \right].$$
(8)

As  $\mu_N = (\lambda \langle Q_t \rangle + \lambda^2 \langle Q_t^2 \rangle_c / 2 + \lambda^3 \langle Q_t^3 \rangle_c / 6 + ...)/t$ , one simply needs to expand (7) and (8) in powers of K and to eliminate K to obtain  $\mu_N$  as a power series of  $\lambda$  and the cumulants such as (3-5).

Note that (7) and (8) are only valid for  $\rho_a \neq \rho_b$  and in the range of values of  $\lambda$  where K is large enough for the argument of the square root in the integrants not to vanish. We checked that they can also be analytically continued to cover the other ranges of  $\lambda$  and the case  $\rho_a = \rho_b$ .

Our derivation of (7) and (8) is based on an additivity principle that we are going to formulate now. The probability  $P_N(q, \rho_a, \rho_b, t)$  of observing an integrated current  $Q_t = qt$  is exponential in t for large t

$$P_N(q, \rho_a, \rho_b, t) \sim \exp[t F_N(q, \rho_a, \rho_b, \text{contacts})], \quad (9)$$

where  $F_N(q, \rho_a, \rho_b, \text{contacts})$  depends on the length N of the system, on q, on the densities  $\rho_a$  and  $\rho_b$  in the two reservoirs, and on the nature of the contacts of the system with the two reservoirs. ( $F_N$  is negative and vanishes only when q takes its most likely value  $\langle Q_t \rangle / t$ ). When N is large and q is of order 1/N, the effect of the contacts becomes negligible and asymptotically  $F_N(q, \rho_a, \rho_b)$ depends only on  $q, \rho_a, \rho_b$ , on the length N and on the bulk properties of the system. We then assume that, for large N and q of order 1/N, the large deviation function  $F_N(q, \rho_a, \rho_b)$  satisfies the following additivity principle:

$$F_{N+N'}(q,\rho_a,\rho_b) \simeq \max_{\rho} \left\{ F_N(q,\rho_a,\rho) + F_{N'}(q,\rho,\rho_b) \right\} .$$
(10)

This property simply means that the two subsystems are independent, except that they try to ajust the density  $\rho$  at their contact to maximize the following product

$$P_{N+N'}(q, \rho_a, \rho_b, t) \sim \max_{\rho} [P_N(q, \rho_a, \rho, t) P_{N'}(q, \rho, \rho_b, t)].$$

We make also the following scaling hypothesis

$$F_N(q,\rho_a,\rho_b) \simeq N^{-1} G(Nq,\rho_a,\rho_b).$$
(11)

This hypothesis, which is valid in particular for the symmetric simple exclusion process, means that  $\mu_N$  defined by (6) is of order 1/N for large N (see [13]).

If we write N = (N + N')x, i.e. we split a system of macroscopic unit length into two parts of lengths x and 1 - x, then (10,11) lead to

$$G(q, \rho_a, \rho_b) = \max_{\rho} \left\{ \frac{G(qx, \rho_a, \rho)}{x} + \frac{G(q(1-x), \rho, \rho_b)}{1-x} \right\} 12$$

If we keep dividing the system into smaller and smaller pieces and we use that for a piece of small (macroscopic) size  $\Delta x$  (i.e. of  $N\Delta x$  sites) one has (1, 2,10,11)

$$\frac{1}{\Delta x}G(q\Delta x,\rho,\rho+\Delta\rho) \simeq -\frac{\left[q\Delta x+D(\rho)\,\Delta\rho\right]^2}{2\sigma(\rho)\Delta x}\,.$$
 (13)

one finds a variational form for  ${\cal G}$ 

$$G(q, \rho_a, \rho_b) = -\min_{\rho(x)} \left[ \int_0^1 \frac{\left[q + D(\rho(x))\rho'(x)\right]^2}{2\sigma(\rho(x))} \, dx \right] (14)$$

where the minimum is over all the functions  $\rho(x)$  with boundary conditions  $\rho(0) = \rho_a$  and  $\rho(1) = \rho_b$ .

The optimal  $\rho(x)$  in (14) satisfies

$$q^{2}a'(\rho) - c'(\rho)\left(\frac{d\rho}{dx}\right)^{2} - 2c(\rho)\frac{d^{2}\rho}{dx^{2}} = 0,$$

where  $a(\rho) = (2\sigma(\rho))^{-1}$  and  $c(\rho) = D^2(\rho) a(\rho)$ . Multiplying the above equation by  $d\rho(x)/dx$ , one obtains after one integration

$$D^{2}(\rho) \left(\frac{d\rho}{dx}\right)^{2} = q^{2}(1 + 2K\sigma(\rho)), \qquad (15)$$

where K is a constant of integration.

To proceed further one needs to determine the sign of  $\frac{d\rho}{dx}$ . The simplest case is when  $\rho(x)$  is monotone, and this happens when q is close enough to its average value for  $\rho_a \neq \rho_b$  (this corresponds to values of K small enough for the right hand side of (15) not to vanish). The investigation of this regime is enough to determine all the cumulants. If for example  $\rho_a > \rho_b$ , the optimal  $\rho(x)$  is decreasing for small K

$$\frac{d\rho}{dx} = -\frac{q}{D(\rho)}\sqrt{1 + 2K\sigma(\rho)},\qquad(16)$$

and this leads to the following expression for G:

$$G = q \int_{\rho_b}^{\rho_a} \frac{D(\rho)}{\sigma(\rho)} \left[ 1 - \frac{1 + K\sigma(\rho)}{\sqrt{1 + 2K\sigma(\rho)}} \right] d\rho , \qquad (17)$$

where the constant K is determined by:

$$q = \int_{\rho_b}^{\rho_a} d\rho \frac{D(\rho)}{\sqrt{1 + 2K\sigma(\rho)}} \,. \tag{18}$$

One can then show that

$$\frac{\partial G}{\partial q} = \frac{G}{q} + Kq = \int_{\rho_b}^{\rho_a} d\rho \frac{D(\rho)}{\sigma(\rho)} \left[ 1 - \frac{1}{\sqrt{1 + 2K\sigma(\rho)}} \right]$$

where the derivative is taken keeping  $\rho_a$  and  $\rho_b$  fixed, and using the fact that  $\mu_N = N^{-1} \max_q [\lambda q + G(q, \rho_a, \rho_b)]$ , one obtains (7,8).

When the optimal  $\rho(x)$  is no longer monotone, i.e. K is negative enough for the right hand side of (15) to vanish, the expressions (7,8,17,18) of  $\mu_N$ ,  $\lambda$ , G, q are modified. We checked that their new expressions are simply the analytic continuations of (7,8,17,18).

In general when the system is in equilibrium ( $\rho_a = \rho_b = \rho$ ) the fluctuations given by (14) are non Gaussian. However when  $\rho_a = \rho_b = \rho^*$  where  $\rho^*$  is the density for which  $\sigma(\rho)$  is maximum, the optimal  $\rho(x)$  in (14) satisfies  $\rho'(x) = 0$  and the fluctuations become Gaussian  $(G(q, \rho^*, \rho^*) = -q^2/(2\sigma(\rho^*)))$  in agreement with the conjecture made in [13] for a specific model, the symmetric simple exclusion process.

It is also easy to check from (14) that the optimal profile  $\rho(x)$  is the same for q and -q. This implies that

$$G(-q, \rho_a, \rho_b) = G(q, \rho_a, \rho_b) - 2q \int_{\rho_b}^{\rho_a} \frac{D(\rho)}{\sigma(\rho)} d\rho$$

which is the Gallavotti-Cohen relation [13, 14, 15].

FIG. 1: The system connecting the two reservoirs contains a loop with two arms of unequal lengths.

Consider now a system composed of 4 parts as in Figure 1. The left reservoir is connected to C by a chain of length  $Nx_1$ . Between C and D there is a loop made of two chains in parallel of lengths  $Ny_1$  and  $Ny_2$  and Dis connected to the right reservoir by a chain of length  $Nx_2$ . According to the *additivity principle*, one should have

$$G_{\text{loop}}(q, \rho_a, \rho_b) = \max_{\rho_c, \rho_d, q'} \left[ \frac{G(qx_1, \rho_a, \rho_c)}{x_1} + \frac{G(q'y_1, \rho_c, \rho_d)}{y_1} + \frac{G((q-q')y_2, \rho_c, \rho_d)}{y_2} + \frac{G(qx_2, \rho_d, \rho_b)}{x_2} \right]$$

The optimum is achieved when  $q'(y_1 + y_2) = qy_2$ , thus

$$G_{\text{loop}}(q, \rho_a, \rho_b) = G(qu, \rho_a, \rho_b)/u$$

with  $u = x_1 + (y_1^{-1} + y_2^{-1})^{-1} + x_2$ . So the current fluctuations for the system with a loop are the same as for

We consider now two specific examples of stochastic dynamics on a 1*d* lattice, the symmetric simple exclusion process (SSEP) and the zero range process (ZRP). The number of particles at site  $i \in \{0, N\}$  is denoted by  $\eta_i$ .

In the SSEP, each site is is either empty or occupied by a single particle ( $\eta_i = 0 \text{ or } 1$ ) and each particle attempts to jump to its right or to its left at rate 1 if there is no other particle at the corresponding neighboring site. Let  $Q_t^i$  be the integrated current through bond (i, i+1) during time t. As in the steady state the integrated current is independent of the bond and  $\partial_t \langle Q_t^i \rangle = \langle \eta_i - \eta_{i+1} \rangle$ ,

$$N\partial_t \langle Q_t \rangle = \sum_i \langle \eta_i - \eta_{i+1} \rangle = \rho_a - \rho_b$$

thus  $D(\rho) = 1$ . As  $N^2 \partial_t \langle (Q_t)^2 \rangle = \partial_t \langle \left( \sum_i Q_t^i \right)^2 \rangle$ , we write

$$\sum_{i,j} \partial_t \langle Q_t^i Q_t^j \rangle = \sum_{j,i} \langle Q_t^j (\eta_i (1 - \eta_{i+1}) - \eta_{i+1} (1 - \eta_i)) \rangle$$
$$+ \sum_i \langle \eta_i (1 - \eta_{i+1}) \rangle + \langle \eta_{i+1} (1 - \eta_i) \rangle$$
$$= \sum_{j,i} \quad \langle Q_t^j (\eta_i - \eta_{i+1}) \rangle + 2 \sum_i \langle \eta_i (1 - \eta_{i+1}) \rangle.$$

The first term simplifies

$$\sum_{j,i} \langle Q_t^j (\eta_i - \eta_{i+1}) \rangle = \langle (\sum_j Q_t^j) \eta_0 \rangle - \langle (\sum_j Q_t^j) \eta_N \rangle.$$

and it vanishes for  $\rho = \rho_a = \rho_b$ . For  $\rho_a = \rho_b$ , the stationary measure is product so that  $\sigma(\rho) = 2\rho(1-\rho)$  according to (2). The cumulants derived in [13] as well as the expression conjectured for  $\mu_N$ 

$$\mu_N(\lambda) = -N^{-1} [\sin^{-1}(\sqrt{-\omega})]^2$$
, for  $\omega \le 0$ . (19)

where  $\omega = (1 - e^{-\lambda})(e^{\lambda}\rho_a - \rho_b - (e^{\lambda} - 1)\rho_a\rho_b)$  can be recovered from (7,8). This can be seen by noticing that the optimal profile solution of (16) is

$$\rho(x) = \frac{1}{2} \left( 1 + \frac{\sin\left(2\left(\theta_a + (\theta_b - \theta_a)x\right)\right)}{\sin(2f)} \right) \,,$$

where the parameters  $f, \theta_a, \theta_b$  are fixed by  $K = \tan^2(2f), \rho(0) = \rho_a, \rho(1) = \rho_b$ . In terms of these parameters,  $\lambda$  and  $\mu_N$  take the form

$$\lambda = \log \left[ \frac{\cos(f + \theta_a) \, \cos(f - \theta_b)}{\cos(f - \theta_a) \, \cos(f + \theta_b)} \right], \quad \mu_N = -(\theta_a - \theta_b)^2.$$

For the ZRP the number of particles on each site can be arbitrary and the jump rate  $\Phi(\eta_i)$  from site *i* to each of its neighbors is an increasing function of the number of particles  $\eta_i$  at this site. We choose  $\Phi(0) = 0$ . We have

$$\sum_{i} \partial_t \langle Q_t^i \rangle = \sum_{i} \langle \Phi(\eta_i) - \Phi(\eta_{i+1}) \rangle = \Psi(\rho_a) - \Psi(\rho_b) \,,$$

where the expectation of  $\Phi$  under the stationary measure at density  $\rho$  is denoted by  $\Psi(\rho)$ . We also have

$$N^{2}\partial_{t}\langle (Q_{t})^{2}\rangle = \sum_{j,i} \langle Q_{t}^{j} \left( \Phi(\eta_{i}) - \Phi(\eta_{i+1}) \right) \rangle + 2\sum_{i} \langle \Phi(\eta_{i}) \rangle$$

As for the SSEP the 1st term in the rhs of the above equation vanishes when  $\rho_a = \rho_b$ . We finally obtain  $D(\rho) = \sigma'(\rho)/2$  and  $\sigma(\rho) = 2\Psi(\rho)$  according to (1,2). Therefore

$$\mu_N(\lambda) = (1 - e^{-\lambda}) \left( e^{\lambda} \sigma(\rho_a) - \sigma(\rho_b) \right) / 2N \; .$$

This generalizes the case of non-interacting particles for which  $\sigma(\rho) = 2\rho$ . The optimal profile is obtained by the change of variables

$$\sigma(\rho(x)) = \frac{(\theta_a + (\theta_b - \theta_a)x)^2 - 1}{2K},$$

where  $\theta_a, \theta_b$  are fixed by  $\rho(0) = \rho_a$  and  $\rho(1) = \rho_b$ . In particular, the expression of  $\mu_N$  follows from

$$\lambda = \log\left(\frac{1+\theta_b}{1+\theta_a}\right), \quad \mu_N(\lambda) = -\frac{(\theta_a - \theta_b)^2}{4K}$$

The *additivity principle* (9,10) formulated here and its variational expression (14) can be derived (work in progress) from the hydrodynamic large deviation theory [17, 18, 19]. This theory was extended recently by Bertini et al. [16] who could calculate the density large deviation functional of the steady state as the optimal cost per unit time for a space/time density fluctuation. For diffusive systems, the exponential cost of observing an atypical space/time density profile over a time t can be estimated by a functional depending only on  $D(\rho), \sigma(\rho)$  and on the density  $\{\rho(x,s)\}_{x \in [0,1], 0 \le s \le t}$  (see eg [17, 18, 19]). The optimal strategy to create a fluctuation of the current  $Q_t = qt$  over a very long time t is to create a fixed density profile  $\rho(x)$  in order to facilitate the deviation of the current and (14) can be understood as the cost for maintaining this atypical density profile. The optimal profile controling here the current fluctuations is time independent, in contrast to the one which controls the steady state density fluctuations that Bertini et al. [16] had to calculate. This is why our task here was easier and the additivity principle (10) is simpler than the one obtained in [20] for the steady state fluctuations of the density.

It would be interesting to see whether the Bertini et al. macroscopic fluctuation theory satisfies a generalized additivity principle for time-dependent densities in the reservoirs. Other interesting extensions of the present work include the study of the effect of asymmetry in the bulk dynamics (i.e. of a field which favors jumps of particles from left to right) [21, 22, 23, 24, 25] or the analysis of more complex networks, in particular of systems in contact with three or more reservoirs [26, 27, 28, 29, 30].

Of course, a very challenging issue would be to see whether the additivity principle could be valid for some mechanical systems satisfying (1,2) without the need of an intrinsic source of noise as the stochastic systems considered here.

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