PastedGraphic-1.png

LDP @ LONG TIMES

$$dX_t = b(X_t)\,dt + \sigma\, dW_t$$

$X_t \in \mathbb{R}^d$ $\quad \sigma \in \mathbb{R}$

$W_t \in \mathbb{R}^d$

$$I_T = \mathbb{E}^x\big(\underbrace{\exp\big(\textstyle\int_0^T f(X_t)\,dt\big)}\big) \equiv A_T$$

$$\Rightarrow \quad dA_t = f(X_t)\, A_t\, dt. \qquad + \; g(X_t)\, dW_t$$

- $\rho_t(x,a)$ = PDF of (X_t, A_t)

$$\Rightarrow \quad I_t = \int_{\mathbb{R}} da \int_{\mathbb{R}^d} dx \; e^{a} \rho_t(x,a)$$

$$\boxed{\partial_t \rho_t = -\nabla\cdot\big(b(x)\rho_t\big) + \tfrac{1}{2} D\, \Delta \rho_t - f(x)\,\partial_a\big(a\rho_t\big)}$$

$$+ \tfrac{1}{2} |g(x)|^2 \partial_a^2 \rho_t$$

$$+ \sigma \partial_a \nabla_x\cdot\big(g(x)\, a \rho_t\big)$$

- $\hat{\rho}_t(x) = \int_{\mathbb{R}} e^{a} \rho_t(x,a)\, da$

$$\Rightarrow \quad I_t = \int_{\mathbb{R}^d} \hat{\rho}_t(x)\, dx.$$

not conserved.

$$\boxed{\partial_t \hat{\rho}_t = -\nabla\cdot\big(b(x)\hat{\rho}_t\big) + \tfrac{1}{2} D \Delta \hat{\rho}_t + f(x)\,\hat{\rho}_t}$$

$$+ \tfrac{1}{2} |g(x)|^2 \hat{\rho}_t - \sigma \nabla\cdot\big(g(x)\,\hat{\rho}_t\big)$$

$$\rho_t(x) = \frac{\hat{\rho}_t(x)}{\int_{\mathbb{R}^d} \hat{\rho}_t(x)\,dx}$$

$$\partial_t \rho_t = -\nabla\cdot(b\,\rho_t) + \tfrac{1}{2} D\,\Delta\rho_t + f(x)\,\rho_t - \rho_t \left(\frac{\frac{d}{dt}\int \hat{\rho}_t(x)\,dx}{\int \hat{\rho}_t(x)\,dx} \right)$$

$$\| $$

$$\int f(x)\,\rho_t(x)\,dx \equiv \lambda_t$$

$$\downarrow$$

Lag. mult. enforcing normalization

note that:

$$\frac{\frac{d}{dt}\int \hat{\rho}_t(x)\,dx}{\int \hat{\rho}_t(x)\,dx} = \frac{d}{dt}\log\int \hat{\rho}_t(x)\,dx = \frac{d}{dt}\log I_t .$$

i.e. $I_t = \exp\left(\int_0^t \lambda_s\,ds\right)$

Summarizing:

$$I_t = \mathbb{E}^x \exp\left(\int_0^t f(X_s)\, ds \right).$$

$$\partial_t \rho_t(x) = -\nabla\cdot(b(x)\rho_t) + \frac{1}{2} D \Delta \rho_t + f(x)\rho_t - \lambda_t \rho_t$$

$$\lambda_t : \frac{d}{dt}\int \rho_t(x)\, dx = 0$$

$$\& \quad I_t = \exp\left(\int_0^t \lambda_s\, ds \right).$$

let $t \uparrow \infty$ if $\rho_t \to \rho_*$ then $\lambda_t \to \lambda_*$

$$I_t = \exp(t \lambda_*)$$

$$\Rightarrow \lim_{t\uparrow\infty} \frac{1}{t} \log I_t = \lambda_*.$$

LDP.

$$-\nabla\cdot(b(x)\rho_*) + \frac{1}{2} D \Delta \rho_* + f(x)\rho_* = \lambda_* \rho_*$$

eigenvalue problem.

NB: adjoint: $b(x)\cdot \nabla w_* + \frac{1}{2} D \Delta w_* + f(x) w_* = \lambda_* \rho_*$

NB : Simulations

$$\partial_t \rho_t(x) = -\nabla\cdot(b(x)\rho_t) + \frac{1}{2}D\Delta\rho_t + f(x)\rho_t - \lambda_t\rho_t$$

$$\lambda_t : \frac{d}{dt}\int \rho_t(x)\,dx = 0$$

$$\rightsquigarrow \quad dX^i_t = b(X^i_t)\,dt + \sigma\,dW^i_t$$

+ birth/death with rate $f(X^i_t) - \lambda^i_t$

particles preserved.

Small noise

$$dX_t = b(X_t)\,dt + \sqrt{\varepsilon}\,\sigma\, dW_t$$

$$I_T = \mathbb{E}^x \exp\left(\frac{1}{\varepsilon}\int_0^T f(X_t)\right)$$

$$\lim_{\varepsilon\downarrow 0}\ \lim_{t\uparrow\infty} \frac{\varepsilon}{t}\log I_t = \lambda_* .$$ LDP.

! order of lim. matters

$$-\nabla\cdot\left(b(x)\,\rho_*\right) + \frac{\varepsilon}{2}D\Delta\rho_* + \frac{1}{\varepsilon}f(x)\,\rho_* - \frac{\lambda_*}{\varepsilon}\rho_* = 0$$

$\partial_t \rho$

eigenvalue problem.

$$\lambda_* = \int f(x)\rho_*(x)\,dx$$

$$\int \rho_*(x)\,dx = 1.$$

$$\rightarrow \quad \rho_* = e^{-\frac{1}{\varepsilon}\phi_*}$$

$$\boxed{b(x)\cdot\nabla\phi_* + \frac{1}{2}D|\nabla\phi_*|^2 + f(x) - \lambda_* = 0} \quad -\partial_t\phi$$

Hamiltonian: $H = b\cdot\theta + \frac{1}{2}D|\theta|^2 + f - \lambda_*$

Slow / Fast Systems

$$\begin{cases} dX_t^\varepsilon = a(X_t^\varepsilon, Y_t^\varepsilon)\,dt & \text{slow} \\ dY_t^\varepsilon = \frac{1}{\varepsilon}\, b(X_t^\varepsilon, Y_t^\varepsilon)\,dt + \frac{\sigma}{\sqrt{\varepsilon}}\, dW_t \, . & \text{fast.} \end{cases}$$

- $$A^\varepsilon = \mathbb{E}^{x,y} \exp\left(\frac{1}{\varepsilon} f(X_t^\varepsilon) \right)$$

- $$\partial_t u^\varepsilon + L^\varepsilon u^\varepsilon = 0 \qquad u^\varepsilon(t=T) = e^{\frac{1}{\varepsilon} f(x)}$$

$$L^\varepsilon u^\varepsilon = \underbrace{a(x,y) \cdot \partial_x u^\varepsilon}_{L_0 u^\varepsilon} + \underbrace{\frac{1}{\varepsilon}\left[b(x,y)\, \partial_y u^\varepsilon + \frac{\sigma^2}{2} \partial_y^2 u^\varepsilon \right]}_{\frac{1}{\varepsilon} L_1 u^\varepsilon}$$

$$\Longrightarrow \quad u^\varepsilon(t=0,x,y) = A^\varepsilon$$

$$u^\varepsilon(t,x,y) = [w(t,x,y) + O(\varepsilon)]\, e^{\frac{1}{\varepsilon} v(t,x,y)}.$$

$$\partial_t u^\varepsilon = \left[\partial_t w + \frac{1}{\varepsilon} w\, \partial_t v + O(\varepsilon)\right] e^{\frac{1}{\varepsilon} v}$$

$$L^0 u^\varepsilon = \left[a\cdot\partial_x w + \frac{1}{\varepsilon} w\, a\cdot\partial_x v\right] e^{\frac{1}{\varepsilon} v}$$

$$\frac{1}{\varepsilon} L^1 u^\varepsilon = \Big[\frac{1}{\varepsilon} b\cdot\partial_y w + \frac{1}{\varepsilon^2} w\, b\cdot\partial_y v$$

$O(\varepsilon^{-1})$

$$+ \frac{\sigma^2}{2\varepsilon}\partial_y^2 w + \frac{\sigma^2}{\varepsilon^2}\partial_y w\cdot\partial_y v + \frac{\sigma^2}{2\varepsilon^2} w\,\partial_y^2 v$$

$O(\varepsilon^{-2})$

$O(\varepsilon^{-3})$

$$+ \frac{\sigma^2}{2\varepsilon^3} w\,|\partial_y v|^2\Big]\, e^{\frac{1}{\varepsilon} v}$$

Since:

$$\partial_y^2\left(w\, e^{\frac{1}{\varepsilon} v}\right) = \partial_y\cdot\left[\partial_y w\, e^{\frac{1}{\varepsilon} v} + \frac{1}{\varepsilon} w\,\partial_y v\, e^{\frac{1}{\varepsilon} v}\right]$$

$$= \Big[\partial_y^2 w + \frac{1}{\varepsilon}\partial_y w\cdot\partial_y v$$

$$+ \frac{1}{\varepsilon}\partial_y w\cdot\partial_y v + \frac{1}{\varepsilon} w\,\partial_y^2 v + \frac{1}{\varepsilon^2} w\,|\partial_y v|^2\Big]\, e^{\frac{1}{\varepsilon} v}$$

$O(\varepsilon^{-3})$: $\quad \partial_y v = 0 \quad \Rightarrow \quad v(t,x,y) = v(t,x)$.

$O(\varepsilon^{-2})$: $\quad (\cdots)\cdot \partial_y v = 0$

$O(\varepsilon^{-1})$: $\quad w\,\partial_t v + w\, a(x,y)\cdot \partial_x v + b(x,y)\cdot \partial_y w + \frac{\sigma^2}{2}\partial_y^2 w = 0$

eigenvalue problem in y with x = parameter.

we know from before that:

$$d Y^x_\tau = b(x, Y^x_\tau)\, d\tau + \sigma\, dW_\tau$$

$$\partial_t v = \lim_{T\uparrow\infty} \frac{1}{T} \log \mathbb{E} \exp\left(\int_0^T a(x, Y^x_\tau)\cdot \partial_x v \, d\tau \right)$$

$$\equiv H(x, \partial_x v)$$

HJE

Assuming limit exist!

NB: $\underline{\text{LLN}}$ $\quad \partial_\theta H(x, \theta = 0) = \lim_{T\uparrow\infty} \frac{1}{T} \mathbb{E} \int_0^T a(x, Y^x_\tau)\, d\tau$

ex:

$$\begin{cases} dX_t^\varepsilon = |Y_t^\varepsilon|^2\, dt - \lambda X_t^\varepsilon\, dt \\ dY_t^\varepsilon = -\frac{1}{\varepsilon}\gamma(X_t^\varepsilon)\, dt + \frac{\sigma}{\sqrt{\varepsilon}}\, dW_t \end{cases} \qquad \gamma(x) > 0.$$

$$\boxed{w\,\partial_t v + w\, a(x,y)\cdot\partial_x v + b(x,y)\cdot\partial_y w + \frac{\sigma^2}{2}\partial_y^2 w = 0}$$

$$\rightsquigarrow \quad y^2\theta\, w - \gamma(x)\, y\, \partial_y w + \frac{\sigma^2}{2}\partial_y^2 w = -(H + \lambda x\theta)\, w$$

$$\Rightarrow \quad \boxed{H(x,\theta) = \frac{1}{2}\left[\gamma(x) - \sqrt{\gamma(x)^2 - 2\sigma^2\theta}\,\right] - \lambda x\theta \qquad \text{if } \theta \leq \frac{\gamma(x)}{2\sigma^2}}$$

Since: $w = e^{by^2/2} \Rightarrow \partial_y w = by\, w$

$$\partial_y^2 w = b w + (by)^2 w$$

$$\Rightarrow \quad y^2\theta - \gamma b y^2 + \frac{\sigma^2}{2}[b + b^2y^2] = -H$$

$O(y^2)$: $\theta - \gamma b + \frac{\sigma^2}{2}b^2 = 0 \Rightarrow b = \frac{-\gamma \pm \sqrt{\gamma^2 - 2\theta\sigma^2}}{\sigma^2}$

$O(y^0)$: $H = -\frac{\sigma^2 b}{2}$

LLN: $\partial_\theta H(x,\theta=0) = -\lambda x + \frac{1}{2}\frac{\sigma^2}{\sqrt{\gamma^2(x)}} = -\lambda x + \frac{\sigma^2}{2\gamma(x)}$

2D Variant:

$$\begin{cases} dX_i^\varepsilon = -\beta_i X_i^\varepsilon \, dt + \kappa \sum_{j=1}^{2} D_{ij} X_j^\varepsilon \, dt + |Y_i^\varepsilon|^2 \, dt \\ dY_i^\varepsilon = -\frac{1}{\varepsilon} \gamma(X_i^\varepsilon) Y_i^\varepsilon \, dt + \frac{1}{\sqrt{\varepsilon}} \sigma \, dW_i \end{cases}$$

$D = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$

$\beta_1 = 0.6$, $\beta_2 = 0.3$

$\sigma = \sqrt{10}$

$\kappa = 0.2$

$\gamma(x) = (x-5)^2 + 10$