Theory of large deviations & applications- Les Houches 2024

HIGH-DIMENSIONAL RANDOM LANDSCAPES

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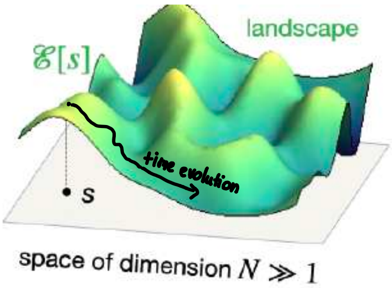
[If you find typos
please let me know!]

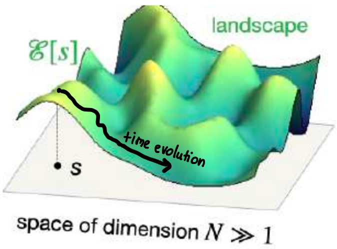
WHAT: High-D random landscapes are functions of many variables $\mathcal{E}[\vec{s}]$, $\vec{s}=(s_1,..,s_N)$ with $N \gg 1$, which are random, with given $\mathbb{P}[\mathcal{E}[\vec{s}]]$ (in the following, Gaussian)

WHY: Many 'complex systems' are inherently high-dimensional. They evolve trying to optimize some function (fitness, energy, cost...). Function encodes complex interactions between constituents, often modelled with random variables.

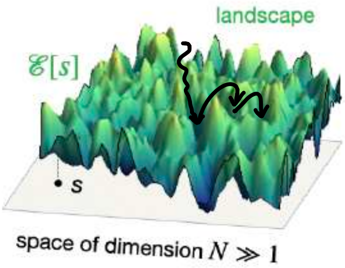


What to expect from this optimization processes in high-D, typically (i.e., with high probability)?

HOW: Characterize landscapes structure & its dynamical exploration using tools of stat physics ($N \gg 1$) of disordered systems: random matrix theory, saddle-point & large-N limits, large-deviations, replica tricks, Kac-Rice counting formulas....



Scenario 1: "Smooth" landscape.



Scenario 2: "rugged" landscape.

HIGH-D RANDOM LANDSCAPES

PART I: QUADRATIC HIGH-D LANDSCAPES

WHY: an example from high-D inference

An 'easy' inference problem – From denoising to landscapes – Questions & strategy

HOW: Random Matrix Theory

From landscapes back to random matrices – Basic RMT facts

WHAT: Ground state, landscape, dynamics

Recovering the signal – A landscape of saddles – DMFT & beyond

PART II : RUGGED HIGH-D LANDSCAPES

WHY: another example from high-D inference

A 'hard' inference problem: noisy tensors – Landscape problem, & complexity

HOW: Kac-Rice formalism

Averages vs typical values, and replicas – Kac-Rice formula(s) –
Computing the complexity: 3 steps – The annealed complexity

WHAT: Ground state, landscape, dynamics

Recovering the signal – A landscape of minima – DMFT. And beyond?

Large deviations theory matters here

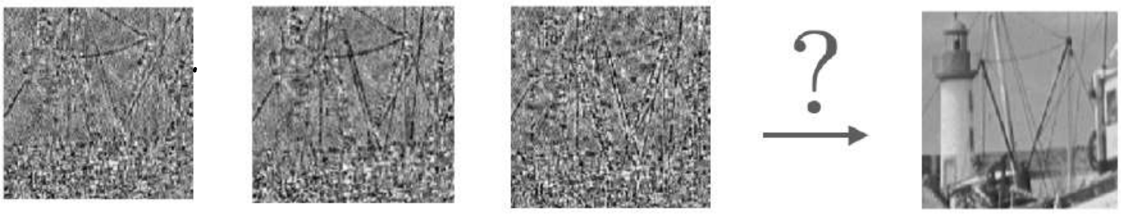
PART I

Quadratic high-D landscapes

I.1 WHY: AN INFERENCE EXAMPLE

▦ An 'easy' inference problem: noisy matrices

Inference problem: measure a 'signal' corrupted by noise. Combining measurements, can recover information on signal?



(figure adapted from the web)

Denoising of matrices ("spiked" matrices): JOHNSTONE 2001

$$\hat{M} = \underset{\text{signal strength}}{r}\,\underset{\text{signal}}{\frac{\vec{U}\vec{U}^T}{N}} + \underset{\substack{\text{noise}\\ \text{(randomness)}}}{\hat{J}}$$

size N×N, $r \geq 0$

$N >> 1$

▨ SIGNAL $\vec{v}$: vector of norm $\|\vec{v}\|^2 = N$, $\vec{v} = (v_1, \dots, v_N)$
Unknown. Quenched (fixed). Independent of $\hat{J}$.

▨ NOISE $\hat{J}$: matrix with random, symmetric ($J_{ij} = J_{ji}$) entries, N×N. Gaussian statistics:

$$\langle J_{ij} \rangle = 0, \quad \langle J_{ij}^2 \rangle = \frac{\sigma^2}{N}\left(1 + \delta_{ij}\right)$$

Probability to observe one instance of $\hat{J}$:

$$P_N(\hat{J})\, d\hat{J} = A_N\, e^{-\frac{N}{2\sigma^2}\sum_{i<j} J_{ij}^2 - \frac{N}{4\sigma^2}\sum_{i=1}^{N} J_{ii}^2} \prod_{i\leq j} dJ_{ij}$$

$$= A_N\, e^{-\frac{N}{4\sigma^2}\mathrm{Tr}(\hat{J}^2)} \prod_{i\leq j} dJ_{ij} \qquad A_N = \frac{1}{2^N}\left(\frac{N}{2\pi\sigma^2}\right)^{\frac{N(N+1)}{2}}$$

"GAUSSIAN ORTHOGONAL ENSEMBLE" = rotationally invariant ensemble. $\hat{O}$ rotation ($\hat{O}\hat{O}^T = \hat{1}$). Matrix $\hat{J}$ in new basis: $\hat{J}_R = \hat{O}\hat{J}\hat{O}^T$. Rotationally invariant means: $\hat{J}$ has same prob. as $\hat{J}_R = \hat{O}\hat{J}\hat{O}^T$: $\hat{J} \overset{\text{in law}}{\sim} \hat{J}_R$

Notice: Same eigenvalues, eigenvectors $\vec{u}_R = \hat{O}\vec{u}$. The eigenbasis of $\hat{J}$ has same distribution as any other vector basis obtained with rotation $\Rightarrow$ uniform orthogonal vectors on sphere.

From denoising to landscapes

Estimator (guess) of $\vec{v}$: $\vec{S}_{gs} = \underset{\|\vec{s}\|^2=N}{\operatorname{argmax}} \; \vec{s}^T \cdot \hat{M}\vec{s}$

this is "maximum likelihood estimator" of the signal $\vec{v}$.

Maximum-likelihood

$$\hat{M} = r\frac{\vec{v}\vec{v}^T}{N} + \hat{J}$$

$\hat{M}$ - observation
$\vec{v}$ – unkown signal
$\hat{J}$ - iid gaussians

Bayes formula:

$$\underbrace{P(\vec{s}|\hat{M})}_{\text{Posterior}} = \underbrace{P_0(\vec{s})}_{\text{PRIOR}} \underbrace{P(\hat{M}|\vec{s})}_{\text{LIKELIHOOD}} \frac{1}{P(\hat{M})} = P_0(\vec{s}) \frac{e^{-\frac{N}{4\sigma^2}\sum_{i,j}(M_{ij}-\frac{r}{N}s_is_j)^2}}{Z(\hat{M})}$$

$$\mathcal{L}(\vec{s}|\hat{M}) = \log P(\hat{M}|\vec{s}) = -\frac{N}{2\sigma^2}\sum_{i\leq j}\left(M_{ij}-\frac{r}{N}s_is_j\right)^2\left(\frac{1}{1+\delta_{ij}}\right) + \ell(\hat{M})$$

"log-likelihood"

The maximum-likelihood estimator is the vector that maximizes the log-likelihood.

If we Know $\|\vec{v}\|^2 = N$, we can assume $\|\vec{s}\|^2 = N$ and thus the estimator is minimizing

$$\sim \sum_{i,j=1}^{N}\left(M_{ij} - \frac{r}{N} s_i s_j\right)^2 = \sum_{i,j=1}^{N} M_{ij}^2 + \frac{r^2}{N^2}\|\vec{s}\|^4 - \frac{2r}{N}\sum_{ij} M_{ij} s_i s_j$$

$$\Longrightarrow \vec{s}_{ls} = \underset{\|\vec{s}\|^2 = N}{\arg\max}\ \vec{s}\cdot\hat{M}\vec{s}$$

S_{ls} is also the ground state of the energy landscape:

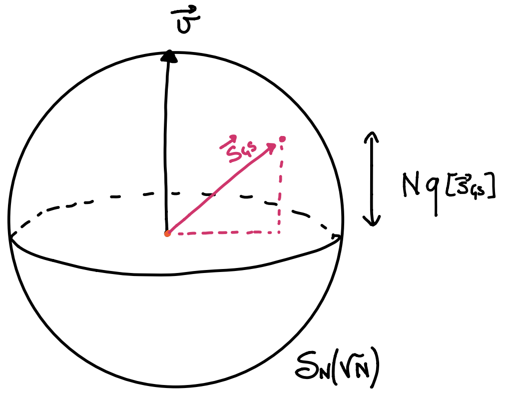
$$\mathcal{E}[\vec{s}] = -\frac{1}{2}\sum_{i,j=1}^{N} s_i M_{ij} s_j = -\frac{1}{2}\sum_{i,j=1}^{N}\Big[\underbrace{J_{ij} s_i s_j}_{\text{random isotropic}} + \underbrace{rN\left(\frac{\vec{v}\cdot\vec{s}}{N}\right)^2}_{\text{deterministic biased towards } \vec{v}}\Big]$$

defined on $S_N(\sqrt{N}) = \{\vec{s} : \|\vec{s}\|^2 = N\}$

Finding the estimator $\Longleftrightarrow$ solving optimization problem for random landscape $\mathcal{E}[\vec{s}]$.

Notice: for $r=0$, this is same landscape introduced by P. Vivo in his lecture 2!

Define the "OVERLAP WITH SIGNAL"



$$q_N[\vec{s}] = \frac{\vec{s}\cdot\vec{v}}{N}$$

$$q[\vec{s}] = \lim_{N\to\infty} q_N[\vec{s}]$$

$r=0$: random, fully-connected interactions between s_i : "pure spherical $p=2$ model"
Isotropic statistics

$$\langle \varepsilon[\vec{s}] \rangle = 0$$

$$\langle \varepsilon[\vec{s}]\, \varepsilon[\vec{s}\,'] \rangle = \frac{N\sigma^2}{2}\left(\frac{\vec{s}\cdot\vec{s}\,'}{N}\right)^2 \Longrightarrow$$

by entropy, expect $\vec{s}_{GS} \perp \vec{v}$ for $r=0$ (see below)

$\sigma=0$: the points in the vicinity of $\vec{v}$ are favored energetically, $\vec{s}_{GS} = \vec{v}$

Competition leads to transitions in r/σ (signal-to-noise ratio) when $N\to\infty$.

High-D geometry: typical values of overlaps

Let $\vec{v}$ be fixed vector $\|\vec{v}\|^2 = N$. Assume $\vec{S}$ uniformly taken on sphere. Then typical value of $\left(\frac{\vec{v}\cdot\vec{S}}{N}\right) \xrightarrow{N\to\infty} 0.$ With overwhelming probability, two vectors are orthogonal when $N \to \infty$.

Indeed:

Basis-independent. Choose basis in which $\vec{v} = \sqrt{N}(0,0,0,1)$

$$\left\langle \left(\frac{\vec{v}\cdot\vec{S}}{N}\right)^2 \right\rangle = \int_{S_N(\sqrt{N})} \prod_{i=1}^{N} dS_i \left(\frac{\vec{v}\cdot\vec{S}}{N}\right)^2 = \int_{S_N(\sqrt{N})} \prod_{i=1}^{N} dS_i \, \frac{S_N^2}{N} = N^{N/2} \int_{S_N(1)} \prod_{i=1}^{N} d\sigma_i \, \sigma_N^2$$

rescale : $\vec{\sigma} = \vec{S}/\sqrt{N}$

Sphere : $\sum_{i=1}^{N-1} \sigma_i^2 + \sigma_N^2 = 1.$

$$= \frac{N^{N/2}}{N} \int d\sigma_N \, \sigma_N^2 \underbrace{\left[\int \prod_{i=1}^{N-1} d\sigma_i \, \delta\left(\sum_{i=1}^{N-1} \sigma_i^2 - [1 - \sigma_N^2] \right) \right]}$$

Volume of sphere of radius $R = \sqrt{1-\sigma_N^2}$

$$V_N = \frac{2\pi^{N/2}}{\Gamma(N/2)} \left(1 - \sigma_N^2\right)^{\frac{N-1}{2}}$$

$$\overset{N\gg1}{\approx} \frac{1}{N} \int d\sigma_N \, \sigma_N^2 \, e^{\frac{N}{2}\log\left[2\pi e(1-\sigma_N^2)\right] + o(N)} \xrightarrow[N\to\infty]{\text{SADDLE POINT: } \sigma_N^{SP} = 0} 0$$

Could do this for all components by rotational invariance: all σ_i^2 are statistically equivalent $\Rightarrow \sum_{i=1}^{N} \sigma_i^2 \approx N \cdot \langle \sigma_i^2 \rangle \equiv 1 \Rightarrow \langle \sigma_i^2 \rangle \approx 1/N$

$\Rightarrow \left\langle \left(\frac{v\cdot S}{N}\right)^2 \right\rangle \approx 1/N$

■ Questions & strategy

▶ Three questions:

[Q1] RECOVERY QUESTION ($T=0$ EQUILIBRIUM)

for which values of r is $\vec{s}_{gs}$ informative of signal $\vec{v}$, i.e. "close" to $\vec{v}$ in configuration space?

For $N \to \infty$, $q[\vec{s}_{gs}] > 0$ ("magnetization")

[Q2] LANDSCAPE QUESTION (METASTABLE STATES)

are there many local minima / stationary points at higher energy? How far from $\vec{s}_{gs}$? How far from $\vec{v}$?

[Q3] ALGORITHMIC QUESTION (DYNAMICS)

founding $\vec{s}_{gs}$ with (local) optimization algorithms (gradient descent / Langevin: $\frac{d\vec{s}(t)}{dt} = -\nabla_\perp \mathcal{E}[\vec{s}'] + \sqrt{2T}\,\vec{\eta}(t)$) is

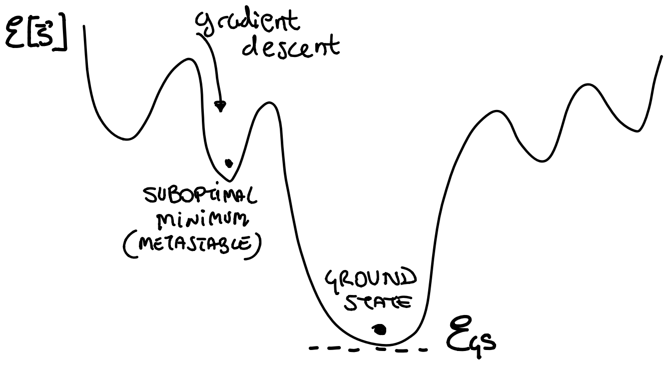
gradient on the sphere

easy: timescales $\tau_{typ} \sim \mathcal{O}(N^\alpha)$, or

hard: timescales $\tau_{typ} \sim \mathcal{O}(e^N)$?

Q2 and Q3 related: optimization hard when many metastable states/local minima in which system gets stuck!

↳ glassiness



▶ The strategy:

Study the typical distribution of stationary points $\vec{s}^*$: $\nabla_\perp \mathcal{E}[\vec{s}^*]=0$ as a function of:

(i) energy density $\epsilon_N[\vec{s}^*] = \mathcal{E}[\vec{s}^*]/N$

(ii) stability / local curvature (minima, saddles)

curvature = eigenvalues of Hessian $\nabla_\perp^2 \mathcal{E}[\vec{s}]$ $\left(\sim \frac{\partial^2 \mathcal{E}[\vec{s}]}{\partial s_i \partial s_j}\right)$

index $K_N[\vec{s}] = \{\#$ negative evalues $\nabla_\perp^2 \mathcal{E}[\vec{s}]\}$

(minima: all evalues positive, $K=0$)

(iii) geometry: overlap with signal $q_N[\vec{s}^*] = (\vec{s}^* \cdot \vec{v})/N$

Q1. Properties of global minimum

Q2/Q3. Properties of local minima

Notice: here "typical" means: happening with probability $\mathbb{P} \longrightarrow 1$ when $N \to \infty$.
"rare" means happening with $\mathbb{P} \xrightarrow{N\to\infty} 0$.

▶ We shall see:

Quadratic landscape $\mathcal{E}[\vec{s}]$: Can aswer all the questions when $N \to \infty$,
using Random matrix theory. Describe what happens typically (= with large probability) when N large.
More complicated landscapes: PART II.

I.2 HOW: RANDOM MATRIX THEORY

▦ From landscapes back to random matrices

Consider a fixed realization of $\hat{M} \to$ of landscape $\mathcal{E}[\vec{s}]$

KOSTERLITZ, THOULESS, JONES 1976

Implement spherical constraint:

$$\mathcal{E}_\lambda[\vec{s}] = -\frac{1}{2}\sum_{i,j=1}^{N} M_{ij} s_i s_j + \frac{\lambda}{2}\left(\sum_{i=1}^{N} s_i^2 - N\right)$$

Stationary points $(\vec{s}^*, \lambda^*)$ satisfy:

$$\begin{cases} \dfrac{\partial \mathcal{E}_\lambda[\vec{s}^*]}{\partial s_i} = -\sum_{j=1}^{N} M_{ij} s_j^* + \lambda^* s_i^* = 0 \qquad \forall i = 1, .., N \\ \dfrac{\partial \mathcal{E}_\lambda[\vec{s}^*]}{\partial \lambda} = \sum_i [s_i^*]^2 - N = 0 \end{cases}$$

The first equation is evalue equation for $\hat{M}$: $\hat{M}\vec{s}^* = \lambda^* \vec{s}^*$

If $\{\vec{u}_\alpha, \lambda_\alpha\}$ are evectors/evalues of $\hat{M}$ for $\alpha = 1, .., N$, then:

$\vec{s}_\alpha = \pm\sqrt{N}\,\vec{u}_\alpha$ are stationary points of $\mathcal{E}[\vec{s}]$: 2N of them!

(notice symmetry bc. quadratic function)

Properties:

(i) Energy. Multiply first equation by s_i^*, sum & use second one:

$$\sum_{i,j} s_i^* M_{ij} s_j^* = \lambda^* \cdot N \Rightarrow \lambda^* = -\frac{2\mathcal{E}[s^*]}{N} = -2\underset{\substack{\uparrow \\ \text{energy density}}}{\epsilon[s^*]}$$

$\Rightarrow$ The $\vec{s}_\alpha$ have energy density

$$\epsilon_N[\vec{s}_\alpha] = -\frac{\lambda_\alpha}{2}$$

(ii) Stability. minima, saddles?

Hessian: $\nabla^2 \mathcal{E}_\lambda[s^*] = -M_{ij} + \lambda^*$

At stationary point $\vec{s}^\alpha$: $\nabla^2 \mathcal{E}_\lambda[\vec{s}^\alpha] = -(\hat{M} - \lambda_\alpha \hat{1})$

The eigenvalues of $\hat{M}$ are $\lambda_1 \leq .. \leq \lambda_N$. The eigenvalues of $\nabla^2 \mathcal{E}_\lambda[\vec{s}^\alpha]$ are $-\underbrace{(\lambda_1 - \lambda_\alpha)}_{\text{positive if } \alpha > 1}, -(\underbrace{\lambda_2 - \lambda_\alpha}_{\text{positive if } \alpha > 2}) \cdots$

One zero eigenvalue (due to spherical constraint), $(\alpha-1)$ positive and $N-\alpha$ negative: Stationary points $\vec{s}_\alpha$ are saddles of index $\kappa_N[s_\alpha] = N-\alpha$

Ground state: $\alpha = N$. Global minimum $(\kappa = 0)$

$\Rightarrow$ For each realization of randomness $\hat{J}$, $\mathcal{E}[\vec{s}]$ has 2N stationary points; their energy distribution is related to eigenvalue distribution of $\hat{M}$.

Statistical properties when $N \gg 1$ determined by Random Matrix Theory (RMT).

Notation: gradients & Hessians on sphere

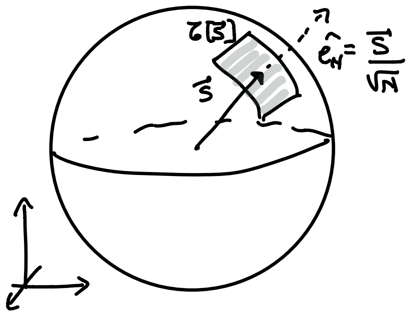
$$\nabla\mathcal{E}[s] = \left(\frac{\partial \mathcal{E}}{\partial s_i}\right)_{i=1}^{N} \quad \text{gradient in } \mathbb{R}^N$$

Lagrange multiplier λ^* subtracts the radial component:

$$\lambda^* = -\nabla\mathcal{E}[\vec{s}]\cdot\vec{s}\,, \quad \nabla\mathcal{E}_\lambda[\vec{s}] = \nabla\mathcal{E}[\vec{s}] - \underbrace{\left(\frac{\nabla\mathcal{E}[\vec{s}]}{N}\cdot\vec{s}\right)\vec{s}}_{\text{radial component}}$$

Choose basis vectors such that

$$\vec{e}_\alpha = \begin{cases} \perp \vec{s} & \alpha = 1, \dots, N-1 \\ \vec{e}_N = \vec{s}/\sqrt{N} & \end{cases} \longleftarrow \text{Spanning tangent plane } \tau[\vec{s}]$$



In this basis:

$$\nabla\mathcal{E}_\lambda[\vec{s}] = \begin{pmatrix} \nabla_\perp \mathcal{E}[\vec{s}] \\ 0 \end{pmatrix}$$

(N-1)-dim "gradient on the sphere".

Similarly, Hessian on the sphere $\nabla_\perp^2 \mathcal{E}[\vec{s}]$ is the $(N-1)\times(N-1)$ matrix $\frac{\partial^2 \mathcal{E}[\vec{s}]}{\partial s_i \partial s_j} + \lambda^*[\vec{s}]\hat{1}$ projected on $\tau[\vec{s}]$

III Some facts in Random Matrix Theory (RMT)

- The results below hold true for rank-1 perturbed GOE matrices of the type:

$$\hat{M} = \hat{J} + \hat{R} = \hat{J} + r\,\vec{w}\,\vec{w}^T \qquad \left(\vec{w} = \vec{v}/\sqrt{N},\ \|\vec{w}\| = 1\right)$$

$\hat{J}$ = GOE matrix: both a Wigner matrix (real, symmetric, iid entries) & rotationally-invariant $\left(J \overset{\text{in law}}{\sim} J_R = O J O^T\right)$
Normalized so that spectrum in bounded interval when $N \to \infty$.

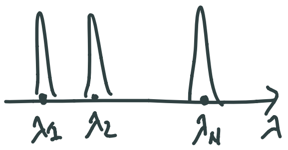
$\hat{R}$ = deterministic, rank-one matrix with 1 evalue equal to r, and $(N-1)$ zero eigenvalues. Independent of $\hat{J}$
Perturbation to GOE! "spike"

- Some results have some degree of universality: can be generalized to other matrix ensembles, or perturbations of higher rank (finite in N)

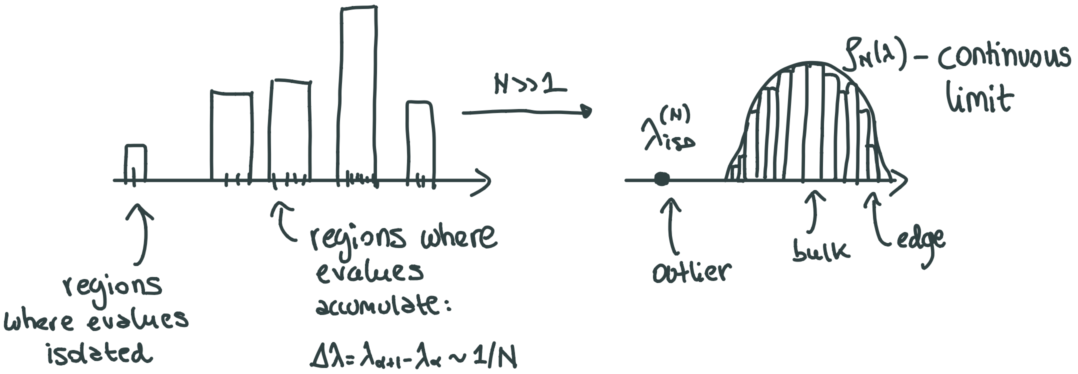
- Eigensystem: $\{\lambda_\alpha, u_\alpha\}_{\alpha=1}^N$. In this section, averages are w.r.t. distribution of $\hat{M}$: $\langle\cdot\rangle = \int d\hat{M}\, P(\hat{M})\,\cdot$
Assume $\lambda_1 \leq \dots \leq \lambda_N$, and $\|u_\alpha\| = 1$.

The eigenvalue distribution: density, & outliers

N finite: $\nu_N(\lambda) = \frac{1}{N}\sum_{\alpha=1}^{N} \delta(\lambda - \lambda_\alpha)$



Typical scenario when N increases:



$$\boxed{\nu_N(\lambda) \overset{N\gg1}{\simeq} \underbrace{\rho_N(\lambda)}_{\text{Density}} + \underbrace{\frac{1}{N}\delta\left(\lambda - \lambda_{iso}^{(N)}\right)}_{\text{Outliers}}}$$

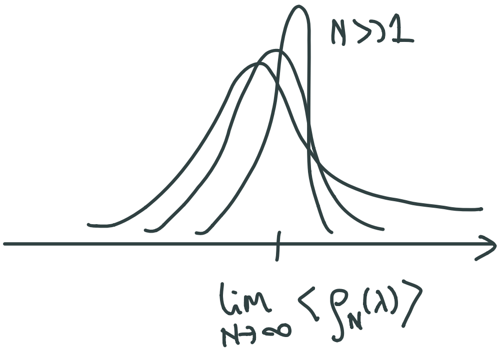
The density. Where evalues accumulate, distribution described by continuous density: $\rho_N(\lambda)$.

$$\mathbb{P}\left(\lambda_\alpha \in [\lambda, \lambda + \delta\lambda]\right) \overset{N\gg1}{\simeq} \rho_N(\lambda)\, d\lambda$$

have $O(N)$ evalues around λ, separated by $O(1/N)$ in bulk, or $O(1/N^\alpha)$ at edges.

Facts: (a) Density $\rho_N(\lambda)$ is self-averaging

$$\lim_{N\to\infty} \underbrace{\rho_N(\lambda)}_{\text{random function}} = \underbrace{\rho_\infty}_{\text{deterministic}}(\lambda) = \lim_{N\to\infty} \langle \rho_N(\lambda) \rangle$$



(b) Can be obtained from Stieltjes transform:

$$g_N(z) = \int \frac{d\nu_N(\lambda)}{z-\lambda} = \frac{1}{N} \sum_\alpha \frac{1}{z-\lambda_\alpha} = \frac{1}{N} \operatorname{tr} \underbrace{\left(\frac{1}{z-M}\right)}_{\text{resolvent}}$$

This function is singular when $z \to \lambda_\alpha$ (poles)
Define it away from real axis, e.g. $z \in \mathbb{C}^-$, $z = E - i\eta$
(then: analytically continue).
$\lim_{N\to\infty} g_N(z) = g_\infty(z)$ also self-averaging

(c) When $N\to\infty$, poles accumulate into branch-cut. The discontinuity at the cut is related to $\rho_\infty(\lambda)$:

$$\rho_\infty(\lambda) = \lim_{\eta \downarrow 0} \frac{1}{\pi} \mathrm{Im}\left\{ g_\infty(\lambda - i\eta) \right\}$$

<u>Isolated eigenvalues</u>. Isolated poles of $g_N(z)$, contributing to order $1/N$.

They also "concentrate": $\lim_{N\to\infty} \lambda_{iso}^{(N)} = \lambda_{iso}^{\infty}$

Questions:

▨ $\rho_\infty(\lambda)$, typical value of $\lambda_{iso}^{(N)}$ when $N\to\infty$?

▨ typical fluctuations at N large & finite?

▨ atypical fluctuations: large deviations

LIVAN, NOVAES, VIVO - Introduction to random matrices, 2017
POTTERS, BOUCHAUD - A first course in random matrix theory, 2021
MEHTA - Random Matrices, 2004

▶ Typical values: the density $\rho_\infty(\lambda)$.

Can be studied with REPLICA METHOD → EXERCISE 1

One finds that:

(1) The finite rank perturbation $\hat{R}$ does not affect the density of $\hat{M}$, that is the same as the one of $\hat{J}$.

$$\lim_{N\to\infty} \rho_N(\lambda;r) = \rho_\infty(\lambda;r=0) \quad \left(\begin{array}{l}\text{effect of rank-1 perturbation}\\ \text{disappears for } N\to\infty\end{array}\right)$$

(2) When $\hat{J}$ is Gaussian, $\langle J_{ij}^2\rangle = \frac{\sigma^2}{N}\left(1+\delta_{ij}\right)$, then:

The Stieltjes transform satisfies a self-consistent equation: $\boxed{\sigma^2 g_\infty^2(z) - z\,g_\infty(z) + 1 = 0}$ $\quad z \notin$ spectrum

(3) This is solved by: $g_{sc}(z) = \dfrac{z - z\sqrt{1-4\sigma^2/z}}{2\sigma^2}$ choice of branch?

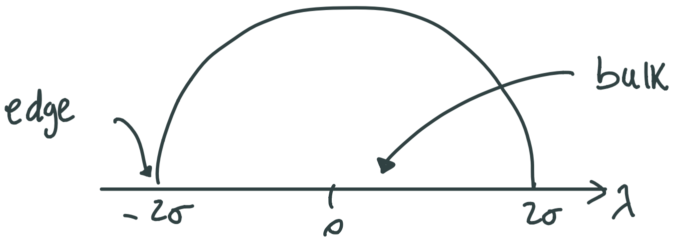
Continuation to real axis: $z \longrightarrow \lambda$

$$g_{sc}(\lambda) = \frac{\lambda - \text{sign}(\lambda)\sqrt{\lambda^2-4\sigma^2}}{2\sigma^2} \qquad \lambda \notin [-2\sigma, 2\sigma]$$

Choise of branch guarantees $\lim_{|\lambda|\to\infty} g_{sc}(\lambda) = 0 \quad \left(g_{sc}(z) \sim \frac{1}{z}\right)$

By inversion formula:

$$\boxed{\rho_\infty(\lambda) = \rho_{sc}(\lambda) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2-\lambda^2}\ \mathbb{1}_{\lambda\in[-2\sigma,2\sigma]}}$$



- ▩ Universality of $\rho_{sc}(\lambda)$: it is the limiting density for a large class of matrices of the Wigner type: symmetric, with iid entries not necessarily Gaussian, finite second moment.

  ERDÖS – Universality of Wigner random matrices: a survey of results, 2010

  Also spectrum of Laplacian of random graphs. (adjacency matix), Burgers equation...

- ▩ $\hat{R}$ can have larger rank, not scaling with N (finite rank)

▷ Typical values: the isolated evalue(s)\ evector(s)

The 1/N contibutions to $g_N(\lambda)$ can be studied in a large-N expansion → EXERCISE 2

One finds that:

(1) For $\hat{R}=0$, There are no isolated eigenvalues

$$\lim_{N\to\infty} \lambda_1 = -2\sigma \quad \text{minimal eigenvalue}$$

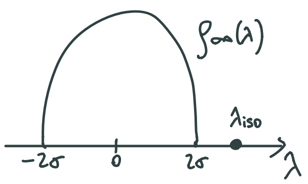
$$\lim_{N\to\infty} \lambda_N = 2\sigma \quad \text{maximal eigenvalue}$$

(almost surely)

(2) When $N\to\infty$, a transition in maximal evalue when $r=r_c=\sigma$ (notice: smaller than radius 2σ)

$$\boxed{\lim_{N\to\infty} \lambda_N = \begin{cases} 2\sigma & r\leq r_c=\sigma \\ \frac{\sigma^2}{r}+r & r>r_c=\sigma \end{cases} \quad \text{(almost surely)}}$$

For $r<r_c$, same behavior as for $r=0$: largest evalue sticks to boundary. For $r>r_c$, the largest eigenvalue is isolated:

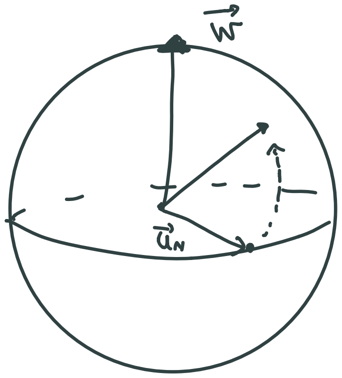


$r > r_c = \sigma$
$(N \to \infty)$

KOSTERLITZ, THOULESS, JONES 1976
PÉCHÉ 2006

(3) The eigenvector $\vec{u}_N$ when $r \geqslant r_c$ acquires macroscopic projection on $\vec{w} = \vec{v}/\sqrt{N}$

Then:
$$\boxed{\lim_{N\to\infty} \left(\vec{u}_N \cdot \vec{w}\right)^2 = \begin{cases} 0 & \text{if} \quad r \leq r_c \\ 1-(\sigma/r)^2 & \quad r \geqslant r_c \end{cases}}$$



While all other eigenvectors such that $(\vec{u}_\alpha \cdot \vec{w})^2 = 0 \quad \alpha \neq N$

This can be seen as a "LOCALIZATION" transition.

- For $r=0$, consistent with rotational invariance:

eigenvectors of $\hat{J}$ like random vectors on sphere (statistically), and $\vec{w}$ is independent of $\hat{J}$.

As in calculation above,

$$\langle(\vec{u}_\alpha\cdot\vec{w})^2\rangle = \int \prod_{i=1}^{N} du_\alpha^i \; \delta(\|\vec{u}_\alpha\|-1)\,(\vec{u}_\alpha\cdot\vec{w})^2 \sim \frac{1}{N} \xrightarrow{N\gg 1} 0$$

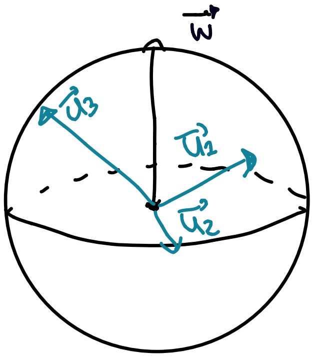
(Use this in Exercise 2): two arbitrary vectors on sphere are typically orthogonal when $N\to\infty$.

$\vec{w}$ is DELOCALIZED in basis $\vec{u}_\alpha$: overlap is of same order of magnitude for all α, no special direction.

Terminology from quantum problems, where $\vec{u}_\alpha$ and $\vec{w}$ eigenvectors of local operators (QM is linear).

[CONNECTED NOTIONS: QUANTUM CHAOS, FREE PROBABILITY.]

- When $r>0$, isotropy broken in direction $\vec{w}$. For $r>r_c$, $\vec{w}$ localized in basis $\vec{u}_\alpha$!



Measure of localization in a basis $\vec{u}_\alpha$: IPR, or HERFINDAHL INDEX:

$$\text{IPR} = \sum_{\alpha=1}^{N}(\vec{w}\cdot\vec{u}_\alpha)^4 \Big/ \sum_{\alpha=1}^{N}(\vec{w}\cdot\vec{u}_\alpha)^2$$

non-zero in localized phase

$$\mathrm{IPR} = \begin{cases} \sum_{\alpha=1}^{N} \left(\frac{1}{N}\right)^2 \sim \frac{1}{N} \xrightarrow{N\to\infty} 0 & r \leq r_c \\ \sum_{\alpha=1}^{N-1} \left(\frac{1}{N}\right)^2 + \mathcal{O}(1) \xrightarrow{N\to\infty} \mathcal{O}(1) & r > r_c \end{cases}$$

It is also an instance of CONDENSATION (sum over many elements dominated by $\mathcal{O}(1)$ terms) → see EXERCISE 3

Generalizations:

- The above is true if $\hat{J}$ is extracted from a rotationally invariant ensemble (not necessarily Gaussian), with density $\rho_\infty(\lambda)$ supported in $[a,b]$. Then one can show that almost surely:

$$\lim_{N\to\infty} \lambda_N = \begin{cases} b & r \leq r_c = 1/g_\infty(b) \\ g_\infty^{-1}\left(\frac{1}{r}\right) & r > r_c = 1/g_\infty(b) \end{cases}$$

$$\lim_{N\to\infty} (\vec{w}\cdot\vec{u}_N)^2 = \begin{cases} 0 \quad \text{if} & r \leq r_c \\ \frac{-1}{r^2 g'_\infty(\lambda_{iso})} & r \geq r_c \end{cases}$$

PÉCHÉ 2006

BENAYCH-GEORGES & NADAKUDITI 2011

One can recover the GOE expressions from these general ones

Important thing: $\hat{R}$ is independent ("Free") of $\hat{J}$.

CAPITAINE, DONATI-MARTIN 2016

- can be generalized to perturbations $\hat{R}$ with rank $n > 1$: n transitions, potentially n isolated eigenvalues. One r_c for each of them.

▸ Finite-N fluctuations: small deviations

Above results describe $N \to \infty$ limit, when things are self-averaging / concentrate.

At finite N: fluctuations. Things are distributed.

Fluctuations of smallest eigenvalue?

- Transition at $r = r_c$ becomes a crossover.

  Critical regime: $\tau = N^{1/3}(r - r_c)$

  $$\begin{cases} \text{If } (r_c - r) \gg N^{-1/3} : \text{subcritical} \\ \text{If } (r - r_c) \gg N^{-1/3} : \text{supercritical} \end{cases}$$

BLOEMENTAL, VIRÁG, 2013
DUBACH, ERDÖS 2022

(See example in figure below.)

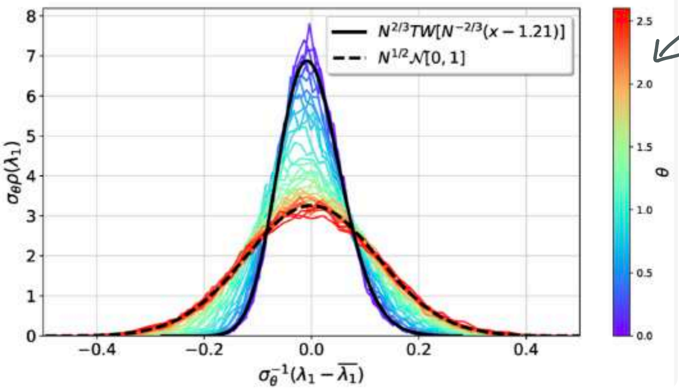


Figure 1. Scaled probability density distributions of an ensemble of 10^4 spike random matrices with $N = 100$. The distributions are centered relative to the ensemble average $\overline{\lambda_1}$ and σ_θ stands for the predicted standard deviation when $\theta > 1$. The centered TW distribution TW (15) and the normal distribution $\mathcal{N}[0,1]$ (16) have been scaled similarly to the data.

Crossover in distribution of λ_N, from Tracy-Widom to gaussian [here θ denotes r/σ]

PIMENTA, STARIOLO 2023

■ For $r>0$:

$$\lambda_N \overset{N\gg 1}{\approx} \begin{cases} 2\sigma + N^{-2/3}\sigma\, \xi_{TW} & \text{subcritical} \\ \lambda_{iso}^{\infty} + N^{-1/2}\sqrt{2\sigma^2\left(1-\frac{\sigma^2}{r^2}\right)}\,\xi_{Gauss} & \text{supercritical} \end{cases}$$

▶ ξ_{TW} = random variable with GOE Tracy-Widom distribution P_{TW}

This means that in subcritical regime:

TRACY, WIDOM 1994

FORRESTER 1993

$$\lim_{N\to\infty} p\left(\frac{N^{2/3}(\lambda_N - 2\sigma)}{\sigma}\right) = P_{TW}$$

The gap between eigenvalues at edge is $\mathcal{O}(N^{-2/3})$ in subcritical regime

BAIK, LEE 2017

▶ ξ_{GAUSS} = random variable with GAUSSIAN distribution

$$\lim_{N\to\infty} P\left(N^{1/2} \frac{(\lambda_{iso} - \lambda_{iso}^{\infty})}{\sqrt{2\sigma^2(1-\sigma^2/r^2)}} \right) = P_{gauss}$$

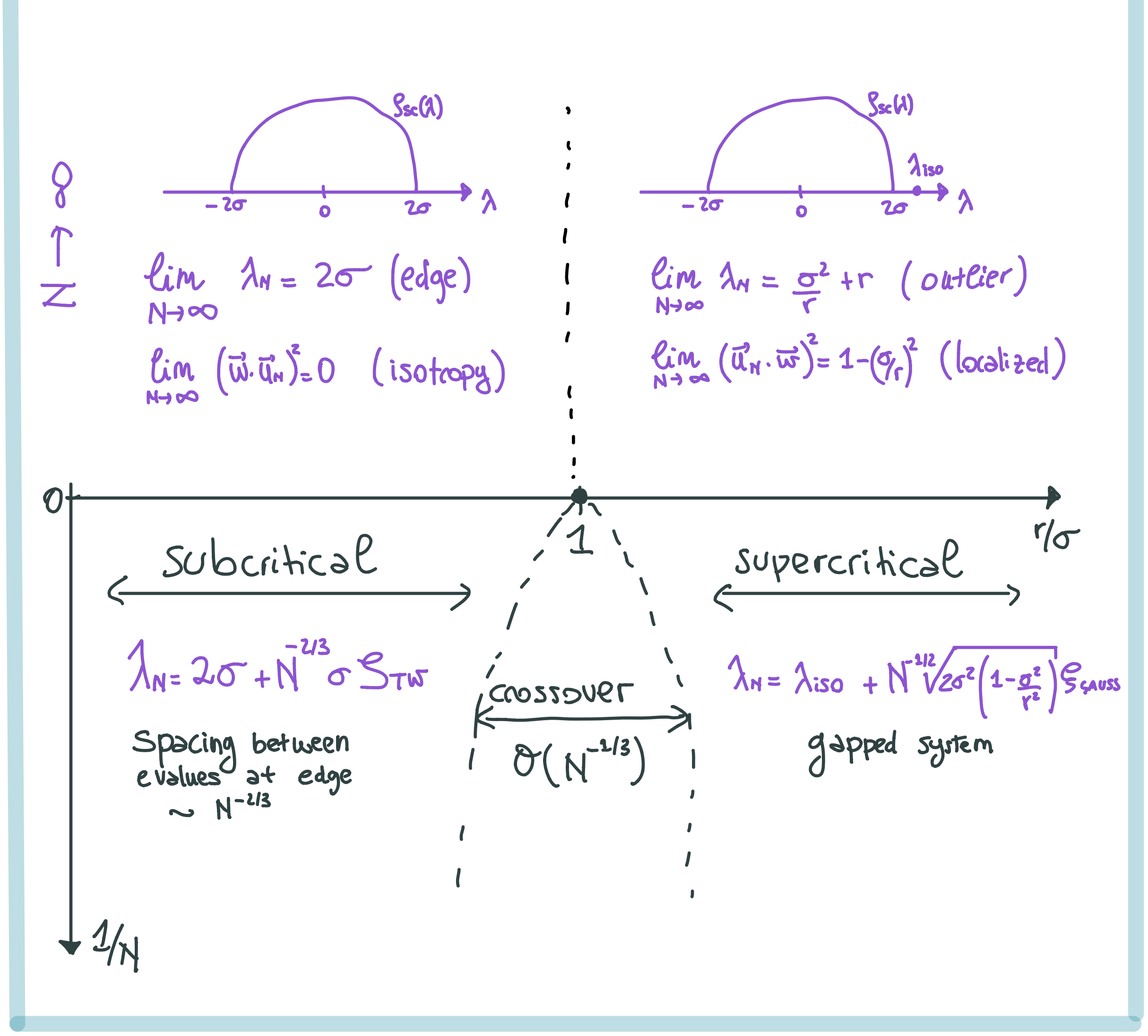
▦ At r_c, also a transition on the scaling of the fluctuations of largest evalue, not just on its typical value ⇒ "BBP transition".

BAIK, BEN AROUS, PÉCHÉ 2005

▨ The Tracy-Widom distribution appears in a huge variety of contexts: universality.

"KPZ (Kardar, Parisi, Zhang) universality class".

In summary:



$$\lim_{N\to\infty} \lambda_N = 2\sigma \quad \text{(edge)}$$

$$\lim_{N\to\infty} (\vec{w}\cdot\vec{u}_N)^2 = 0 \quad \text{(isotropy)}$$

$$\lim_{N\to\infty} \lambda_N = \frac{\sigma^2}{r} + r \quad \text{(outlier)}$$

$$\lim_{N\to\infty} (\vec{u}_N\cdot\vec{w})^2 = 1-(\sigma/r)^2 \quad \text{(localized)}$$

$$\lambda_N = 2\sigma + N^{-2/3}\sigma\,\xi_{TW}$$

Spacing between evalues at edge $\sim N^{-2/3}$

$$\mathcal{O}(N^{-1/3})$$

$$\lambda_N = \lambda_{iso} + N^{-1/2}\sqrt{2\sigma^2\left(1-\frac{\sigma^2}{r^2}\right)}\,\xi_{GAUSS}$$

gapped system

▶ Finite N fluctuations: large deviations

Joint evalue-evector projection distribution

$$\mathcal{P}_N\left(\{\lambda_\alpha, \xi_\alpha\}\right) = \frac{e^{-N\sum_\alpha f(\lambda_\alpha, \xi_\alpha)}}{Z_N} \prod_{\alpha=1}^{N} \theta\left(\lambda_{\alpha+1} - \lambda_\alpha\right) \underbrace{\prod_{\alpha<\beta} |\lambda_\beta - \lambda_\alpha|}_{\text{Vandermonde}} \times$$

$$\times\, \delta\left(\sum_{\alpha=1}^{N} \xi_\alpha - 1\right) \frac{1}{\sqrt{\xi_\alpha}}$$

where $f(\lambda_\alpha, \xi_\alpha) = \frac{1}{4\sigma^2}\left(\lambda_\alpha^2 - 2r\lambda_\alpha \xi_\alpha\right)$

$$\xi_\alpha = \left(\vec{u}_\alpha \cdot \frac{\vec{v}}{\sqrt{N}}\right)^2 = \left(\vec{u}_\alpha \cdot \vec{w}\right)^2$$

- $r=0$ [spectrum $\hat{J}$]: decoupling of evalues & evectors proj.

The eigenvalues alone distibuted as:

$$\mathcal{P}_N\left(\{\mu_1 \leq \cdots \leq \mu_N\}\right) = \frac{N!}{Z_N(\sigma)} \prod_{i=1}^{N} \left(e^{-\frac{N\mu_i^2}{4\sigma^2}} \theta\left(\mu_{i+1} - \mu_i\right)\right) \prod_{i<j} |\mu_i - \mu_j|$$

$$Z_N(\sigma) = \sigma^{\frac{N(N+1)}{2}} e^{\frac{3}{2}N\log 2} \left(\frac{2}{N}\right)^{\frac{N(N+1)}{4}} \prod_{i=1}^{N} \Gamma\left(1 + \frac{i}{2}\right)$$

The eigenvectors have statistics of random unit vectors: setting $q_\alpha = \sqrt{\xi_\alpha}$, then:

$$\mathcal{P}_N\left(\{q_\alpha\}_{\alpha=1}^{N}\right) = C_N\, \delta\left(\sum_{\alpha=1}^{N} q_\alpha^2 - 1\right)$$

| rotational invariance: evectors of $\hat{J}$ and $\hat{J}_R$ are equally probable

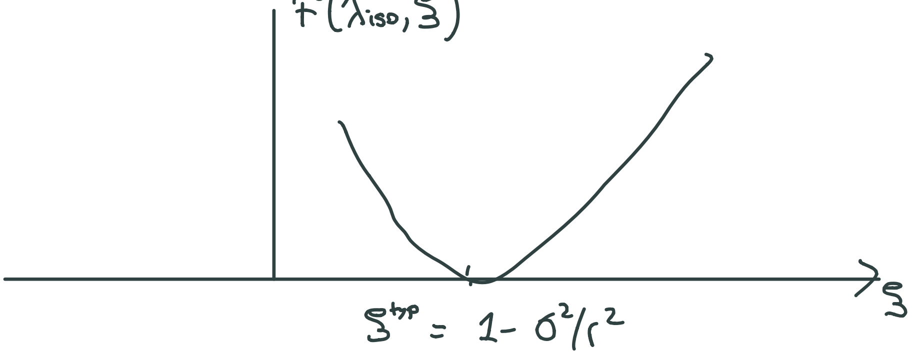
- For $r>0$: coupling of evalues & evector projection! This coupling can "pull" some eigenvector (the extremal) towards $\vec{w}$ when $r>r_c$.

- From $\mathcal{P}_N(\{\lambda_\alpha, \xi_\alpha\})$, can get the joint large deviation probability of λ_N, ξ_N - maximal eigenvalue/vector

$$P_{LD}(\lambda_N, \xi_N) \sim e^{-N F(\lambda_N, \xi_N)}$$

BIROLI, GUIONNET 2019

↖ probability of $\mathcal{O}(1)$ deviations from typical, asyptotic $N\to\infty$ value.



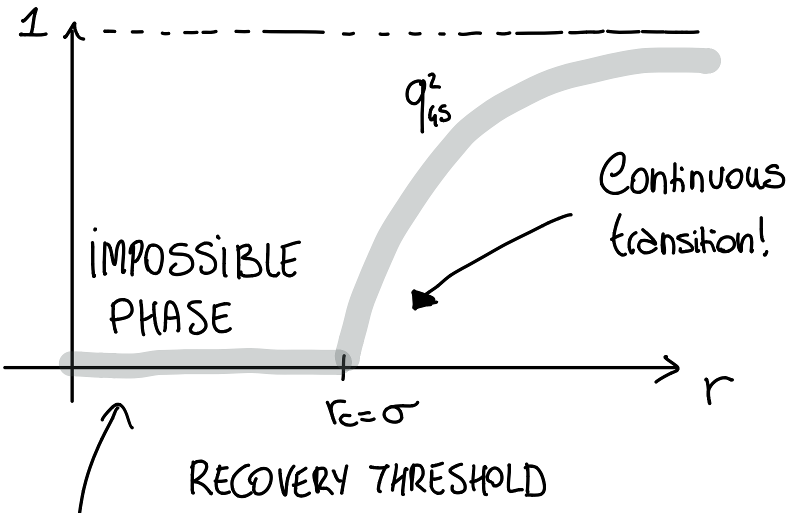
I.3 WHAT: GS, LANDSCAPE, DYNAMICS

Back to the inference problem...

▨ Q1: Recovering the signal

Q1: when is $\vec{S}_{GS}$ informative, i.e. $q_{GS} > 0$?

A sharp transition when $N \to \infty$: informative for $r > r_c$



here, even if I am able to find $\vec{S}_{GS}$, I would get no info on $\vec{v}$ because $\vec{S}_{GS}$ is uncorrelated to it.

Comments:

- The transition in the ground state could be found also from thermodynamics, studying the $\beta \to \infty$ limit of:

$$Z_\beta = \int_{S_N(\sqrt{N})} d\vec{s}\; e^{-\beta \mathcal{E}[\vec{s}]} = \int d\vec{s}\, d\lambda\; e^{-\beta \mathcal{E}[s] - \beta\frac{\lambda}{2}\left(\sum_{i=1}^{N} s_i^2 - N\right)}$$

Thermodynamically, the zero-temperature transition at $r = r_c = \sigma$ is a transition between a spin-glass phase at $r < r_c$, and a ferromagnetic phase at $r > r_c$.

At $T > 0$: phenomenology of condensation ⇒ EXERCISE 3!

KOSTERLITZ, THOULESS, JONES 1976

CUGLIANDOLO LECTURE NOTES CARGESE 2020

- Critical threshold for maximum likelihood is also "detection threshold" when $\vec{v}$ has gaussian or rademacher prior: below r_c, no estimator distinguishes between pure noise (GOE) and spiked matrices.

PERRY, WEIN, BANDEIRA, MOITRA 2018

Q2: A landscape of saddles

- Stationary points above ground state.

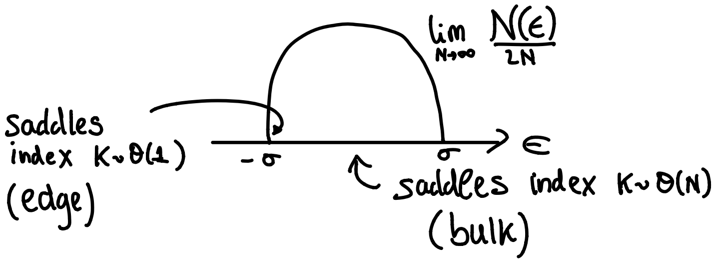
  $N_N(\epsilon) = \#$ stationary points with $\epsilon_N[s^*] = \epsilon$

  is a self-averaging random variable such that:

  $$\boxed{\lim_{N\to\infty} \frac{N_N(\epsilon)}{2N} = \lim_{N\to\infty} \frac{\langle N_N(\epsilon)\rangle}{2N} = \rho_{sc}(2\epsilon)}$$

  GOE density

- All stationary points (except GS) are saddles with negative directions of curvature: most have index $K \sim \Theta(N)$: NO trapping local minima!



- All these saddles have $q_N[s_\alpha] = \left(\frac{s_\alpha \cdot w}{\sqrt{N}}\right) \approx 0$

- Expect optimization not to be "hard" $(z \not\propto e^N)$

▦ Q3: Dynamics: DMFT, & beyond

Consider simplest algorithm: gradient descent.
(Langevin with $T \to 0$)

$$\frac{dS_i(t)}{dt} = -\sum_j J_{ij} S_j(t) - \lambda(t) S_i(t) + \sqrt{2T}\,\eta_i(t)$$

need to enforce to stay on sphere at each time

Gaussian white noise

$$\langle \eta_i(t)\,\eta_j(t') \rangle = \delta_{ij}\,\delta(t-t')$$

Mimicks coupling to degrees of freedom equilibrated at T

When $T \to 0$ (no noise), expect convergence to $T=0$ equilibrium state, the ground state $\vec{S}_{gs} = \pm\sqrt{N}\,\vec{u}_N$, when $t \to \infty$; how to take $N \to \infty$? Relevant timescales?

▨ Large-time and large-N limit: how?

(1) Mean-field dynamics: take $N \to \infty$ before, then $t \to \infty$.

Fully-connected models with randomness can be described by DMFT ('Dynamical Mean Field Theory')

Why? ▶ Dynamics becomes self-averaging when $N \to \infty$: properties of trajectories for different realizations of $\mathcal{E}[\vec{s}]$ become deterministic, converge to average value [can replace average over $\mathcal{E}$ with $N \to \infty$]

e.g. energy density: $\lim_{N\to\infty} \frac{\mathcal{E}[\vec{s}(t)]}{N} = \lim_{N\to\infty} \left\langle \frac{\mathcal{E}[s(t)]}{N} \right\rangle$

▶ These properties are one and two-point functions in time, for which have closed eqs, "DMFT equations"

$\mathcal{E}(t)$ ← time-dependent energy

$C(t,t') = \frac{1}{N} \sum_{i=1}^{N} S_i(t) S_i(t')$ ← correlation function

$R(t,t') = \frac{1}{N} \sum_{i=1}^{N} \left. \frac{\delta S_i(t)}{\delta \varepsilon_i(t')} \right|_{\varepsilon=0}$ ← response function

⇒ Used in many contexts: CUGLIANDOLO 2023 (Annual review of condensed matter physics)

(2) Beyond mean-field: dynamics for N large but finite.

DIFFICULT PROBLEM!

Often, fluctuations matter, no self-averagingness

Quantities are distributed.

Averages & typical values are different.

$\Longrightarrow$ This model (for $r=0$) is a rare case in which dynamics can be studied in both regimes, using Random Matrix Theory.

- $T=0$. In the eigenbasis $S_\alpha = (\vec{S} \cdot \vec{u}_\alpha)$

$$\frac{dS_\alpha}{dt}(t) = -\left[\lambda_\alpha + \underbrace{\lambda(t)}\right] S_\alpha(t)$$

couples all different α.
makes the equations non-linear.

- $T=0$, dynamics should converge to $\vec{S}_{gs} = \pm \sqrt{N}\, \vec{u}_N$.

Study convergence by excess energy:

$$\Delta \epsilon_N(t) = \left(\frac{\mathcal{E}(t)}{N} - \epsilon_{gs}\right) = \frac{1}{2} \frac{\sum_{\alpha \neq N} (\lambda_N - \lambda_\alpha) e^{-2(\lambda_N - \lambda_\alpha)t}}{1 + \sum_{\alpha \neq N} e^{-2(\lambda_N - \lambda_\alpha)t}} = f(\{\lambda_N - \lambda_\alpha\})$$

for random initial conditions $S_\alpha(t=0) = 1 \quad \forall \alpha$.

Short times, large times, dynamical crossovers

$$\Delta\epsilon_N(t) \overset{t\gg1}{\approx} g_N\, e^{-2t g_N} \qquad g_N = \lambda_N - \lambda_{N-1} \quad \text{'gap'}$$

Natural time where "probe" finite-N, energy scales where discreteness of spectrum matters:

$$\tau_{dync} \sim 1/g_N$$

$$\text{such that:} \begin{cases} t \ll \tau_{dync} & : \text{ dynamics looks as if } N\to\infty \text{ (DMFT-like)} \\ t \gg \tau_{dync} & : \text{ finite-N dynamics} \end{cases}$$

The fluctuations of the gap g_N are of the same order as those of maximal eigenvalue. Recall RMT detour:

$$1/g_N \sim \begin{cases} \mathcal{O}(N^{2/3}) & \text{subcritical regime } r \lesssim r_c \text{ (Tracy-Widom)} \\ ?! & \text{critical regime } |r-r_c| \sim \mathcal{O}(N^{-1/3}) \\ \mathcal{O}(N^{0}) & \text{supercritical regime} : r \gtrsim r_c \text{ system is GAPPED!} \end{cases}$$

$\mathcal{O}(N^0)$ is more precisely $\sim \log N$ D'ASCOLI, REFINETTI, BIROLI 2022

The mean-field dynamics: $N\to\infty$

One finds in this time regime:

$$\lim_{N\to\infty} \langle \Delta\epsilon_N(t)\rangle = v_{MF}(t) \overset{t\gg1}{\approx} \begin{cases} \frac{3\sigma}{8t} & r\le r_c \\ \frac{3\sigma}{8t} + \left[-\sigma-\epsilon_{GS}\right] & r\ge r_c \end{cases}$$

Slow, algebraic decay to the energy density of the ground state ($\epsilon_{GS}=-\sigma$) when $r\le r_c$, and to the same energy (which is no longer the ground state) when $r>r_c$.

(1) Dynamics is always out-of-equilibrium in this regime. It is, in fact, glassy:

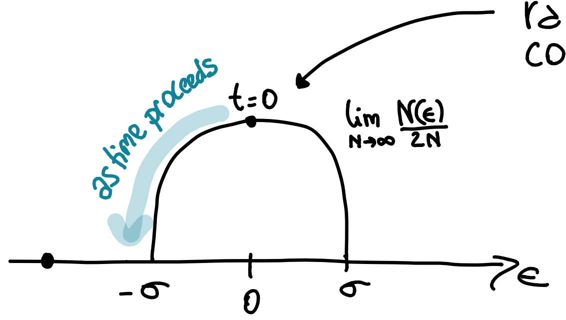
- $C(t,t')\neq c(t-t')$, modified FDT
- separation of timescales in $t-t'$
- aging & weak ergodicity breaking

"Aging": dynamics slower and slower as system becomes older (i.e., as time proceeds)

CUGLIANDOLO & DEAN 1995

BEN AROUS, DEMBO, GUIONNET 2001 (math)

(2) Landscape interpretation: in these timescales, Probe landscape at extensive energies above ϵ_{gs}, $\Delta\epsilon_N(t) \sim O(1)$. Region of landscape dominated by saddles, with density described by $\rho_{sc}(-2\epsilon)$:



random initial conditions: typically $\epsilon \simeq 0$

Why slowing down? no trapping by local minima (there are not!), but slow decay due to decreasing number of negative directions of saddles (decreasing index).

DMFT, $N\to\infty$ dynamics probes the bulk on $\rho_{sc}(2\epsilon)$.

▨ The finite-N dynamics, $N \gg 1$

▶ The subcritical regime ($r \leq r_c$): dynamics as for $r=0$. Crossover time $\tau_{dync} \sim N^{2/3}$.

$$\langle \Delta\epsilon_N(t) \rangle \sim \begin{cases} v_{MF}(t) & t \ll N^{2/3} = \tau_{dync} \\ N^{-2/3} v_{NMF}(t\, N^{-2/3}) & t \gg N^{2/3} = \tau_{dync} \end{cases}$$

For $t \ll N^{2/3}$, the system explores extensive energies above ϵ_{gs}. Dynamics is self-averaging, captured by mean-field (DMFT)

For $t \gg N^{2/3}$, system explore intensive energies above ϵ_{gs}. Dynamics not self-averaging, not captured by mean-field.

Consider $t \gg N^{2/3}$

- System explores intensive energies on top of ϵ_{gs}: Sensitive to statistics of extreme values and gaps g_N.
- Dynamics not self-averaging: $\langle \Delta\epsilon_N(t) \rangle$ dominated by realization where gap atypically small.
- The distribution of g_N is known! PERRET, SCHEHR 2015

$$\begin{cases} P(N^{2/3} g_N) \sim b\, N^{2/3} g_N & N^{2/3} g_N \to 0 \quad \text{(small gaps)} \\ P(N^{2/3} g_N) \sim e^{-2/3 (N^{2/3} g_N)^{3/2}} & N^{2/3} g_N \to \infty \quad \text{(large gaps)} \end{cases}$$

$$\Longrightarrow \langle \Delta_N \epsilon(t) \rangle \sim N^{-2/3} f_0(t\, N^{-2/3}) \qquad f_0(x) \sim \begin{cases} \dfrac{3\sigma}{8x} & x \to 0 \\ \dfrac{a\sigma}{x^3} & x \to \infty \end{cases}$$

scaling function known!

FYODOROV, PERRET, SCHEHR 2015

BARBIER, PIMENTA, CUGLIANDOLO, STARIOLO 2021

- **The supercritical regime**: in this case system is gapped: for $t \gg \tau_{dync} \sim \log N$, the system is able to reach the vicinity of S_{gs} and to relax to it exponentially (as in ferromagnetic systems):

$$\langle \Delta\epsilon_N(t) \rangle \sim \begin{cases} \sigma_{MF}(t) & t \ll \log N = \tau_{dync} \\ \frac{C_r}{t^{3/2}} e^{-2t\underbrace{\left|\frac{\sigma^2}{r} + r - 2\sigma\right|}_{\text{gap}}} & t \gg \log N = \tau_{dync} \end{cases}$$

- **The critical regime** $(|r - r_c| \sim \mathcal{O}(N^{-2/3})$: open problem!

$$\langle \Delta\epsilon_N(t) \rangle \overset{t \gg 1}{\sim} \int_0^\infty dg_N \, p(g_N) \, g_N e^{-2g_N t}$$

λ_N and λ_{N-1} strongly corelated, distribution $p(g_N)$ unknown.

From numerics, $p(g_N) \overset{g \ll 1}{\sim} g^{a(r,N)}$ PIMENTA, STARIOLO 2023

giving: $$\langle \Delta\epsilon_N(t) \rangle \overset{t \gg 1}{\sim} \begin{cases} \sigma_{MF}(t) \sim e^{-2t|\lambda_{iso}^\infty - 2\sigma|} & t \ll N^{2/3} \\ N^{2/3} f_a(t N^{-2/3}) & t \gg N^{2/3} \end{cases}$$

PART II

Rugged high-D landscapes

$\boxed{\text{II.1}}$ WHY: A HIGH-D INFERENCE EXAMPLE

▦ A 'hard' inference problem: noisy tensors

Beyond matices? Tensors! MONTANARI, RICHARD 2014

$$M_{i_1 i_2 i_3 \ldots i_p} = \frac{r}{N^{p-1}} v_{i_1} \ldots v_{i_p} + J_{i_1 \ldots i_p} \qquad (p>2)$$

$J_{i_1 \ldots i_p}$ symmetic, iid gaussian $\langle J^2_{i_1 \ldots i_p} \rangle = \frac{p!\,\tilde{\sigma}^2}{N^{p-1}}$

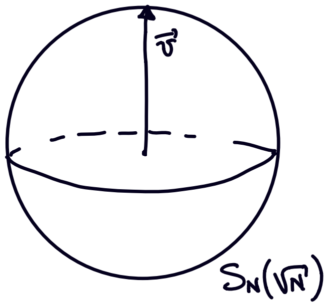
Energy landscape:

$$\boxed{\mathcal{E}[\vec{S}] = -\sum_{i_1 \le i_2 \ldots \le i_p} J_{i_1 \ldots i_p} S_{i_1} \ldots S_{i_p} - r N \left(\frac{\vec{v} \cdot \vec{S}}{N}\right)^p}$$

Again, fully-connected random interactions.

$$\langle \mathcal{E}[\vec{S}] \rangle = -r N \left(\frac{\vec{S} \cdot \vec{v}}{N}\right)^p$$

$$\langle \mathcal{E}[\vec{S}]\, \mathcal{E}[\vec{S}'] \rangle_c = \tilde{\sigma}^2 N \left(\frac{\vec{S} \cdot \vec{S}'}{N}\right)^p$$



Here: no spectrum. Also, landscape at r=0 much different...

▦ Landscape problem & complexity

Same questions as above, same approach: study stationary points.

$$\mathcal{E}_\lambda[\vec{s}] = -\sum_{i_1 \le i_2 \dots \le i_p} M_{i_1 \dots i_p} s_{i_1} \cdots s_{i_p} + \frac{\lambda}{2}\left(\sum_{i=1}^N s_i^2 - N\right)$$

$$\begin{cases} \dfrac{\partial \mathcal{E}_\lambda[\vec{s}^*]}{\partial s_i} = -\sum\limits_{i_2 \le \dots \le i_p} M_{i i_2 \dots i_p} s_{i_2}^* \cdots s_{i_p}^* + \lambda^* s_i^* \\ \dfrac{\partial \mathcal{E}_\lambda[\vec{s}^*]}{\partial \lambda} = \sum\limits_{i=1}^N (s_i^*)^2 - N = 0 \end{cases}$$

As before, multiply first equation by s_i, sum & use second equation:

$$\lambda^* = -\frac{1}{N}\left(\sum_i \frac{\partial \mathcal{E}[\vec{s}^*]}{\partial s_i} \cdot s_i\right) = -p \frac{\mathcal{E}[\vec{s}^*]}{N} = -p\,\epsilon[\vec{s}^*]$$

However, first equation non-linear: how many solutions? Introduce the random variable

$\mathcal{N}_N(\epsilon, q) =$ # stationary points $\vec{s}^*$ with $\epsilon_N[s^*] = \epsilon$ and $q_N[\vec{s}^*] = \frac{\vec{s} \cdot \vec{v}}{N} = q$.

▨ Quadratic landscape $(p=2)$:

$\mathcal{N}_N(\epsilon)$ is $\mathcal{O}(N)$ when $N \gg 1$

Self-averaging: $\lim_{N\to\infty} \frac{\mathcal{N}_N(\epsilon)}{2N} = \lim_{N\to\infty} \frac{1}{2N} \langle \mathcal{N}_N(\epsilon) \rangle = \rho_{sc}(-2\epsilon)$

▨ Landscape for $p>2$:

$\mathcal{N}_N(\epsilon,q)$ is $\mathcal{O}(e^N)$: $\mathcal{N}_N(\epsilon,q) \sim e^{N \Sigma_N(\epsilon,q)}$

$\mathcal{N}_N(\epsilon,q)$ not self-averaging

but $\Sigma_N(\epsilon,q)$ is:

$$\lim_{N\to\infty} \Sigma_N(\epsilon,q) = \lim_{N\to\infty} \langle \Sigma_N(\epsilon,q) \rangle = \Sigma_\infty(\epsilon,q)$$

$$\Rightarrow \Sigma_\infty(\epsilon,q) = \lim_{N\to\infty} \frac{1}{N} \langle \log \mathcal{N}_N(\epsilon,q) \rangle \quad \text{"COMPLEXITY"}$$

▦ Averages vs typical values, and replicas

Means that typically when $N \to \infty$ (with probability $\to 1$):

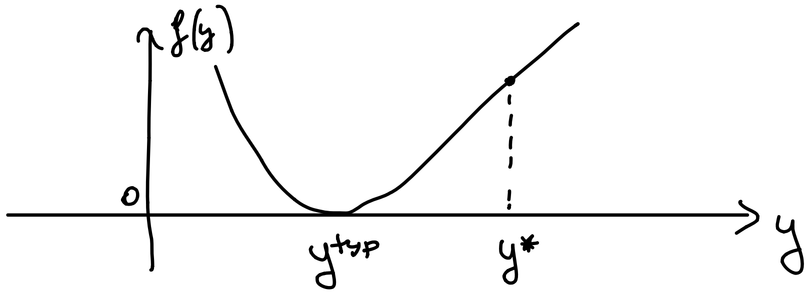
$[\mathcal{N}(\epsilon,q)]_{typ} \sim e^{N \Sigma_\infty(\epsilon,q)}$ (most probable value of $\mathcal{N}$)

But most-probable value is different from the average value: $\langle \mathcal{N}(\epsilon,q) \rangle \nsim e^{N \Sigma_\infty(\epsilon,q)}$

Average vs typical values: example.

Assume X_N is a random variable scaling as $X_N \sim e^N$: means that $Y_N = \frac{\log X_N}{N}$ has a limiting distibution when $N \to \infty$.

Assume that when $N \gg 1$, distribution of Y_N takes large-deviation form: $P_{Y_N}(y) \sim e^{-N f(y) + o(N)}$.



Then, typical value of X_N is:

$$[X_N]^{typ} \sim e^{N y^{typ}} \quad \text{where } y^{typ} \text{ such that } f'(y^{typ}) = 0 = f(y^{typ}).$$

On the other hand:

$$\langle X_N \rangle \simeq \int dy \, P_{Y_N}(y) e^{Ny} = \int dy \; e^{N[y - f(y)] + o(N)} \underset{\uparrow}{\simeq} e^{N[y^* - f(y^*)]}$$

and y^* such that $f'(y^*) = 1$. saddle point approximation

Since $y^* \neq y^{typ}$, $f(y^*) > 0$: y^* is exponentially rare, but controls the average: average "dominated" by rare realizations of random variable!

Message: to characterize what happens typically (with large probability) when $N \gg 1$ need:

"QUENCHED CALCULATION" $\Sigma_\infty(\epsilon, q) = \lim_{N \to \infty} \frac{1}{N} \langle \log \mathcal{N}_N(\epsilon, q) \rangle$

But this is hard; requires tricks like REPLICAS:

$$\langle \log \mathcal{N} \rangle = \lim_{\omega \to 0} \frac{\langle \mathcal{N}^\omega \rangle - 1}{\omega}$$

ω-th moment of $\mathcal{N}$

analytic continuation

In the following, we perform instead:

"ANNEALED APPROXIMATION" $\Sigma_A(\epsilon, q) = \lim_{N \to \infty} \frac{1}{N} \log \langle \mathcal{N}(\epsilon, q) \rangle$

It holds $\Sigma_A(\epsilon, q) \geq \Sigma_\infty(\epsilon, q) \Rightarrow \langle \mathcal{N}_N \rangle \gg [\mathcal{N}_N]_{typ}$

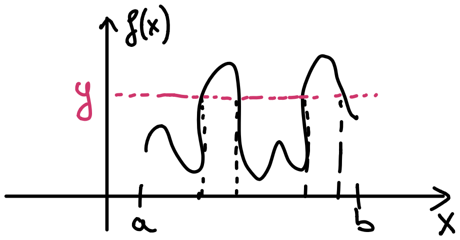
For the quenched calculation of the complexity in this model: ROS, BEN AROUS, BIROLI, CAMMAROTA 2018

II 2. HOW: KAC-RICE FORMALISM

Kac-Rice formula(s)

Kac-Rice formula = formula for average (or higher moments) of number of solutions of random equations.

Counting formulas: example.



$f(x)$ random function in $[a,b]$.
How many solutions of $f(x)=y$?

$$N(y) = \int_a^b dx\, \delta\left(x - f^{-1}(y)\right) = \int_a^b dx \frac{1}{\left|\frac{d}{dy} f^{-1}(y)\right|} \delta\left(y - f(x)\right)$$

$$= \int_a^b dx\, |f'(x)|\, \delta\left(y - f(x)\right) \qquad |f'(x)| = \text{Jacobian}$$

In higher dimension: $\vec{x} \in \mathcal{I} \subset \mathbb{R}^d$, $f(\vec{x}) = \vec{y} \in \mathbb{R}^d$

$$N(\vec{y}) = \int_{\mathcal{I}} d\vec{x} \prod_{i=1}^{d} \delta\left(f_i(\vec{x}) - y_i\right) \left| \det \left(\frac{\partial f_i(\vec{x})}{\partial x_j}\right)_{ij} \right|$$

- **Kac-Rice formula: stationary points of landscapes**

Count solutions of $\bar{\nabla}_\perp \mathcal{E}[\vec{s}]=0$, $\mathcal{E}[\vec{s}]=N\epsilon$, $\vec{s}\cdot\vec{v}=Nq$

Then:

$$\mathcal{N}(\epsilon, q)=\int_{S_N(\sqrt{N})} d\vec{s}\, \left|\det \nabla_\perp^2 \mathcal{E}[\vec{s}]\right| \delta(\bar{\nabla}_\perp \mathcal{E}[\vec{s}])\, \delta(\mathcal{E}[\vec{s}]-N\epsilon)\, \delta(\vec{s}\cdot\vec{v}-Nq)$$

Take average $\Rightarrow$ Kac-Rice formula.

$$\langle \mathcal{N}(\epsilon,q)\rangle=\int_{S_N(\sqrt{N})} d\vec{s}\, \delta(\vec{s}\cdot\vec{v}-Nq) \left\langle \left|\det \nabla_\perp^2 \mathcal{E}[\vec{s}]\right| \right\rangle_{\substack{\nabla_\perp \mathcal{E}=0 \\ \mathcal{E}=N\epsilon}} P_{\nabla_\perp \mathcal{E}, \mathcal{E}}(\vec{0}, N\epsilon)$$

average conditioned to $\nabla_\perp \mathcal{E}[\vec{s}]=0$ and $\mathcal{E}[\vec{s}]=N\epsilon$

joint density of $(\nabla_\perp \mathcal{E}, \mathcal{E})$ evaluated at $(\vec{0}, N\epsilon)$

BRAY, MOORE 1980

CAVAGNA, GIARDINA, PARISI 1998

FYODOROV 2013

BEN AROUS, AUFFINGER, CERNY 2010 (math)

Computing the complexity: 3 steps

The calculation is done in 3 steps, & uses 3 main ingredients:

(1) GAUSSIANITY

The functions $\mathcal{E}[\vec{s}]$, $\frac{\partial \mathcal{E}}{\partial s_i}[\vec{s}]$, $\frac{\partial^2 \mathcal{E}}{\partial s_i \partial s_j}[\vec{s}]$ are Gaussian: to get distribution, need only averages & covariances.
Can be computed explicitly (TRY! see below for hints)

Doing so, one finds:

(F1) $\vec{\nabla}_\perp \mathcal{E}[\vec{s}]$ independent of $\mathcal{E}[\vec{s}]$ and $\nabla_\perp^2 \mathcal{E}[\vec{s}]$.

Consequences:

- $P_{\nabla_\perp \mathcal{E}[\vec{s}], \mathcal{E}[\vec{s}]}(\vec{0}, N\epsilon) = P_{\nabla_\perp \mathcal{E}[\vec{s}]}(\vec{0})\, P_{\mathcal{E}[\vec{s}]}(N\epsilon)$

  factorization: two gaussians, known explicitly.

- $\left\langle \left| \det \nabla_\perp^2 \mathcal{E}[\vec{s}] \right| \right\rangle_{\substack{\nabla_\perp \mathcal{E}[\vec{s}]=0 \\ \mathcal{E}=N\epsilon}} = \left\langle \left| \det \nabla_\perp^2 \mathcal{E}[\vec{s}] \right| \right\rangle_{\mathcal{E}=N\epsilon}$

  Statistics of Hessian at stationary point is same as at any point of same energy.

(F2) The $(N-1)\times(N-1)$ matrix $\nabla_\perp^2 \mathcal{E}$ conditioned to $\mathcal{E}=N\epsilon$ has the same statistics as matrices:

$$\hat{M}[\vec{s}] = \hat{J} - p\epsilon \hat{1} - r_{\text{eff}}\left[q_N[\vec{s}]\right] \vec{w}_\perp \vec{w}_\perp^T$$

spherical constraint

rank-1 perturbation
Vector $\vec{w}$ is projection of $\vec{v}$ on tangent plane $\tau[\vec{s}]$

where $\hat{J}$ is a GOE: $\langle J_{ij}\rangle = 0$, $\langle J_{ij}^2\rangle = \frac{p(p-1)\vec{\sigma}^2}{N}(1+\delta_{ij})$

$r_{\text{eff}}(q) = r\,p(p-1)\,q^{p-2}(1-q^2)$, $\|w_\perp\|^2 = 1$.

(2) ISOTROPY

There is only one special direction in the sphere, that is $\vec{v}$. All averages & convariances, and so the joint distribution of $\mathcal{E}[\vec{s}]$, $\nabla_\perp \mathcal{E}[\vec{s}]$, $\nabla_\perp^2 \mathcal{E}[\vec{s}]$ depend on $\vec{s}$ only via $q_N[\vec{s}] = \left(\frac{\vec{s}\cdot\vec{v}}{N}\right) \longrightarrow$ (see above!)

Consequencies: for all $\vec{s}$ such that $q_N[\vec{s}] = q$

$$P_{\nabla_\perp \mathcal{E}[\vec{s}]}(\vec{0}) \to P_1(q) = (2\pi p\,\vec{\sigma}^2)^{-\left(\frac{N-1}{2}\right)} e^{-\frac{N}{2\vec{\sigma}^2} p r^2 q^{2p-2}(1-q^2)}$$

$$P_{\mathcal{E}}(N\epsilon) \to P_2(\epsilon, q) = \sqrt{\frac{N}{2\pi\vec{\sigma}^2}}\, e^{-\frac{N}{2\sigma^2}(\epsilon + rq^p)^2}$$

And $\left\langle \left|\det \nabla_\perp^2 \mathcal{E}[\vec{s}]\right| \right\rangle_{\mathcal{E}[\vec{s}']=N\epsilon} := \mathcal{D}_N(\epsilon, q)$

Therefore:

$$\langle \mathcal{N}(\epsilon, q)\rangle = \int_{S_N(\sqrt{N})} d\vec{s}\, \delta(\bar{s}.\bar{v} - Nq) \left\langle \left|\det \nabla_\perp^2 \mathcal{E}[\vec{s}]\right| \right\rangle_{\mathcal{E}=N\epsilon} P_{\nabla_\perp \mathcal{E}}(\vec{0})\, p_{\mathcal{E}}(N\epsilon)$$

$$= \mathcal{D}_N(\epsilon, q)\, p_1(q)\, p_2(\epsilon, q)\, V_N(q)$$

where $V_N(q) = \int_{S_N(\sqrt{N})} d\vec{s}\, \delta(Nq - \bar{s}\cdot\bar{v})$ Volume of the sub-sphere

Can show that $V_N(q) \overset{N \gg 1}{\sim} e^{N/2 \log[2\pi e(1-q^2)] + o(N)}$

Example: 2d rotationally-invariant function

$$I = \int_{\mathbb{R}^2} ds_1\, ds_2\, f\left(\sqrt{s_1^2+s_2^2}\right) \delta\left(\sqrt{s_1^2+s_2^2} - q\right) = \int_0^{2\pi} d\theta \int_0^{\infty} dr\, r\, f(r)\, \delta(r-q)$$

$$= (2\pi q)\, f(q) = V(q)\cdot f(q)$$

(3) LARGE-N AND RANDOM MATRIX THEORY

$$D_N(\epsilon, q) = \left\langle \left| \det\left(\hat{J} - p\epsilon \hat{1} - r_{eff}(q) \vec{w}_1 \vec{w}_1^T \right) \right| \right\rangle$$

Call $\lambda_2 \leq \ldots \leq \lambda_N = \{\lambda_\alpha\}_{\alpha=1}^{M=N-1}$ evalues of $\hat{J} - r_{eff}(q) \vec{w}_1 \vec{w}_1^T$

Then:

$$D_N(\epsilon, q) = \left\langle \prod_{\alpha=1}^{M} |\lambda_\alpha - p\epsilon| \right\rangle = \left\langle e^{\sum_{\alpha=1}^{M} \log|\lambda_\alpha - p\epsilon|} \right\rangle$$

$$= \left\langle e^{M \int d\nu_M(\lambda) \log|\lambda - p\epsilon|} \right\rangle$$

where $\nu_M(\lambda) = \frac{1}{M} \sum_{\alpha=1}^{M} \delta(\lambda - \lambda_\alpha)$ $\qquad M = N-1$

Recall facts in RMT (part I):

- The leading order contibution when $N \gg 1$ is given by the continuous part of $\nu_N(\lambda)$, the density:

$$D_N \approx \left\langle e^{N \int d\lambda\, \rho_N(\lambda) \log|\lambda - p\epsilon| + o(N)} \right\rangle$$

The average $\langle \cdot \rangle$ becomes average over $P_N(\{\rho\}) D\rho$

▥ The density $\rho_N(\lambda)$ is self-averaging, and $\rho_\infty(\lambda)$ does not depend on r and is the semicircular law $\rho_{sc}(\lambda)$ with $\sigma^2 \to \tilde{\sigma}^2 p(p-1)$.

$$D_N \overset{N\gg 1}{\simeq} e^{N\int d\lambda\, \rho_{sc}(\lambda)\, \log|\lambda - p\epsilon| + o(N)}$$

$\Longrightarrow$ This integral can be done explicitly:

$$\int d\lambda\, \frac{1}{2\pi\, p(p-1)\tilde{\sigma}^2} \sqrt{4p(p-1)\tilde{\sigma}^2 - \lambda^2}\, \log|\lambda - p\epsilon| =$$

$$= \int \frac{d\mu}{\pi} \sqrt{2-\mu^2}\, \log\left| \sqrt{2p(p-1)\tilde{\sigma}^2}\,\mu - p\varepsilon \right|$$

$$= \log\sqrt{2p(p-1)\tilde{\sigma}^2} + I\left(\frac{p\epsilon}{\sqrt{2p(p-1)\tilde{\sigma}^2}} \right)$$

$$I(y) = \int d\mu\, \frac{\sqrt{2-\mu^2}}{\pi} \log|\mu - y|$$

$$= \begin{cases} \frac{y^2-1}{2} + \frac{y}{2}\sqrt{y^2-2} + \log\left(\frac{-y+\sqrt{y^2-2}}{2} \right) & y \leq -\sqrt{2} \\ \frac{y^2}{2} - \frac{1}{2}(1+\log 2) & y > -\sqrt{2} \end{cases}$$

Computing distributions: example

Consider the unconstrained gradient: $\nabla\mathcal{E}[s] = \left(\frac{\partial \mathcal{E}}{\partial s_i}\right)_{i=1}^{N}$

Then:

$$\left\langle \frac{\partial \mathcal{E}}{\partial s_i}[\vec{s}]\right\rangle = -r\,N\,p\left(\frac{v\cdot s}{N}\right)^{p-1}\frac{v_i}{N}$$

while:

$$\left\langle \frac{\partial \mathcal{E}[\vec{s}]}{\partial s_i}\frac{\partial \mathcal{E}[\vec{s}']}{\partial s'_j}\right\rangle_c \textcircled{=} \sum_{k_1=1}^{p}\sum_{k_2=1}^{p}\sum_{\substack{i_1\leq\dots\leq i_p\\ j_1\leq\dots\leq j_p}}\langle J_{i_1 i_2\dots i_p} J_{j_1 j_2\dots j_p}\rangle\,\delta_{i_{k_1},i}\,\delta_{j_{k_2},j}\; s_{i_1}\cdots\cancel{s_{i_{k_1}}}\cdots s_{i_p}\times$$

$$\times\, s'_{j_1}\cdots\cancel{s'_{j_{k_2}}}\cdots s'_{j_p}$$

Using that $\langle J_{i_1\dots i_p} J_{j_1\dots j_p}\rangle = \frac{p!\,\tilde{\sigma}^2}{N^{p-1}}\prod_{n=1}^{p}\delta_{i_n j_n}$,

$$\textcircled{=}\sum_{k_1=1}^{p}\sum_{k_2=1}^{p}\frac{p!\,\tilde{\sigma}^2}{N^{p-1}}\,\frac{1}{p!}\sum_{i_1,\dots,i_p}\delta_{i_{k_1},i}\,\delta_{i_{k_2},j}\; s_{i_1}\cdots\cancel{s_{i_{k_2}}}\cdots s_{i_p}\times$$

$$\times\, s'_{i_1}\cdots\cancel{s'_{i_{k_2}}}\cdots s'_{i_p}$$

Distinguishing the case $k_1=k_2$ (p of them) and $k_1\neq k_2$ ($p\cdot(p-1)$ of them) one gets:

$$\left\langle \frac{\partial \mathcal{E}[\vec{s}]}{\partial s_i}\frac{\partial \mathcal{E}[\vec{s}']}{\partial s'_j}\right\rangle_c = \tilde{\sigma}^2\left\{p\,\delta_{ij}\left(\frac{s\cdot s'}{N}\right)^{p-1} + p(p-1)\frac{s'_i s_j}{N}\left(\frac{s\cdot s'}{N}\right)^{p-2}\right\}$$

Now, $\nabla_\perp\mathcal{E}[\vec{s}]$ is the projections of $\nabla\mathcal{E}[\vec{s}]$ on the space orthogonal to $\vec{s}$, i.e. on the tangent plane $\tau[\vec{s}]$. Choosing $\vec{e}_\alpha[\vec{s}]$ a basis of $\tau[\vec{s}]$, one has $\vec{e}_\alpha\cdot\vec{s}=0$. Thus:

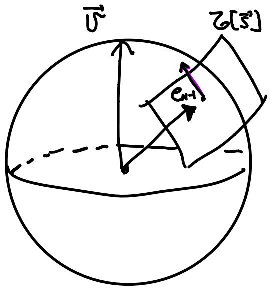
$$\langle(\nabla_\perp\mathcal{E}[\vec{s}])_\alpha\rangle = \langle(\nabla\mathcal{E}[\vec{s}]\cdot\vec{e}_\alpha)\rangle = -r\,N\,p\left(\frac{v\cdot s}{N}\right)^{p-1}\left(\frac{v\cdot e_\alpha}{N}\right)$$

And:

$$\langle (\nabla_\perp \mathcal{E}[\vec{s}])_\alpha (\nabla_\perp \mathcal{E}[\vec{s}\,])_\beta \rangle_c = \langle (\nabla \mathcal{E}[\vec{s}] \cdot e_\alpha[\vec{s}]) (\nabla \mathcal{E}[s] \cdot e_\beta[\vec{s}'])\rangle =$$

$$= \tilde{\sigma}^2 p\, \delta_{\alpha\beta} + p(p-1) \underbrace{\left(\frac{s \cdot e_\alpha}{\sqrt{N}}\right)}_{=0} \underbrace{\left(\frac{s \cdot e_\beta}{\sqrt{N}}\right)}_{=0} = p \tilde{\sigma}^2 \delta_{\alpha\beta}$$

In the annealed calculation, all distributions depend on $\vec{s}$ only via $q[\vec{s}] = \frac{\vec{v} \cdot \vec{s}}{N}$: $\vec{v}$ is the only 'special direction' on the sphere, that breaks isotropy. It is convenient to choose, for each $\vec{s}$, this basis on the tangent plane:



$$\vec{e}_{N-1}[\vec{s}] = \frac{1}{\sqrt{N(1-q^2)}} \left(\vec{v} - q[\vec{s}]\,\vec{s}\right)$$

$$\vec{e}_\alpha[\vec{s}] \perp \{\vec{v}, \vec{s}\} \qquad \alpha = 1, .., N-2$$

$\Rightarrow$ only $(\vec{\nabla}_\perp \mathcal{E})_{N-1}$ and $(\nabla^2_\perp \mathcal{E})_{\alpha\, N-1}$ or $(\nabla^2_\perp \mathcal{E})_{N-1, \alpha}$ will have a q-dependent distibution

$$\langle \nabla \mathcal{E}[\vec{s}] \cdot \vec{e}_\alpha \rangle = r N p \left(\frac{Q[\vec{s}]}{N}\right)^{p-1} \left(\frac{\vec{v}}{N} \cdot \vec{e}_\alpha\right) = \begin{cases} 0 & \alpha < N-1 \\ \neq 0 & \alpha = N-1 \end{cases}$$

$$\langle (\nabla_\perp \mathcal{E}[\vec{s}])_\alpha (\nabla_\perp \mathcal{E}[\vec{s}])_\beta \rangle_c = p \tilde{\sigma}^2 \delta_{\alpha\beta}$$

The annealed complexity

Combine all terms:

$$\langle \mathcal{N}(\epsilon, q) \rangle = V_N(q)\, D_N(\epsilon, q)\, P_1(q)\, P_2(N\epsilon) = e^{N \Sigma_A(\epsilon, q) + o(N)}$$

$$\boxed{\begin{aligned}\Sigma_A(\epsilon, q) = \frac{1}{2} \log\left[2e(p-1)(1-q^2)\right] - {}^{p}/_{2\tilde{\sigma}^2}\, r^2 q^{2p-2}(1-q^2) \\ - \frac{1}{2\tilde{\sigma}^2}\left(\epsilon + r q^p\right)^2 + I\left(\sqrt{{}^{p}/_{2(p-1)\tilde{\sigma}^2}}\; \epsilon\right)\end{aligned}}$$

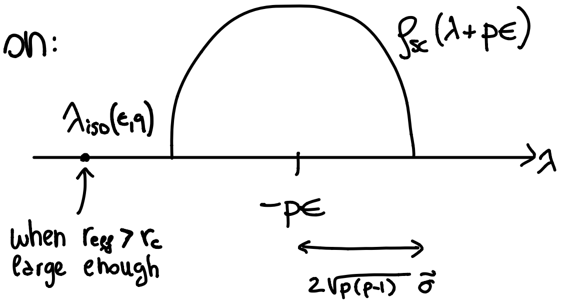
This gives distribution of stationary points in energy and geometry (overlap with $\vec{v}$), on average.

What about stability?

- The Hessian at a stationary point with (ϵ, q) is a rank-1 perturbed, shifted GOE:

$$\nabla_\perp^2 \mathcal{E}\Big|_{\epsilon, q} \overset{\text{law}}{\sim} \hat{J} - p\epsilon\, \hat{1} - r_{\text{eff}}(q)\, \vec{\omega}_\perp \vec{\omega}_\perp^T$$

The eigenvalue distribution:



■ Local minima have all eigenvalues positive. For bulk, need:

$$-p\epsilon > 2\sqrt{p(p-1)}\,\tilde{\sigma} \Rightarrow \epsilon < \epsilon_{th} = -2\tilde{\sigma}\sqrt{\frac{p-1}{p}}$$

ϵ_{th} = "threshold energy". Also, $\lambda_{iso}(\epsilon,q) > 0$.

The p→2 limit of $\Sigma_A(\epsilon,q)$

The annealed complexity is maximal at $q=0$. We set $\Sigma_A(\epsilon) = \Sigma_A(\epsilon, q=0)$.

Recall that $\langle J^2_{i_1 \ldots i_p} \rangle = \frac{p!\,\tilde{\sigma}^2}{N}$ while in PART I we set $\langle J_{ij}^2 \rangle = \frac{\sigma^2}{2}(1+\delta_{ij})$. To be consistent, $\tilde{\sigma}^2 = \frac{\sigma^2}{2}$

$$\Sigma_A(\epsilon) \xrightarrow{p=2} \frac{1}{2}\log(2e) - \frac{\epsilon^2}{\sigma^2} + I\left(\sqrt{2/\sigma^2}\,\epsilon\right)$$

Then, given that $\epsilon > -\sigma$:

$$\longrightarrow \frac{1}{2}\log(2e) - \frac{\epsilon^2}{\sigma^2} + \frac{\epsilon^2}{\sigma^2} - \frac{1}{2} - \frac{\log 2}{2} = 0$$

Consistently with the fact that for $p=2$ there are <u>not</u> exponentially-many stationary points. One can use the Kac-Rice formula to get the results of PART I: exercise 4!

▨ The quenched calculation: what would change?

One needs to compute higher moments $\langle \mathcal{N}_N^{w}(\epsilon,q)\rangle$ with $w=2,3,4\dots$ and $w\to 0$!

One can use Kac-Rice formulas, too, for higher moments: need to consider w points on sphere: $\vec{s}^{a}$ with $a=1,\dots,w$. The fields $\mathcal{E}[\vec{s}^{a}], \nabla_\perp\mathcal{E}[\vec{s}^{a}], \nabla_\perp^2\mathcal{E}[\vec{s}^{a}]$ are correlated.

$$\langle \mathcal{N}^{w}(\epsilon,q)\rangle = \int_{S_N(\sqrt{N})} \prod_{a=1}^{w} d\vec{s}^{a}\, \delta\left(\vec{s}^{a}\cdot\vec{v}-Nq\right)\, P_{\nabla_\perp\mathcal{E}[\vec{s}^{a}],\,\mathcal{E}[\vec{s}^{a}]}(\vec{0},N\epsilon)\, \Big\langle \prod_{a=1}^{w}\left|\det \nabla_\perp^2\mathcal{E}[\vec{s}^{a}]\right|\Big\rangle_{\substack{\nabla_\perp\mathcal{E}[\vec{s}^{a}]=0\\ \mathcal{E}[\vec{s}^{a}]=N\epsilon}}$$

↑ Joint distribution of w $(N-1)$-dim vectors $\nabla_\perp\mathcal{E}[\vec{s}^{a}]$ and w scalars $\mathcal{E}[\vec{s}^{a}]$

↑ joint expectation value, conditioned to w vectors and functions

Some consequences of correlations:

(i) No decoupling: $\nabla_\perp\mathcal{E}[\vec{s}^{a}]$ for fixed a is independent of $\mathcal{E}[\vec{s}^{a}], \nabla_\perp^2\mathcal{E}[\vec{s}^{a}]$, but not of $\mathcal{E}[\vec{s}^{b}], \nabla_\perp^2\mathcal{E}[\vec{s}^{b}]$ at $b\neq a$.

Consequences: (1) need to compute joint distributions, (2) the expectation of Hessians is a problem of coupled random matrices.

What helps: still Gaussian for (1), and large-N for (2).

(ii) Distributions depend not only on $q_N[\vec{s}^a] = \left(\frac{\vec{s}^a \cdot \vec{v}}{N}\right)$,
but also on mutual overlaps $Q_N[\vec{s}^a, \vec{s}^b] = \left(\frac{\vec{s}^a \cdot \vec{s}^b}{N}\right)$:

Consequence: no longer 1 special direction, but w of them.
What helps: Still, huge dimensionality reduction!
From $N \cdot w$ variables s_i^a to $\frac{w(w-1)}{2} + w$ ones,
the $Q_N[\vec{s}^a, \vec{s}^b]$ and $q_N[\vec{s}^a]$. Because fully-connected.

(iii) The conditional distribution of the Hessian at one point $\vec{s}^a$ is still that of a perturbed GOE, but finite-rank perturbations are more complicated: both additive & multiplicative, and not "free" (in the sense of free probability).

WHY: multiplicative perturbations due to conditioning to $\nabla_\perp \mathcal{E}[\vec{s}^b]$ with $b \neq a$.

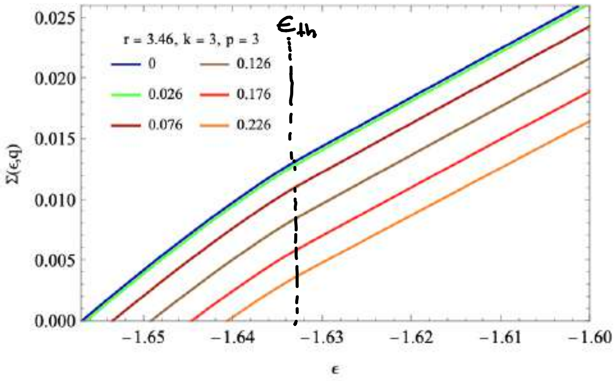
Consequence: calculation of isolated evalues is more involved; what helps: perturbation is still of finite-rank.

To see comparisons between quenched & annealed, see
ROS, BEN AROUS, BIROLI, CAMMAROTA 2018

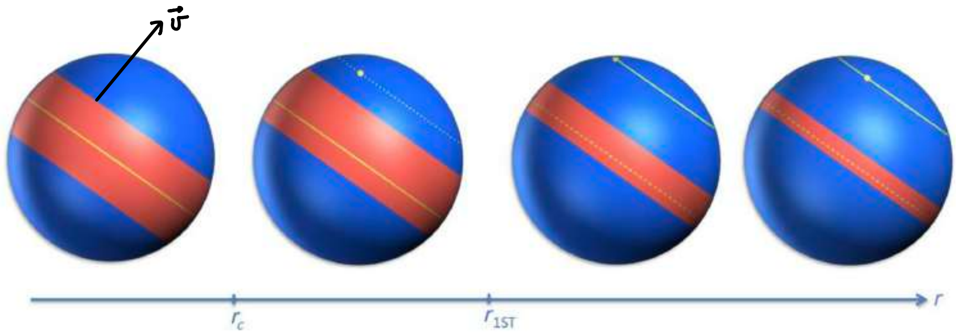
II 3. WHAT: GS, LANDSCAPE, DYNAMICS

Back to the inference problem. Here, summarize results of quenched calculation:

- Quenched complexity curves ($\gamma = 1$)



- Landscape's evolution with r: regions where $\Sigma_\infty(\epsilon,q)>0$ for some ϵ (in red), and $q[S_{GS}]$ (yellow).

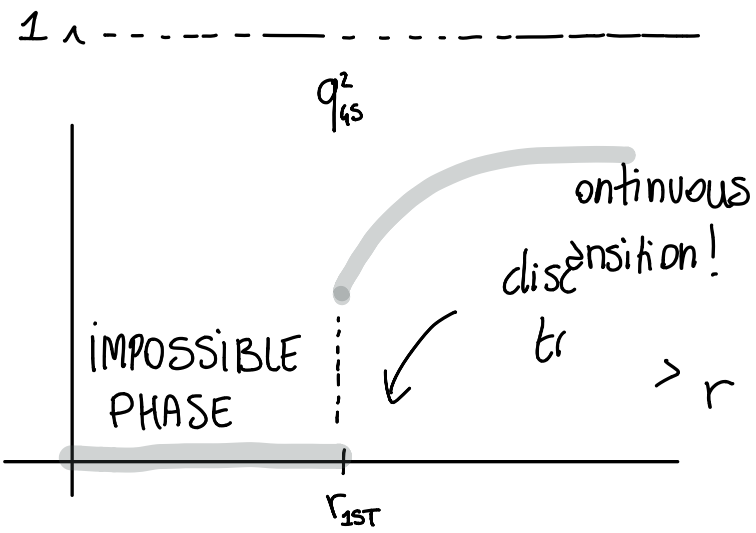


↳ an isolated (local) minimum appears close to signal $\vec{v}$.

Recovering the signal

Q1: when is $\vec{S}_{GS}$ informative, i.e. $q_{GS} > 0$?

A sharp transition when $N \to \infty$ at some $r = r_{1ST}$



RECOVERY THRESHOLD

Differences with respect to $p=2$: the transition is discontinuous, first order!
As for $p=2$: could be obtained with thermodynamic calculation for $\beta \to \infty$

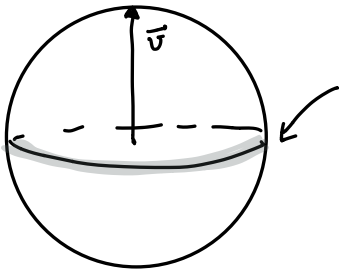
GILLIN SHERRINGTON 2000

▦ A Landscape of minima

- Most stationary points are un-informative of $\vec{v}$:
  (Neglecting isolated minimum at high overlap)
  Optimize over q : $\Sigma_\infty(\epsilon, q)$ maximal at $q=0$:

$$\Sigma_\infty(\epsilon) = \max_q \Sigma_\infty(\epsilon, q) = \frac{1}{2}\log\left[2e(p-1)\right] - \frac{\epsilon^2}{2\tilde{\sigma}^2} + I\left(\sqrt{\frac{p}{2(p-1)\tilde{\sigma}^2}}\,\epsilon\right)$$

does not depend on r! Also, $\Sigma_\infty(\epsilon, q=0) = \Sigma_A(\epsilon, q=0)$



exponential majority of stationary points is orthogonal to the signal! (Not informative)

- Exponentially many local minima!

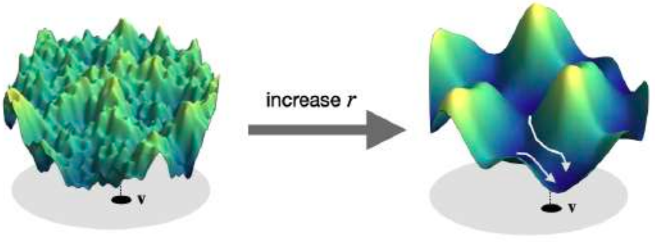
  Recall Hessian (annealed calculation)

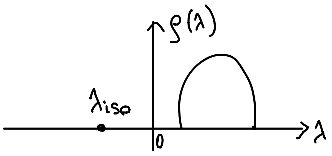
$$\nabla_\perp^2 \mathcal{E}\Big|_{\epsilon, q} \overset{\text{law}}{\sim} \hat{J} - p\epsilon\,\hat{1} - r_{\text{eff}}(q)\,\vec{\omega}_\perp \vec{\omega}_\perp^T$$

Local minima: $\epsilon < \epsilon_{th}$, $\lambda_{iso}(\epsilon, q) > 0$.

$\underline{q=0}$: $r_{\text{eff}}(q=0)=0$. No isolated evalue. Exponentially-many local minima $\epsilon<\epsilon_{th}$: trapping states for dynamics!

$\underline{q>0}$: when r large enough, generate isolated evalue, that can become negative : minima → saddle transitions





Spectrum of rank-1 saddle

- 'Topological trivialization?' How strong should r be to destabilize also minima at equator? Need $r\sim N^{\alpha}$:

$$r_{\text{eff}} = r\,p(p-1)\left(\frac{\underline{s}\cdot\vec{v}}{N}\right)^{p-2}\left(1-\left(\frac{\underline{s}\cdot\vec{v}}{N}\right)^{2}\right)\sim r\left(\frac{1}{\sqrt{N}}\right)^{p-2}$$

$$\Longrightarrow \alpha = \frac{p-2}{2}$$

▦ Dynamics: DMFT. And beyond?

'Easy' phase: for $r \sim N^{\alpha}$ with $\alpha > \alpha_c = \frac{p-2}{2}$, gradient descent converges to $\vec{S}_{GS}$ in times $\mathcal{O}(N^{0})$.

BEN AROUS, GHEISSARI, JAGANNATH 2020

'Hard' phase $r \sim \mathcal{O}(1)$: dynamics from random initial conditions stuck in high-entropy $q=0$ region, the equator. Here landscape is as if $r=0$.

▶ The dynamics at $r=0$: "short times".

Described by DMFT ($N \to \infty$ before $t \to \infty$)

Excess energy does not decay to zero as for $p=2$, but converges to finite value:

$$\lim_{N\to\infty} \Delta_N \epsilon(t) = \lim_{N\to\infty} \left(\epsilon_N(t) - \epsilon_{GS}\right) = -\tilde{\sigma}^2 p \int_0^t C^{p-1}(t,s)\, R(t,s)\, ds - \epsilon_{GS}$$

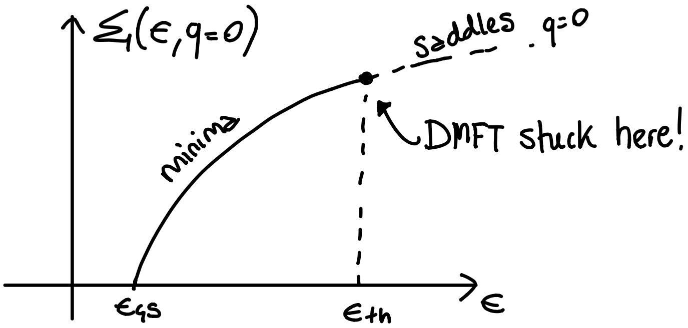
When $t \to \infty$, converges to finite value:

$$\lim_{t\to\infty} \lim_{N\to\infty} \Delta_N \epsilon(t) = \epsilon_{th} - \epsilon_{GS} > 0$$

Never reach the GS energy density in these timescales. Out-of-equilibrium glassy dynamics, aging. CUGLIANDOLO, KURCHAN 1993

BOUCHAUD, CUGLIANDOLO, KURCHAN, MEZARD 1997 (review)

Landscape interpretation?



$\Rightarrow$ gradient descent gets stuck at energies of the highest-energy minima, that are exponentially numerous.

CUGLIANDOLO, KURCHAN 1993

SELLKE 2024 (math)

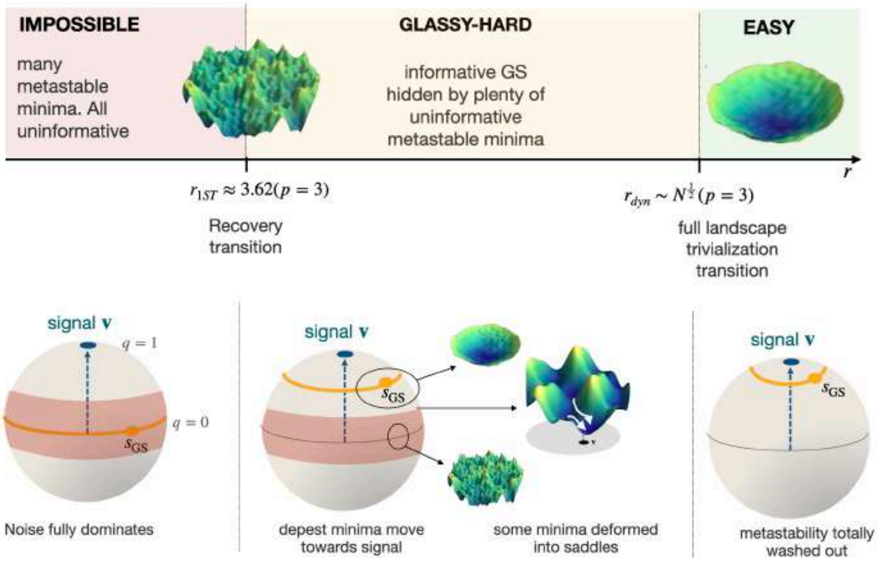
▶ The dynamics at $T=0$: "long times".

For $p=2$, equilibration timescales $\sim \mathcal{O}(N^{2/3})$.

For $p\geq 3$, expect timescales $\sim \mathcal{O}(e^N)$: system has to escape from trapping minima crossing energy barriers $\Delta E \sim \mathcal{O}(N) \Rightarrow$ ACTIVATED DYNAMICS.

This regime of the dynamics is open problem!

In summary



- The ground-state becomes correlated with $\vec{v}$ for $r > r_{1ST}$

- Exponentially-many local minima for all values of r. Those closer to $\vec{v}$ become saddles when r increases, those at equator remain minima.

- Optimization is hard: system trapped by metastable states. Mean-field dynamics studied a lot for $r=0$. Dynamics at finite N is open problem.

Directions: "two beyonds"

▨ Dynamics in complex landscapes, beyond mean-field.

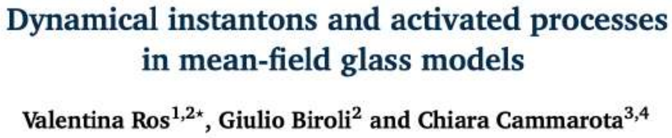
Activated dynamics: at times $t \sim \mathcal{O}(e^N)$, dynamics driven by rare jumps between local minima.

How rare? Arrhenius: $\tau_{jump} \sim e^{+\beta \underset{\substack{\uparrow \\ \text{energy barrier}}}{\Delta E}} \sim e^{\beta N \Delta \epsilon}$

Path Integral Approach Unveils the Role of Complex Energy Landscape for Activated Dynamics of Glassy Systems

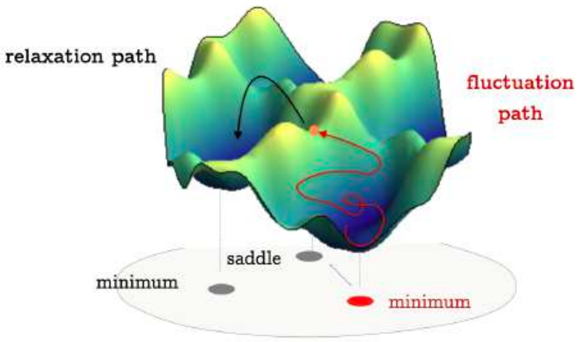
Tommaso Rizzo

ISC-CNR, UOS Rome, Università "Sapienza", Piazzale A. Moro 2, I-00185, Rome, Italy and Dip. Fisica, Università "Sapienza", Piazzale A. Moro 2, I-00185, Rome, Italy



Dynamical instantons and activated processes in mean-field glass models

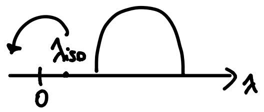
Valentina Ros[1,2⋆], Giulio Biroli[2] and Chiara Cammarota[3,4]



Same problem discussed in many lectures (transition paths, instantons...) but here: HIGH-DIMENSION! Exponentially-many attractors, exponentially-many transition states, entropy....

- **HOW CONNECTED TO THESE LECTURES?** Landscape geometry (which saddles are connected to given minimum?) crucial ingredient to interpret dynamics in activated regime.
  an example: ROS, BIROLI, CAMMAROTA 2021

- **WHY LARGE DEVIATIONS?** Large deviations of evalues of Hessian needed to study saddles:



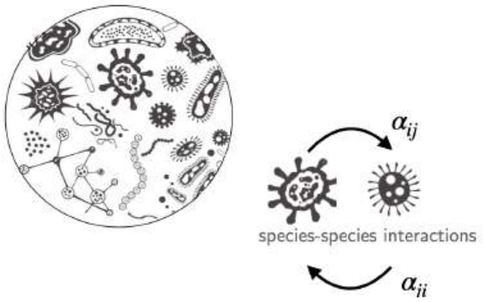
e.g. ROS 2020

Large deviations of dynamics e.g. RIZZO 2020

LARGE DEVIATIONS TO CONTROL STABILITY OF STATIONARY POINTS, OR TO COMPUTE INSTANTONS OF DYNAMICS.

III High-D dynamics, beyond landscapes

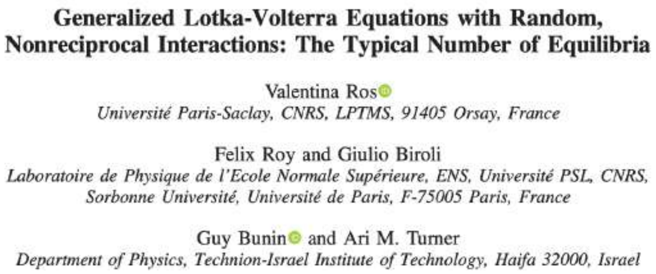
High-D systems with non-reciprocal interactions: dynamics is not optimization! There is no underlying landscape: non-gradient dynamics!



Relevant for modeling interacting species in ecology, neurons in biological networks, agents in society, firms in economy, ...

HOW CONNECTED? Can have multiple equilibria (attractors) of the dynamical equations. Studying their typical properties might help.

an example: ROS, ROY, BIROLI, BUNIN, TURNER 2023



Generalized Lotka-Volterra Equations with Random, Nonreciprocal Interactions: The Typical Number of Equilibria

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Department of Physics, Technion-Israel Institute of Technology, Haifa 32000, Israel

Spiked GOE: eigenvalues density and outliers

[Ref: Bouchaud, Potters, A First Course in Random Matrix Theory, Cambridge University Press 2020].

Take the $N \times N$ matrix $\hat{M} = \hat{J} + \hat{R}$, where $\hat{J}$ is a GOE matrix with $\langle J_{ij} \rangle = 0$ and $\langle J_{ij}^2 \rangle = \frac{\sigma^2}{N}(1 + \delta_{ij})$, while $\hat{R} = r\vec{w}\vec{w}^T$ is a rank-1 perturbation, with $||\vec{w}||^2 = 1$. Call λ_α with $\alpha = 1, \cdots, N$ the eigenvalues of $\hat{M}$, and call $\vec{u}_\alpha$ the corresponding eigenvectors. The resolvent of $\hat{M}$ is

$$\hat{G}_{\hat{M}}(z) = \frac{1}{z\hat{1} - \hat{M}} = \sum_{\alpha=1}^{N} \frac{\vec{u}_\alpha \vec{u}_\alpha^T}{z - \lambda_\alpha}$$

The goal of these two exercises is to derive the self-consistent equations for the Stieltjes transform of $\hat{M}$, and for its isolated eigenvalue.

Exercise 1. Replica calculation of the Stieltjes transform.

The starting point of the calculation is the Gaussian identity :

$$\left(\frac{1}{z\hat{1} - \hat{M}} \right)_{ij} = \frac{1}{\mathcal{Z}} \int \prod_{i=1}^{N} \frac{d\psi_i}{\sqrt{2\pi}} \psi_i \psi_j e^{-\frac{1}{2} \sum_{i,j=1}^{N} \psi_i (z\hat{1} - \hat{M})_{ij} \psi_j}, \qquad \mathcal{Z} = \int \prod_{i=1}^{N} \frac{d\psi_i}{\sqrt{2\pi}} e^{-\frac{1}{2} \sum_{i,j=1}^{N} \psi_i (z\hat{1} - \hat{M})_{ij} \psi_j}$$

We wish to take the average of this expression with respect to the matrix $\hat{M}$. However, averaging the partition function in the denominator makes the calculation potentially difficult; to proceed, we make use of the replica trick to write

$$\mathcal{Z}^{-1} = \lim_{n \to 0} \mathcal{Z}^{n-1}.$$

We then follow the standard steps of replica calculations, see below.

(i) **From randomness to coupled replicas.** Using the replica trick, justify why $(z\hat{1} - \hat{M})^{-1} = \lim_{n \to 0} I_{ij}^{(n)}$ where

$$I_{ij}^{(n)} = \int \prod_{a=1}^{n} \prod_{i=1}^{N} \frac{d\psi_i^a}{\sqrt{2\pi}} \psi_i^1 \psi_j^1 e^{-\frac{1}{2} \sum_{a=1}^{n} \sum_{i,j=1}^{N} \psi_i^a (z\hat{1} - \hat{J} - r\vec{w}\vec{w}^T)_{ij} \psi_j^a}$$

Take the average of this expression with respect to J_{ij}, and show that

$$\langle I_{ij}^{(n)} \rangle = \int \prod_{a=1}^{n} \prod_{i=1}^{N} \frac{d\psi_i^a}{\sqrt{2\pi}} \psi_i^1 \psi_j^1 e^{-\frac{1}{2} \sum_{a=1}^{n} \sum_{i,j=1}^{N} \psi_i^a (z\delta_{ij} - rw_i w_j) \psi_j^a} e^{\frac{\sigma^2}{4N} \sum_{a,b} \left(\sum_{i=1}^{N} \psi_i^a \psi_i^b \right)^2}.$$

Now one has an expression without randomness, in which the replicated variables ψ^a are coupled with each others.

(ii) **Hubbard–Stratonovich.** We would like now to perform the integral over the variables ψ_i^a; however, this integral contains quartic terms in the exponent; in order to turn such an integral into a Gaussian one, we perform a Hubbard-Stratonovich transformation: we introduce the order parameters

$$Q_{ab}[\psi] = \frac{1}{N} \sum_{i=1}^{N} \psi_i^a \psi_i^b \quad a \leq b$$

and write the integral as

$$\int \prod_{a=1}^{n} \prod_{i=1}^{N} \frac{d\psi_i^a}{\sqrt{2\pi}} \cdots \to N^{\frac{n(n+1)}{2}} \int \prod_{a \leq b} dQ_{ab} \int \prod_{a=1}^{n} \prod_{i=1}^{N} \frac{d\psi_i^a}{\sqrt{2\pi}} \prod_{a \leq b} \delta \left(NQ_{ab} - \sum_{i=1}^{N} \psi_i^a \psi_i^b \right) \cdots$$

Show that using the integral representation of the delta distributions

$$\delta\left(NQ_{ab}-\sum_{i=1}^{N}\psi_i^a\psi_i^b\right)=\int\frac{d\lambda_{ab}}{2\pi}e^{i\lambda_{ab}\left[NQ_{ab}-\sum_{i=1}^{N}\psi_i^a\psi_i^b\right]}$$

and introducing the $n\times n$ matrix Λ with components $\Lambda_{ab}=2\lambda_{aa}\delta_{ab}+\lambda_{ab}(1-\delta_{ab})$ and the $N\times N$ matrix A with components $A_{ij}=z\delta_{ij}+rw_iw_j$, the average can be cast in the following form:

$$\langle I_{ij}^{(n)}\rangle=N^{\frac{n(n+1)}{2}}\int\prod_{a\leq b}dQ_{ab}d\lambda_{ab}e^{\frac{N\sigma^2}{4}\text{Tr}_n[Q^2]+\frac{N}{2}\text{Tr}_n[i\Lambda\,Q]}f_N[Q,\vec{w}] \tag{1}$$

with

$$f_N[Q,\vec{w}]=\int\prod_{a=1}^{n}\prod_{i=1}^{N}\frac{d\psi_i^a}{\sqrt{2\pi}}\psi_i^1\psi_j^1e^{-\frac{1}{2}\sum_{a,b}\sum_{i,j}\psi_i^a\left[\hat{1}_N\otimes i\Lambda+A\otimes\hat{1}_n\right]_{ij}^{ab}\psi_j^b}.$$

(iii) **Gaussian integration.** Performing the Gaussian integral, show that

$$\langle I_{ij}^{(n)}\rangle=\delta_{ij}\int\prod_{a\leq b}dQ_{ab}d\lambda_{ab}e^{\frac{N}{2}A_N[Q,i\Lambda]}\left[(A\otimes 1_n+1_N\otimes i\Lambda)^{-1}\right]_{ij}^{11}$$

$$A_N[Q,i\Lambda]=\frac{\sigma^2}{2}\text{Tr}_n[Q^2]+\text{Tr}_n[i\Lambda\,Q]-\frac{1}{N}\text{Tr}_{nN}[\log\left(A\otimes 1_n+1_N\otimes i\Lambda\right)]$$

Hint. Use that $\int\prod_{i=1}^{d}\frac{dx_i}{\sqrt{2\pi}}x_lx_me^{-\frac{1}{2}\vec{x}\cdot\hat{K}\vec{x}}=\hat{K}_{lm}^{-1}|\det K|^{-1}$ and that $\log|\det K|=\text{Tr}\log K$.

(iv) **Saddle point.** The integral can now be computed with a saddle point approximation. Show that the saddle point equations for the matrices Q and $i\Lambda$ read

$$i\Lambda=-\sigma^2Q,\qquad Q=\frac{1}{N}\text{Tr}_{nN}\left[\frac{1}{A\otimes 1_n+1_N\otimes i\Lambda}\right]$$

Show that, plugging the first into the second and assuming that the matrices Λ,Q are diagonal and replica symmetric, i.e. $Q_{ab}=\delta_{ab}g$ and $\lambda_{ab}=\delta_{ab}\lambda$, one reduces to a single equation for g which reads

$$g=\frac{1}{N}\text{Tr}_N\left[\frac{1}{(z-\sigma^2g)\hat{1}_N-r\vec{w}\vec{w}^T}\right]$$

Using that

$$\langle(z\hat{1}-\hat{M})^{-1}\rangle=\lim_{n\to 0}\langle I_{ij}^{(n)}\rangle=\left[\left(A\otimes 1_n-\sigma^2g1_N\otimes 1_n\right)^{-1}\right]_{ij}^{11},$$

justify why g is the Stieljes transform of the matrix M. Show that expanding $g=g_\infty+g_1/N+\cdots$, the leading order term satisfies the equation

$$g_\infty^{-1}=z-\sigma^2g_\infty.$$

Exercise 2. The isolated eigenvalue and eigenvector.

(i) Show that if $\hat{A}$ is a matrix and $\vec{v},\vec{u}$ are vectors, then

$$(\hat{A}+\vec{u}\vec{v}^T)^{-1}=\hat{A}^{-1}-\frac{A^{-1}\vec{u}\vec{v}^TA^{-1}}{1+\vec{v}\cdot A^{-1}\vec{u}}.$$

Use this formula (Shermann-Morrison formula) to get an expression for $\hat{G}_{\hat{M}}(z)$.

(ii) The isolated eigenvalue is a pole of the resolvent operator $\hat{G}_{\hat{M}}(z)$, which is real and such that $\lambda_{\text{iso}}>2\sigma$. Using that λ_{iso} does not belong to the spectrum of the unperturbed matrix $\hat{J}$, show that it solves the equation

$$r\vec{w}\cdot G_{\hat{J}}(\lambda_{\text{iso}})\vec{w}=1.$$

(iii) Using that $\hat{J}$ and $\vec{w}$ are independent and that typically $\vec{w}$ is *delocalized* in the eigenbasis of $\hat{J}$, show that

$$\vec{w} \cdot G_{\hat{J}}(\lambda_{\rm iso})\vec{w} \overset{N\to\infty}{\longrightarrow} g_{\rm sc}(\lambda_{\rm iso})$$

where $g_{\rm sc}(\lambda)$ is the Stieltijes transform of the GOE matrix $\hat{J}$.

(iv) Using the self-consistent equation satisfied by $g_{\rm sc}(\lambda)$, derive the expression of the inverse function $g_{\rm sc}^{-1}$ and determine its domain; use it to show that

$$\lambda_{\rm iso} = \frac{\sigma^2}{r} + r \qquad r \geq \sigma.$$

(v) The eigenvectors projections $\xi_\alpha = (\vec{w} \cdot \vec{u}_\alpha)^2$ can be obtained from the resolvent as residues of the poles:

$$\xi_\alpha = \lim_{\lambda \to \lambda_\alpha} (\lambda - \lambda_\alpha) \vec{w} \cdot G_{\hat{M}}(\lambda)\vec{w}$$

Use this to show that if $\alpha = N$ labels the isolated eigenvalue, then

$$\xi_N = -\frac{1}{r^2 g'_{\rm sc}(\lambda_{\rm iso})} = 1 - \frac{\sigma^2}{r^2}.$$

Hint. Use that if $\lim_{\lambda\to\lambda_0} f(\lambda) = 0 = \lim_{\lambda\to\lambda_0} g(\lambda)$, then $\lim_{\lambda\to\lambda_0} \frac{f(\lambda)}{g(\lambda)} = \lim_{\lambda\to\lambda_0} \frac{f'(\lambda)}{g'(\lambda)}$.

Exercise 1 - solution

Stieltjes transform with replica method

(i) The normalization Z is an integral over the variables ψ_i. Writing:

$$Z^{n-1} = \left[\int \prod_{i=1}^{N} \frac{d\psi_i}{\sqrt{2\pi}} \cdots \right]^{n-1} = \left[\int \prod_{i=1}^{N} \frac{d\psi_i^{(2)}}{\sqrt{2\pi}} \cdots \right] \cdots \left[\int \prod_{i=1}^{N} \frac{d\psi_i^{(n)}}{\sqrt{2\pi}} \cdots \right]$$

we can set:

$$\lim_{n\to 0} Z^{n-1} \int \prod_{i=1}^{N} \frac{d\psi_i}{\sqrt{2\pi}} \psi_i \psi_j \, e^{-\frac{1}{2}\sum_{ij} \psi_i (z-M)_{ij} \psi_j} =$$

$$= \lim_{n\to 0} \int \prod_{i=1}^{N} \frac{d\psi_i}{\sqrt{2\pi}} \psi_i \psi_j \, e^{-\frac{1}{2}\sum_{ij} \psi_i (z-M)_{ij} \psi_j} \int \prod_{a=2}^{n} \prod_{i=1}^{N} \frac{d\psi_i^{(a)}}{\sqrt{2\pi}} \cdots$$

label these variables as $\psi_i^{(1)}$

$$= \lim_{n\to 0} \int \prod_{a=1}^{n} \prod_{i=1}^{N} \frac{d\psi_i^{(a)}}{\sqrt{2\pi}} \psi_i^{(1)} \psi_j^{(1)} \, e^{-\frac{1}{2}\sum_{a=1}^{n}\sum_{ij} \psi_i^{a} (z-M)_{ij} \psi_j^{a}}$$

$$=: \lim_{n\to 0} I_{ij}^{(n)}$$

(ii) Using the integral representation of $\delta(\cdot)$, we obtain:

$$\langle I_{ij}^{(n)} \rangle = \int \prod_{a=1}^{n} \prod_{i=1}^{N} \frac{d\psi_i^{(a)}}{\sqrt{2\pi}} \cdot N^{\frac{n(n+1)}{2}} \int \prod_{a\leq b} dQ_{ab} \int \prod_{a\leq b} \frac{d\lambda_{ab}}{2\pi}\, e^{i\sum_{a\leq b} \lambda_{ab}\left(NQ_{ab} - \sum_i \psi_i^a \psi_i^b\right)} \times$$

$$\times\, \psi_i^1 \psi_j^1\, e^{-\frac{1}{2}\sum_a \sum_{ij} \psi_i^a \underbrace{(z\delta_{ij} - r\omega_i\omega_j)}_{A_{ij}} \psi_j^a} \times$$

$$\times\, e^{\frac{\sigma^2}{4N}\sum_{a,b}\Big(\underbrace{\sum_i \psi_i^a \psi_j^b}_{N Q_{ab}}\Big)^2}$$

$\leftarrow$ exchange order integration

$$\textcircled{=}\, N^{\frac{n(n+1)}{2}} \int \prod_{a\leq b} dQ_{ab} \int \prod_{a\leq b} \frac{d\lambda_{ab}}{2\pi}\, e^{\overbrace{iN\left(\frac{1}{2}\sum_{a\neq b} Q_{ab}\lambda_{ab} + \sum_a Q_{aa}\lambda_{aa}\right)}^{(*)}}$$

$$\times\, e^{\frac{\sigma^2 N}{4}\sum_{a,b} Q_{ab}^2} \times \int \prod_{a=1}^{n} \prod_{i=1}^{N} \frac{d\psi_i^a}{\sqrt{2\pi}}\, \psi_i^1 \psi_j^1$$

$$e^{\underbrace{-i\left(\frac{1}{2}\sum_{a\neq b} \lambda_{ab} \sum_i \psi_i^a \psi_i^b + \sum_a \lambda_{aa} \sum_i \psi_i^a \psi_i^a\right)}_{(**)}}\, e^{-\frac{1}{2}\sum_a \sum_{ij} \psi_i^a A_{ij} \psi_j^a}$$

Introducing $\Lambda_{ab} = 2\lambda_{aa}\delta_{ab} + \lambda_{ab}(1-\delta_{ab})$ and the trace $\mathrm{tr}_n[O] = \sum_{a=1}^{n} O_{aa}$, we can rewrite

$$(*) = \frac{N}{2}\,\mathrm{tr}_n[Q \cdot i\Lambda]$$

and

$$(**) = -\frac{1}{2}\sum_{ij}\sum_{ab}\psi_i^a\left[1_N \otimes i\Lambda\right]_{ij}^{ab}\,\psi_j^b \qquad \text{where } 1_N = \overset{N\times N}{\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1\,. \end{pmatrix}}$$

Moreover, $\sum_{a,b} Q_{ab}^2 = \mathrm{tr}_n[Q^2]$.

(iii) The integral over the ψ_i^a is now gaussian.

Using that for an arbitrary (positive-definite) matrix K_{ij} it holds

$$\int \prod_{i=1}^{N} dx_i\, e^{-\frac{1}{2}\sum_{ij} x_i K_{ij} x_j}\, x_\ell x_m = \frac{(K^{1})_{\ell m}\,(2\pi)^{N/2}}{|\det K|}$$

and that $\log|\det K| = \mathrm{tr}\,\log|K|$,

we get:

$$\int \prod_{a=1}^{n}\prod_{i=1}^{N}\frac{d\psi_i^a}{\sqrt{2\pi}}\,\psi_i^1\psi_j^1\, e^{-\frac{1}{2}\sum_{a,b}\sum_{ij}\psi_i^a\left[A_{ij}\delta_{ab} + \delta_{ij}(i\Lambda)_{ab}\right]\psi_j^b} =$$

$$= (K^{-1})_{ij}^{11}\, e^{-\mathrm{tr}\log K} \qquad \text{where } \overset{Nn\times Nn}{K} = A\otimes 1_n + 1_N \otimes i\Lambda$$

Combining everything, one gets the final expression

(iv) The saddle point equations are obtained taking the variation of

$$A_N[Q, i\Lambda] = \frac{\sigma^2}{2} \sum_{a,b} Q_{ab}^2 + \sum_{a,b} (i\Lambda)_{ab} Q_{ab} - \frac{1}{N} \mathrm{Tr} \log \left(A \otimes 1_n + 1_N \otimes i\Lambda \right)$$

$$\frac{\delta A_N}{\delta Q_{ab}} = \sigma^2 Q_{ab} + i\Lambda_{ab} = 0 \qquad \Rightarrow \quad i\Lambda = -\sigma^2 Q$$

$$\frac{\delta A_N}{\delta \Lambda_{ab}} = Q_{ab} - \frac{1}{N} \mathrm{tr}_N \left(\frac{1}{A \otimes 1_n + 1_N \otimes i\Lambda} \right)_{ab} = 0$$

$$\Longrightarrow Q = \frac{1}{N} \mathrm{tr}_N \left(\frac{1}{A \otimes 1_n + 1_N \otimes i\Lambda} \right) = \frac{1}{N} \mathrm{tr}_N \left(\frac{1}{A \otimes 1_n - \sigma^2 1_N \otimes Q} \right)$$

If $Q = \begin{pmatrix} g & & \\ & \ddots & \\ & & g \end{pmatrix}$, then componentwise:

$$g = \frac{1}{N} \mathrm{tr}_N \left(\frac{1}{z - r w w^T - \sigma^2 g} \right)$$

To compute the trace, one can choose a basis e_α such that $e_1 = w$, $e_\alpha \perp w$ $\forall \alpha = 2, \dots, N$. Then:

$$g = \frac{1}{N} (N-1) \frac{1}{z - \sigma^2 g} + \frac{1}{N} \frac{1}{z - r - \sigma^2 g} = \frac{1}{z - \sigma^2 g} + \mathcal{O}(1/N)$$

$$\Longrightarrow g_\infty = \frac{1}{z - \sigma^2 g_\infty} \Longrightarrow \sigma^2 g_\infty - z g_\infty + 1 = 0.$$

Exercise 2 - Solution

isolated evalue/evector of spiked GOE matrix

(i) One has $(A+uv^T)^{-1} = \left(A\left[1+A^{-1}uv^T\right]\right)^{-1} = (1+A^{-1}uv^T)^{-1}A^{-1}$

Using the formal expansion:

$$(1+A^{-1}uv^T)^{-1} = 1 - A^{-1}uv^T + A^{-1}uv^TA^{-1}uv^T + \dots$$

leads to

$$(A+uv^T)^{-1} = A^{-1} - A^{-1}uv^TA^{-1} + A^{-1}u\underbrace{(v^TA^{-1}u)}_{\text{number}}v^TA^{-1} + \dots$$

Calling $X = v^TA^{-1}u$ and resumming the series:

$$(A+uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1+X}$$

In the case of the rank-1 perturbation with $\vec{u} = -\frac{r}{N}\vec{w}, \vec{v} = \vec{w}$ and $\hat{A} = z\hat{1} - \hat{J}$ we get

$$\hat{G}_M(z) = (z-\hat{M})^{-1} = \hat{G}_J(z) + r\,\frac{\hat{G}_J(z)\vec{w}\vec{w}^T\hat{G}_J(z)}{1-r\,\vec{w}.\hat{G}_J(z)\vec{w}} \quad (*)$$

(ii) The eigenvalues of $\hat{M}$ are poles of $\hat{G}_M(z)$. If λ_{iso} is an outlier, it is not a pole of $\hat{G}_J(z)$, because it does not belong to the spectrum of $\hat{J}$ that is the semicircle in $[-2\sigma, 2\sigma]$.

To be a pole of $\hat{g}_M(z)$ and not of $\hat{G}_J(z)$, λ_{iso} must be a zero of the denominator of the second term in (*):

$$1 - r\,\vec{w}\cdot\hat{G}_J(\lambda_{iso})\vec{w} = 0$$

(iii) The fact that $\vec{w}$ is "delocalized" in the basis of eigenstates of $\hat{J}$, which I call $\vec{e}_\alpha$, implies that typically

$$(\vec{w}\cdot\vec{e}_\alpha)^2 \sim 1/N \qquad N \gg 1.$$

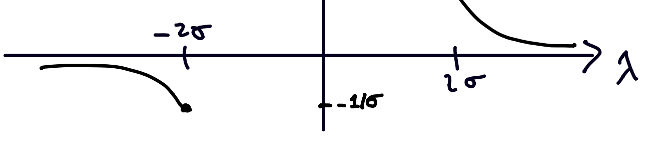
The scalar product $\vec{w}\cdot\hat{G}_J(\lambda)\vec{w}$ can be expanded in the eigenbasis of $\hat{J}$, and one gets:

$$\vec{w}\cdot\hat{G}_J(\lambda)\cdot\vec{w} = \sum_{\beta=1}^{N}(\vec{e}_\beta\cdot\vec{w})^2\left[\hat{G}_J(z)\right]_{\beta\beta} \overset{N\gg1}{\sim} \frac{1}{N}\sum_{\beta=1}^{N}\left[\hat{G}_J(z)\right]_{\beta\beta}$$

The last term is the normalized trace of the resolvent, i.e. the Stieltjes transform. Therefore:

$$\lim_{N\to\infty}\vec{w}\cdot\hat{G}_J(\lambda)\cdot\vec{w} = g_{sc}(\lambda)$$

(iv) The function $g_{sc}(\lambda)$ has the following behavior on the real axis:



The function is invertible only if $y \in [-1/\sigma, 1/\sigma]$.

The expression for g_{sc}^{-1} can be more easily obtained from the self-consistent equation:

$$\sigma^2 g_{sc}^2(z) - z\, g_{sc}(z) - 1 = 0$$

$$\Longrightarrow z = \sigma^2 g_{sc}(z) + \frac{1}{g_{sc}(z)} \Rightarrow g_{sc}^{-1}(y) = \sigma^2 y + 1/y$$

The equation for λ_{iso} reads: $g_{sc}(\lambda_{iso}) = 1/r$.

It admits a solution only for $1/r \in [-\frac{1}{\sigma}, \frac{1}{\sigma}]$.

meaning that $r \geqslant \sigma$ for $r > 0$.

In this case, $\lambda_{iso} = g_{sc}^{-1}(1/r) = \frac{\sigma^2}{r} + r$

(v) Using the decomposition of $g_{\hat{M}}$ in its eigenbasis $(\lambda_\alpha, \vec{u}_\alpha)_{\alpha=1}^N$

$$\hat{g}_M(z) = \sum_{\beta=1}^{N} \frac{\vec{u}_\beta \vec{u}_\beta^T}{z - \lambda_\beta} \Rightarrow \vec{w}^T \hat{g}_M \vec{w} = \sum_{\beta=1}^{N} \frac{\xi_\beta^2}{z - \lambda_\beta}$$

Then obviously if $z \to \lambda_\alpha$ is an isolated pole,

$$\xi_\alpha^2 = \lim_{\lambda \to \lambda_\alpha} \sum_{\beta=1}^{N} \frac{(\lambda - \lambda_\alpha)}{(\lambda - \lambda_\beta)} \xi_\beta^2$$

We use again the expression (*). Since λ_{iso} is not a pole of $\hat{J}$, the first term will not contribute to the residue and so:

$$\xi_N = \lim_{\lambda \to \lambda_{iso}} \frac{r\left(\vec{w}^T \hat{G}_J(\lambda) \cdot \vec{w}\right)^2}{1 - r\,\vec{w} \cdot \hat{G}_J(\lambda)\,\vec{w}} (\lambda - \lambda_{iso})$$

$$\overset{N \gg 1}{\simeq} \lim_{\lambda \to \lambda_{iso}} (\lambda - \lambda_{iso}) \frac{r\, g_{sc}^2(\lambda)}{1 - r\, g_{sc}(\lambda)}$$

$$= \lim_{\lambda \to \lambda_{iso}} \frac{(\lambda - \lambda_{iso})}{1 - r g_{sc}(\lambda)} \cdot g_{sc}(\lambda_{iso})$$

When $\lambda \to \lambda_{iso}$, $1 - r g_{sc}(\lambda) \to 0$ and thus the limit gives $0/0$: One has to compute it by taking the derivative of both numerator & denominator

$$\lim_{\lambda \to \lambda_{iso}} \frac{(\lambda - \lambda_{iso})}{1 - r g_{sc}(\lambda)} g_{sc}(\lambda_{iso}) = g_{sc}(\lambda_{iso}) \lim_{\lambda \to \lambda_{iso}} \frac{-1}{r g'_{sc}(\lambda)}$$

Using that $g_{sc}(\lambda_{iso}) = 1/r$, one gets: $\xi_N = -\frac{1}{r^2 g'_{sc}(\lambda_{iso})}$

TO make this more explicit, convenient to take the self-consistent eq. for $g_{sc}(\lambda)$ and derive it:

$$2\sigma^2 g'_{sc}(z)\, g_{sc}(z) - g_{sc}(z) - z\, g'_{sc}(z) = 0$$

$$(2\sigma^2 g_{sc} - z)\, g'_{sc} = g_{sc} \Rightarrow \frac{g_{sc}}{g'_{sc}} = 2\sigma^2 g_{sc} - z$$

At $z = \lambda_{iso}$,

$$\xi_N = -\frac{1}{r}\left(\frac{g_{sc}}{g'_{sc}}\right) = -\frac{1}{r}\left(2\sigma^2 g_{sc}(\lambda_{iso}) - \lambda_{iso}\right)$$

$$= \frac{-2\sigma^2}{r^2} + \frac{1}{r}\left(\frac{\sigma^2}{r^2} + r\right) = 1 - \sigma^2/r^2$$

Condensation transition

[Ref: Kosterlitz, Thouless, Jones, *Spherical Model of a Spin-Glass*, PRL 36 (1976)].

The matrix denoising problem is formulated in terms of the ground state of the energy lansdcape:

$$\mathcal{E}[\vec{s}] = -\frac{1}{2}\sum_{ij} s_i(J_{ij} + r v_i v_j)s_j, \qquad ||\vec{s}||^2 = N = ||\vec{v}||^2, \qquad \hat{J} \sim GOE$$

The behavior of the ground state can be characterized by studying the thermodynamics of the system in the limit $\beta \to \infty$, through the partition function:

$$\mathcal{Z}_\beta = \int_{S_N(\sqrt{N})} d\vec{s} e^{-\beta \mathcal{E}[\vec{s}]}, \qquad S_N(\sqrt{N}) = \left\{\vec{s} : ||\vec{s}||^2 = N\right\}$$

As a function of temperature, this model exhibits a transition at a critical temperature $T_c(r)$, which can be interpreted as a *condensation transition* (like in BEC physics).

Exercise 3. Thermodynamics of the model

(i) Call λ_α ($\lambda_1 \leq \lambda_2 \leq \cdots \lambda_N$) the eigenvalues of $\hat{M} = \hat{J} + \hat{R}$, and $\vec{u}_\alpha$ the corresponding eigenvectors. Call $s_\alpha = \vec{s} \cdot \vec{u}_\alpha$. Show that the partition function can be written as

$$\mathcal{Z}_\beta = \int d\lambda \int \prod_{\alpha=1}^{N} ds_\alpha e^{\frac{\beta}{2}\left[\sum_\alpha \lambda_\alpha s_\alpha^2 - \lambda(\sum_\alpha s_\alpha^2 - N)\right]}$$

(ii) Show that the thermal expectation value of the mode occupations is

$$\langle s_\gamma^2 \rangle = \frac{1}{\mathcal{Z}_\beta} \int d\lambda \int \prod_{\alpha=1}^{N} ds_\alpha \, s_\gamma^2 \, e^{-\frac{\beta}{2}\left[-\sum_\alpha \lambda_\alpha s_\alpha^2 + \lambda(\sum_\alpha s_\alpha^2 - N)\right]} = \frac{1}{\beta(\lambda^* - \lambda_\gamma)}$$

where $\lambda^* > \lambda_\gamma$ for all γ is fixed by the equation

$$\sum_{\gamma=1}^{N} \langle s_\gamma^2 \rangle = N = \sum_{\gamma=1}^{N} \frac{1}{\beta(\lambda^* - \lambda_\gamma)}$$

(iii) The matrix $\hat{M}$ is a spiked GOE. Take $r < r_c = \sigma$. Justify why for large N the equation for λ^* becomes:

$$\beta = g_{\rm sc}(\lambda^*) \qquad \lambda^* > 2\sigma$$

where $g_{\rm sc}(\lambda^*)$ is the Stieltjies transform of the GOE; show that there is a critical temperature $\beta_c = \sigma^{-1}$ and compute the solution λ^* for $\beta < \beta_c$. Show that at β_c, λ^* attains its maximal possible value. Show that at low temperature $\beta > \beta_c$ the equation can be solved assuming *condensation* of the fluctuations in the lowest-energy mode:

$$\frac{1}{N}\langle s_N^2 \rangle = 1 - \frac{1}{\beta\sigma}$$

This condensation transition corresponds also to a transition between a paramagnet at high temperature, and a spin-glass at low temperature.

(iv) Consider now $r > r_c = \sigma$, when the maximal eigenvalue is $\lambda_N = \lambda_{\rm iso} = \frac{\sigma^2}{r} + r$; justify why now the critical temperature is $\beta_c = 1/r$, and a solution of the equation for λ^* (with $\lambda^* > \lambda_\gamma$) exists for $\beta < \beta_c$. Show that for $\beta > \beta_c$ it must hold

$$\frac{1}{N}\langle s_N^2 \rangle = \frac{1}{N}\langle s_{\rm iso}^2 \rangle = 1 - \frac{1}{\beta r}$$

In this regime, the condensation transition coincides with a transition between a paramagnet at high temperature, and a ferromagnet at low temperature.

Exercise 3 - solution

Thermodynamics and the condensation transition

(i) One has:

$$Z_\beta = \int_{S_N(\sqrt{N})} d\vec{s}\; e^{\frac{\beta}{2}\sum_{i,j=1}^{N} s_i \underbrace{(J_{ij} + r\, v_i v_j)}_{M_{ij}} s_j} = \int d\vec{s}\, d\lambda\; e^{\beta/2 \sum_{i,j} s_i M_{ij} s_j - \frac{\beta\lambda}{2}\left(\sum_i s_i^2 - N\right)}$$

implement spherical constraint.

Performing the change of basis, one gets:

$$Z_\beta = \int d\lambda \int \prod_{\alpha=1}^{N} ds_\alpha \; e^{\frac{\beta}{2}\sum_\alpha \lambda_\alpha s_\alpha^2 - \frac{\beta\lambda}{2}\left(\sum_\alpha s_\alpha^2 - N\right)}$$

(ii) The average:

$$\langle s_\gamma^2 \rangle = \frac{1}{\tilde{Z}_\beta}\int d\lambda\, e^{\frac{\beta\lambda N}{2}} \int \prod_{\alpha \neq \gamma} e^{\frac{\beta}{2}\lambda_\alpha s_\alpha^2 - \frac{\beta\lambda}{2} s_\alpha^2}\, ds_\alpha \int ds_\gamma s_\gamma^2\, e^{\frac{\beta}{2}\lambda_\gamma s_\gamma^2 - \frac{\beta\lambda}{2} s_\gamma^2}$$

$$= \frac{1}{\tilde{Z}_\beta}\int d\lambda\; e^{+\frac{\beta\lambda N}{2}} \left(\frac{2\pi}{\beta}\right)^{N/2} \left(\prod_{\alpha=1}^{N} \frac{1}{\lambda - \lambda_\alpha}\right)^{1/2} \frac{1}{\beta(\lambda - \lambda_\gamma)} \qquad (*)$$

Assuming $\lambda > \lambda_\alpha \ \forall \alpha$.

The integral over λ can be performed with a saddle point when $N \gg 1$, optimizing

$$f(\lambda) = \lambda \beta - \frac{1}{N} \sum_{\alpha=1}^{N} \log(\lambda - \lambda_\alpha)$$

$$f'(\lambda)\Big|_{\lambda=\lambda^*} = 0 \Rightarrow \beta = \frac{1}{N} \sum_{\alpha=1}^{N} \frac{1}{\lambda^* - \lambda_\alpha}$$

Plugging this in $(*)$ and simplifying the exponential terms in numerator with those in Z_β, one gets

$$\langle S_\gamma^2 \rangle = \frac{1}{\beta(\lambda^* - \lambda_\gamma)}$$

with λ^* solving:

$$N = \sum_{\gamma=1}^{N} \frac{1}{\beta(\lambda^* - \lambda_\gamma)} = \sum_{\gamma=1}^{N} \langle S_\gamma^2 \rangle$$

(iii) For $r < r_c = \sigma$, there is no isolated eigenvalue and the spectrum of $\hat{M}$ has an eigenvalue density that tends to the semicircle $\rho_{sc}(\lambda)$ when $N \to \infty$. Thus:

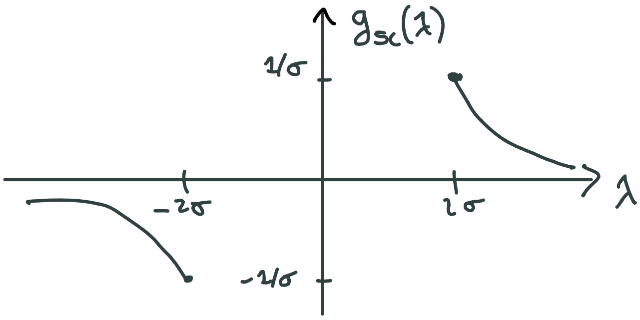
$$\frac{1}{N} \sum_{\gamma} \frac{1}{\lambda^* - \lambda_\gamma} \overset{N \gg 1}{\sim} \int d\lambda \frac{\rho_{sc}(\lambda)}{\lambda^* - \lambda_\gamma} = g_{sc}(\lambda^*)$$

The equation for λ^* becomes:

$$g_{sc}(\lambda^*) = \beta \quad \text{for } \lambda^* > \lambda_N = 2\sigma$$

This can be solved only for $\beta < \beta_c = 1/\sigma$, and in this case

$$\lambda^* = \frac{1}{\beta} + \sigma^2 \beta$$



At $\beta \to \beta_c$, $\lambda^* \to 2\sigma$, that is the boundary value of the domain where the saddle point can be taken. For $\beta > \beta_c$, the saddle point sticks to the boundary: $\lambda^* = 2\sigma$

This is a freezing transition: it signals the transition to a glass phase.

Then the equation for λ^* is solved assuming condensation in the lowest energy mode

$$\langle S_N^2 \rangle \sim \mathcal{O}(N)$$

In particular: $1 = \frac{1}{\beta} \underbrace{g_{sc}(\lambda^* = 2\sigma)}_{1/\sigma} + \frac{1}{N}\langle S_N^2 \rangle$

$\Longrightarrow \frac{1}{N}\langle S_N^2 \rangle = 1 - 1/\sigma\beta.$

(iv) For $r > r_c = \sigma$, $\lambda_N = \lambda_{iso} = \frac{\sigma^2}{r} + r > 2\sigma$. is the maximal value that λ^* can take. Since $g_{sc}(\lambda)$ is monotonically decreasing, the maximal β for which a solution to $\beta = g_{sc}(\lambda^*)$ can be found is the β such that: $\beta = g_{sc}(\lambda_{iso})$

Recalling that $g_{sc}(\lambda_{iso}) = 1/r$, one has $\beta_c = 1/r$.

For $\beta > \beta_c$, it must hold:

$$1 = \frac{1}{\beta} g_{sc}(\lambda_{iso}) + \frac{1}{N}\langle S_N^2 \rangle \Rightarrow \frac{1}{N}\langle S_N^2 \rangle = 1 - 1/\beta r.$$

Phase transitions in temperature: