SDE

$X_t^\varepsilon \in \mathbb{R}^n$; $\mathcal{H}$.

$$dX_t^\varepsilon = b(t, X_t^\varepsilon)\, dt + \sqrt{\varepsilon}\, \sigma(t, X_t^\varepsilon)\, dW_t \qquad X_{t=0}^\varepsilon = x_0$$

det. drift $b \in \mathbb{R}^n$

noise. $\sigma \in \mathbb{R}^{n\times n}$

W_t = Wiener process

Euler-Maruyama.

$$\tilde{X}_0 = x_0$$

$$\tilde{X}_{(k+1)\Delta t} = \tilde{X}_{k\Delta t} + b(k\Delta t, \tilde{X}_{k\Delta t})\Delta t + \sqrt{\varepsilon}\, \sigma(k\Delta t, \tilde{X}_{k\Delta t})(W_{(k+1)\Delta t} - W_{k\Delta t})$$

$$W_{(k+1)\Delta t} - W_{k\Delta t} \overset{d}{=} \sqrt{\Delta t}\, \xi_k$$

X_t continuous, not diff

Hölder 1/2

with: $\xi_k \sim N(0, Id)$ iid.

$$\boxed{\forall T > 0: \quad \mathbb{E}^{x_0}\Big[\sup_{0 \le k \le T/\Delta t} |\tilde{X}_{k\Delta t} - X_{k\Delta t}|\Big] \xrightarrow[\Delta t \downarrow 0]{} 0}$$

Small noise: $\varepsilon \downarrow 0$

1. **LLN** $\forall T > 0$: $\mathbb{E}^{x_0} \sup_{0 \leq t \leq T} |X_t^\varepsilon - X_t| \xrightarrow[\varepsilon \downarrow 0]{} 0$

where: $\dot{X}_t = b(X_t)$ $\quad X_{t=0} = x_0$

$$\mathbb{P}^{x_0}\left(\sup_{0 \leq t \leq T} |X_t^\varepsilon - X_t| > \varepsilon \right) \xrightarrow[\varepsilon \downarrow 0]{} 0$$

2. **CLT:** $X_t^\varepsilon = X_t + \sqrt{\varepsilon}\, \eta_t^\varepsilon \iff \eta_t^\varepsilon = \dfrac{X_t^\varepsilon - X_t}{\sqrt{\varepsilon}}$

derive eq. for $\lim_{\varepsilon \downarrow 0} \eta_t^\varepsilon = \eta_t$

linear eq. with coeff dep. on X_t.

3. LDP Given $f: \mathbb{R}^n \to \mathbb{R}$ & $T > 0$, consider:

$$A_\varepsilon = \mathbb{E}^{x_0}\left[e^{\frac{1}{\varepsilon} f(X_T^\varepsilon)} \right]$$

↳ expect. cond. on $X_{t=0} = x_0$

Use:

$$\tilde{X}_{(k+1)\Delta t} \mid \tilde{X}_{k\Delta t} \sim \mathcal{N}\left(\Delta t\, b(k\Delta t, \tilde{X}_{k\Delta t}),\ \varepsilon \Delta t\, D(k\Delta t, \tilde{X}_{k\Delta t}) \right)$$

$$D(t,x) = \sigma(t,x)\, \sigma^T(t,x)$$

$b(t,x) = b(x)$ $\quad \sigma(t,x) = Id$

$$\tilde{A}_\varepsilon = (2\pi \varepsilon \Delta t)^{-K/2} \int dx_1 \dots dx_K \exp\Big(\frac{1}{\varepsilon} f(x_{k\Delta t}) - \frac{1}{2\varepsilon \Delta t} \sum_{k=1}^{K-1} \left| x_{(k+1)\Delta t} - x_{k\Delta t} - \Delta t\, b(k\Delta t, x_{k\Delta t}) \right|^2 \Big)$$

formally ↓

$$A_\varepsilon \propto \int_{x_{t=0}=x} Dx_t \exp\left(\frac{1}{\varepsilon} f(x_T) - \frac{1}{2\varepsilon} \int_0^T |\dot{x}_t - b(x_t)|^2 dt \right)$$

path integral.

↓ meaning via Girsanov thm.

$\varepsilon \downarrow 0$: **Laplace method:**

$$\lim_{\varepsilon \to 0} \varepsilon \log A_\varepsilon = - \inf_{\substack{x_t \\ x_0 \text{ fixed.}}} \left[\frac{1}{2} \int_0^T |\dot{x}_t - b(x_t)|^2 dt - f(X_T) \right]$$

$$L(x_t, \dot{x}_t) = \frac{1}{2} |\dot{x}_t - b(x_t)|^2$$

1st order opt. cond.:

E.L.:

$$\begin{cases} (\dot{x}_t - b(x_t))^{\circ} = - \nabla b^T(x_t) (\dot{x}_t - b(x_t)) \\ x_0 = 0 \\ \dot{x}_T - b(x_T) = \nabla f(x_T) \end{cases}$$

define: $\theta_t = \dot{x}_t - b(x_t)$

$$\begin{cases} \dot{x}_t = b(x_t) + \theta_t = \partial_x H(x_t, \theta_t) & x_0 = 0 \\ \dot{\theta}_t = -\nabla b^T(x_t)\, \theta_t = -\partial_\theta H(x_t, \theta_t) & \theta_T = \nabla f(x_T) \end{cases}$$

$$\lim_{\varepsilon \downarrow 0} \varepsilon \log A_\varepsilon = -\frac{1}{2}\int_0^T |\theta^*_t|^2\, dt + f(x^*_T).$$

control problem

θ^* = opt. noise

Hamiltonian formulation:

$$\begin{cases} H(x,\theta) = \sup_y \left(y \cdot \theta - L(x,y)\right) \\ L(x,y) = \sup_\theta \left[y \cdot \theta - H(x,\theta)\right] \end{cases}$$

convex in θ

convex in y.

$$\lim_{\varepsilon \downarrow 0} \varepsilon \log A_\varepsilon = -\inf_x \sup_y \left[\int_0^T [\dot{x}_t \cdot \theta_t - H(x_t, \theta_t)]\, dt - f(x_T) \right]$$

Similarly:

$$\lim_{\varepsilon \downarrow 0} \varepsilon \log \mathbb{P}^{x_0}\left(X_T^\varepsilon \in B\right) = -\frac{1}{2} \int_0^T |\theta_t^*|^2 \, dt$$

$$\begin{cases} \dot{x}_t^* = \partial_\theta H(x_t^*, \theta_t^*) & x_0^* = x_0 \\ \dot{\theta}_t^* = -\partial_x H(x_t^*, \theta_t^*) & x_1^* = b^* \in B \end{cases}$$

etc.

ex :

$$dX_t^\varepsilon = -X_t^\varepsilon\, dt + \sqrt{\varepsilon}\, dW_t \quad , \quad X_{t=0}^\varepsilon = 0$$

$$A_\varepsilon = \mathbb{E}^{X_0=0} \exp\left(\frac{\lambda}{\varepsilon} X_T^\varepsilon\right).$$

$$X_t = \sqrt{\varepsilon} \int_0^t e^{-t+s}\, dW_s \overset{d}{=} \sqrt{\frac{\varepsilon}{2}}\, (1-e^{-t})^{1/2}\, \xi$$

$$\xi = N(0,1).$$

$$A_\varepsilon = \exp\left(\frac{\lambda^2}{4\varepsilon^{\cancel{2}}} \cancel{\varepsilon} (1-e^{-T}) \right)$$

$$\boxed{\begin{cases} \dot{X}_t = -X_t + \theta_t & X_{t=0} = 0 \\ \dot{\theta}_t = \theta_t & \theta_{t=T} = \lambda \end{cases}}$$

$$\Rightarrow \boxed{\theta_t^* = \lambda\, e^{t-T}}$$

$$X_t^* = \int_0^t e^{-t+s}\, \theta_s^*\, ds = \lambda \int_0^t e^{-t+2s-T}\, ds$$

$$\boxed{X_t^* = \frac{\lambda}{2} e^{-T} (e^t - e^{-t})}$$

$t-T \to t$

$$\begin{cases} \theta_t^* = \lambda e^t \\ X_t^* = \frac{\lambda}{2} [e^t - e^{-t-2T}] \xrightarrow[T \downarrow -\infty]{} \frac{\lambda}{2} e^t \end{cases}$$

$$\lim_{\varepsilon \downarrow 0} \varepsilon \log A_\varepsilon = - \frac{1}{2}\int_0^T |\theta_t^*|^2 dt + \lambda x_T^*$$

$$= -\frac{\lambda^2}{2}\int_0^T e^{2t-2T} dt + \frac{\lambda^2}{2} e^{-T}(e^T - e^{-T})$$

$$= -\frac{\lambda^2}{4}(1-e^{-2T}) + \frac{\lambda^2}{2}(1-e^{-2T})$$

$$= +\frac{\lambda^2}{4}(1-e^{-2T})$$

SDE: $$dX_t^\varepsilon = b(X_t^\varepsilon)\,dt + \sqrt{\varepsilon}\,dW_t$$

Ito calc: 2 key formula:

1) $$d\phi(t,X_t^\varepsilon) = [\partial_t\phi(t,X_t^\varepsilon) + b(X_t^\varepsilon)\cdot\nabla\phi(t,X_t^\varepsilon) + \tfrac{1}{2}\Delta\phi(t,X_t^\varepsilon)]dt$$
$$+ \sqrt{\varepsilon}\,\nabla\phi(t,X_t^\varepsilon)\cdot dW_t.$$

Ito formula

$$\equiv [\partial_t\phi + (L\phi)]dt + \sqrt{\varepsilon}\,\nabla\phi\cdot dW_t$$

$$(Lf)(x) = b(x)\cdot\nabla f(x) + \tfrac{1}{2}\Delta f(x)$$
$$= \lim_{t\downarrow 0}\frac{1}{t}\mathbb{E}^x[f(X_t^\varepsilon) - f(x)].$$

generator.

2) $$\mathbb{E}\int_0^T \nabla\phi(t,X_t^\varepsilon)\cdot dW_t = 0$$

$$\mathbb{E}\left|\int_0^T \nabla\phi(t,X_t^\varepsilon)\cdot dW_t\right|^2 = \int_0^T \mathbb{E}|\nabla\phi(t,X_t^\varepsilon)|^2 dt.$$

Ito isometry.

Backward Kolmogorov eq.

$$\partial_t u(t,x) + (Lu)(t,x) = 0 \qquad u(T,x) = F(x).$$

$$\Rightarrow \quad u(0,x) = \mathbb{E}^x F(X_T^\varepsilon)$$

PROOF:

$$du(t,X_t^\varepsilon) = \overbrace{\partial_t u(t,X_t^\varepsilon) + (Lu)(t,X_t^\varepsilon)}^{=0}\, dt + \sqrt{\varepsilon}\, \nabla u(t,X_t^\varepsilon)\, dW_t$$

$$= \sqrt{\varepsilon}\, \nabla u(t,X_t^\varepsilon)\cdot dW_t$$

$$F(X_T^\varepsilon) = u(T,X_T^\varepsilon) = u(0,X_0^\varepsilon) + \sqrt{\varepsilon}\int_0^T \nabla u(t,X_t^\varepsilon)\cdot dW_t$$

$\mathbb{E}^x$

$$\mathbb{E}^x F(X_T^\varepsilon) = \mathbb{E}^x u(0,X_0^\varepsilon) = u(0,x).$$

scaling things

$$\partial_t u^{\varepsilon}(t,x) + b(x)\cdot\nabla u^{\varepsilon}(t,x) + \frac{\varepsilon}{2}\Delta u^{\varepsilon}(t,x) = 0.$$

$$u^{\varepsilon}(t=T,x) = e^{\frac{1}{\varepsilon}f(x)}$$

let $u^{\varepsilon}(t,x) = e^{\frac{1}{\varepsilon}\sigma^{\varepsilon}(t,x)}$

$$\frac{1}{\varepsilon}\partial_t \sigma^{\varepsilon} + \frac{1}{\varepsilon}b\cdot\nabla\sigma^{\varepsilon} + \frac{1}{2}\Delta\sigma^{\varepsilon} + \frac{1}{2\varepsilon}|\nabla\sigma^{\varepsilon}|^2 = 0$$

$$\sigma^{\varepsilon}(t=T,x) = f(x).$$

$\downarrow$

$\sigma^{\varepsilon} \xrightarrow[\varepsilon\downarrow 0]{} \sigma$

$$\partial_t\sigma + \underbrace{b\cdot\nabla\sigma + \frac{1}{2}|\nabla\sigma|^2}_{H(x,\nabla\sigma)} = 0. \qquad \sigma(t=T,x) = f(x)$$

HJE.

$$\sigma(0,x) = \lim_{\varepsilon\downarrow 0}\sigma^{\varepsilon}(0,x) = \lim_{\varepsilon\downarrow 0}\varepsilon\log u^{\varepsilon}(0,x) = \lim_{\varepsilon\downarrow 0}\varepsilon\log \mathbb{E}\, e^{\frac{1}{\varepsilon}f(X_t)}$$

let, $$\begin{cases} \dot{x}_t = b(X_t) + \theta_t & X_{t=0} = 0 \\ \dot{\theta}_t = -(\nabla b)^T(X_t) & \theta_{t=T} = \nabla f(X_{t=T}) \end{cases}$$

$\Downarrow$ $\quad\Longrightarrow.\ \theta_t = \nabla S(t, X_t)$

1) $\theta_t = \nabla S(t, X_t)$. $\qquad \theta_T = \nabla f(X_{t=T}) = \nabla S(T, X_T)$.

$$\partial_t S(t,x) + b(t,x)\cdot\nabla S(t,x) + \frac{1}{2}|\nabla S(t,x)|^2 = 0$$

$\nabla \downarrow$

$$\partial_t \nabla S(t,x) + (\nabla b)^T\cdot\nabla S(t,x) + b(t,x)\cdot\nabla\nabla S + \nabla S\cdot\nabla\nabla S = 0$$

$$\frac{d}{dt}\nabla S(t,X_t) = \partial_t \nabla S(t,X_t) + b\cdot\nabla\nabla S + \theta_t\cdot\nabla\nabla S$$

$$= -(\nabla b)^T \nabla S(t,X_t) \underbrace{- \nabla S\cdot\nabla\nabla S + \theta_t\cdot\nabla\nabla S}_{=0 \text{ if } \theta_t = \nabla S(t,X_t)}$$

2) $$\frac{d}{dt} S(t,X_t) = \partial_t S + \dot{x}_t\cdot\nabla S = \partial_t S + b\cdot\nabla S + \theta_t\cdot\nabla S$$

$$= \frac{1}{2}|\theta_t|^2$$

$$\sigma(T, X_T) - \sigma(0, X_0) = \frac{1}{2}\int_0^T |\theta_t|^2 dt .$$

$$\underset{f(X_T)}{\parallel} \qquad \sigma(0,x_0) = \lim_{\varepsilon \downarrow 0} \varepsilon \log \mathbb{E}^{x_0} \exp\left(\frac{1}{\varepsilon} f(X_T)\right)$$

$$\Downarrow$$

$$\lim_{\varepsilon \downarrow 0} \varepsilon \log \mathbb{E}^{x_0} \exp\left(\frac{1}{\varepsilon} f(X_T^{\varepsilon})\right)$$

$$= f(x_T) - \frac{1}{2}\int_0^T |\theta_t|^2 dt .$$

as before.

$$\partial_t \sigma + H(x, \nabla \sigma) = 0 \qquad \sigma(T, x) = f(x).$$

$$\begin{cases} \dot{x}_t = \partial_\theta H(x_t, \theta_t) \\ \dot{\theta}_t = -\partial_x H(x_t, \theta_t) \end{cases}$$

Girsanov formula:

$$dX_t^\varepsilon = b(X_t^\varepsilon)\,dt + \sqrt{\varepsilon}\,dW_t$$

$$dY_t^\varepsilon = \dot{x}_t^*\,dt + \sqrt{\varepsilon}\,dW_t$$

$$Y_t^\varepsilon = x_t^* + \sqrt{\varepsilon}\,W_t .$$

sol. to $\dot{x}_t^* = \partial_x H(x_t^*, \theta_t^*)$

$\dot{\theta}_t^* = -\partial_\theta H(x_t^*, \theta_t^*)$

$+ BC$

$$\mathbb{E}^{x_0}\left[e^{\frac{1}{\varepsilon} f(X_t^\varepsilon)}\right] = \mathbb{E}^{x_0}\left[e^{\frac{1}{\varepsilon} f(Y_t^\varepsilon)} M_T .\right]$$

$$M_T = \exp\left(-\frac{1}{2\varepsilon}\int_0^T |\dot{x}_t^* - b(Y_t^\varepsilon)|^2 dt + \frac{1}{\sqrt{\varepsilon}}\int_0^T (\dot{x}_t^* - b(Y_t^\varepsilon))\cdot dW_t\right).$$

$\mathbb{E}\, M_T = 0.$

$$M_T = \exp\left(-\frac{1}{2\varepsilon}\int_0^T |\dot{x}_t^* - b(x_t^*)|^2 dt + O(1)\right).$$

cf.
$$-\frac{1}{2}|\dot{X}-b|^2+\frac{1}{2}|\dot{X}-\dot{X}_*|^2$$

$$= -(\dot{X}_*-b)\underset{\dot{X}_*+\sqrt{\varepsilon}\,\dot{W}}{\underset{\shortparallel}{\dot{X}}} - \frac{1}{2}|b|^2+\frac{1}{2}|\dot{X}_*|^2$$

$$= -|\dot{X}_*|^2 + b\cdot X_* - \frac{1}{2}|b|^2+\frac{1}{2}|\dot{X}_*|^2$$
$$+\sqrt{\varepsilon}(\dot{X}_*-b)\dot{W}$$

$$= -\frac{1}{2}|\dot{X}^*-b|^2+\sqrt{\varepsilon}(\dot{X}_*-b)\dot{W}$$