Large Deviation Functionals from Integrable Many-body Systems

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2024 "Hydrodynamic Scales..." WS

"mostly" Toda chain

1. Toda lattice

integrable

Calogero fluid $\quad H_{Ca} = \sum_j \frac{1}{2} p_j^2 + \sum_{i<j} \frac{1}{\sinh^2(q_i - q_j)}$ on $\mathbb{R}$

- low density, n.n. only

$\Rightarrow$ Toda $\quad H = \sum_j \left\{ \frac{1}{2} p_j^2 + e^{-(q_{j+1} - q_j)} \right\}$

$q_j \in \mathbb{R}$ | dense phase

$$\ddot{q}_j = e^{-(q_j - q_{j-1})} - e^{-(q_{j+1} - q_j)}$$

discrete, nonlinear wave equation

$j = 1, \dots, N$

closed chain $\quad q_{j+N} = q_j + \ell$

free volume $\frac{\ell}{N} = \nu$

|| hydrodynamics ||

open chain

$$H = \sum_{j=1}^{N} \frac{1}{2} p_j^2 + \sum_{j=1}^{N-1} e^{-(q_{j+1} - q_j)}$$

scattering

2. Integrability

N particles $\Rightarrow$ N conservation laws

local density

Example: $\sum_{j=1}^{n} p_j^2$ ✓, $\left(\sum_{j=1}^{n} p_j^2\right)^2$ NO

N. Shiraishi 2024

- quantum spin $\frac{1}{2}$ chain on $\mathbb{Z}$, n.n.n. interactions

local conservation laws: 1 or ∞

Toda Flaschka variables:

$$p_j, \quad a_j = e^{-(q_{j+1}-q_j)/2}$$

$$\frac{d}{dt} a_j = \frac{1}{2} a_j (p_{j+1} - p_j)$$

$$\frac{d}{dt} p_j = a_{j-1}^2 - a_j^2$$

closed

$$L = \begin{pmatrix} p_1 & a_1 & & a_N \\ a_1 & \ddots & \ddots & 0 \\ & 0 & \ddots & \ddots \\ a_N & & \ddots & p_N \end{pmatrix}$$

tri-diagonal $N \times N$

(BIG) Lax matrix

$$\frac{d}{dt} L = [B, L]$$

$\rightsquigarrow$ eigenvalues $\lambda_1, \dots, \lambda_N$ are conserved! NOT local

local $\qquad Q^{[n]} = \mathrm{tr}\, L^n = \sum_{j=1}^{N} (L^n)_{jj} = Q_j^{[n]}$ $\qquad$ local

RW expansion

$n = 1, 2, \dots$ in addition

$$Q_j^{[0]} = \log a_j^2$$

momentum $\quad \mathrm{tr}\, L$

energy $\quad \mathrm{tr}\, L^2$

3. GGE (generalized Gibbs)

long time limit $\qquad \frac{1}{Z} e^{-\sum_{n\geq 1} \mu_n Q^{[n]}} e^{-P Q^{[0]}} d^N q\, d^N p$ $\quad$ grand canonical

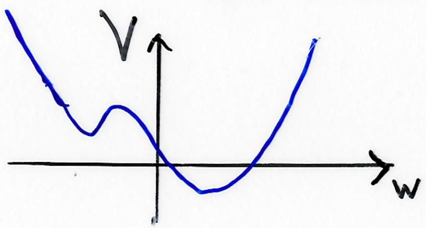
pressure $\quad P > 0$

Boltzmann weight $\qquad \sum_{n\geq 1}^{\infty} \mu_n Q^{[n]} = \sum_{n=1}^{\infty} \mu_n \,\mathrm{tr}\, L^n = \mathrm{tr}\, V(L)$

initial conditions

confining potential

$$V(w) = \sum_{n=1}^{\infty} \mu_n w^n$$



$$\rightsquigarrow \quad \frac{1}{Z} e^{-\mathrm{tr}\, V(L)} \prod_{j=1}^{N} \frac{1}{a_j} (a_j)^{2P} da_j\, dp_j$$

independent of N

$$a_j > 0, \quad p_j \in \mathbb{R}$$

4. Density of states DOS

- L_N is random under GGE

central object DOS $\rho^{(N)}(w) = \frac{1}{N}\sum_{j=1}^{N} \delta(\lambda_j - w)$ random

law of large numbers, large deviations

thermal GGE

$V(w) = \beta w^2$

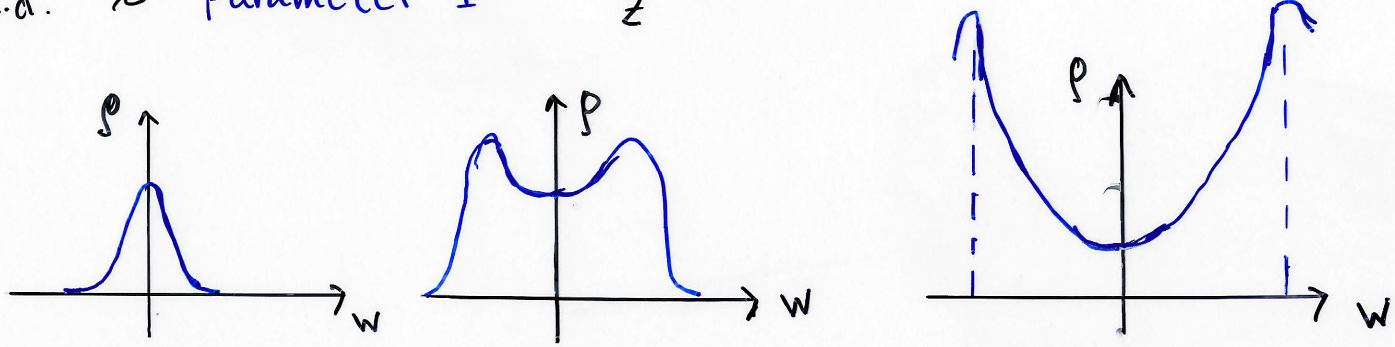
- since 2022 Guionnet, Mazzuca, Memin

$\{p_j, j=1,\dots,N\}$ i.i.d. Gaussian $\langle p_j^2 \rangle = \frac{1}{\beta}$

$\{a_j, j=1,\dots,N\}$ i.i.d. x parameter P $\frac{1}{Z}\, x^{2P-1} e^{-x^2/2}$

$\frac{1}{\sqrt{1-w^2}}$

exact solution $\lim_{N\to\infty} \rho^{(N)}(w)$



$P \ll 1$ $\quad$ $P \cong 1$ $\quad$ $P \gg 1$

Opper 1975, Guionnet 2012

5. Large deviations

fixed P, V, $N \to \infty$

replaced by $\nu > 0$ free volume

$$T(w,w') = \log|w-w'|^2$$

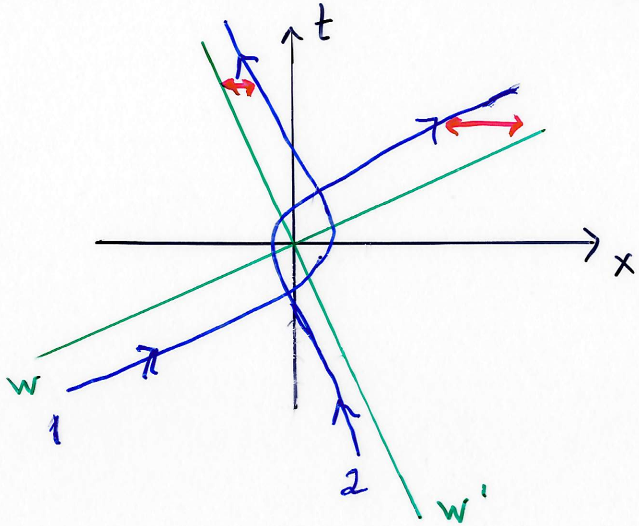
$\rho \geq 0$, $\langle \rho \rangle = 1$, $\rho_s = \nu + T\rho$ $\qquad \langle V, \rho \rangle = \int dw\, \rho(w) V(w)$

$$\left\| \; \mathcal{F}(\rho) = \langle V, \rho \rangle + S(\rho | \rho_s) \; \right\|$$

unique minimizer ρ^* limit DOS

relative entropy

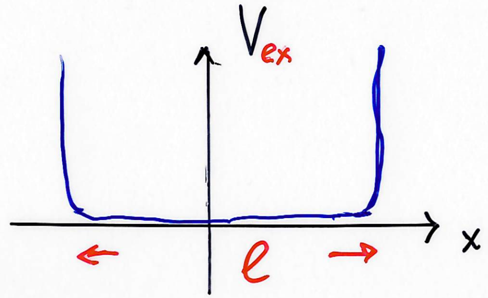
$\Rightarrow$ T is the 2-particle scattering shift $\Leftarrow$ (Bethe ansatz)



quasi-particle velocity w, w'

How? scattering coordinates $p_j(t) = \lambda_j t + \phi_j \qquad t \to \infty$

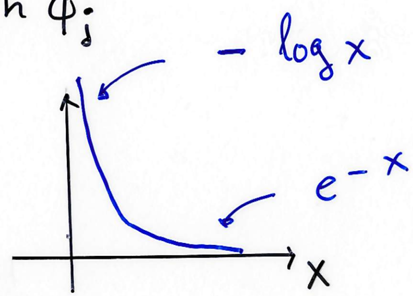
- canonical $\quad d^N q \, d^N p = d^N \lambda \, d^N \phi \quad$ explicit



- $V_{ex}(q)$, box potential

$$\frac{1}{Z} e^{-\operatorname{tr} V(L) - V_{ex}(\vec{q})} d^N q \, d^N p = \frac{1}{Z} e^{-\sum_{j=1}^{N} V(\lambda_j) - V_{ex}(\vec{q})} d^N \lambda \, d^N \phi$$

special choice $\quad V_{ex}(\vec{q}) = e^{-\ell/2}\left(e^{q_1} + e^{-q_N}\right) = 2 \sum_{j=1}^{N} e^{-\ell/2} Y_j \cosh \phi_j$



integral $\; d^N \phi \qquad K_0(x) = \int_0^\infty ds \, e^{-x \cosh s}$

$$\frac{1}{Z} e^{-\sum_{j=1}^{N} V(\lambda_j)} \prod_{j=1}^{N} K_0\left(e^{-\ell/2} Y_j\right) \qquad \log Y_j = \sum_{\substack{i=1 \\ i \neq j}}^{N} \log |\lambda_j - \lambda_i|^2$$

|| mean field ||

6. Log gas

based on Dumitriu, Edelman (2002)

open chain, linear pressure ramp

$$\prod_{j=1}^{N} \frac{1}{a_j} (a_j)^{2\underbrace{(j/N)}_{}P}$$

slope $\frac{1}{N}$

⇝ eigenvalues

$$\frac{1}{Z} e^{-\sum_{j=1}^{N} V(\lambda_j) + \frac{P}{N} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \log|\lambda_i - \lambda_j|}$$

⇒ large deviations $\tilde{\mathcal{F}}(\rho) = \langle V, \rho \rangle - S(\rho) - P \langle \rho, T\rho \rangle$

$$\rho \geq 0 \quad , \quad \langle \rho \rangle = 1$$

Toda

$$\tilde{\mathcal{F}} \neq \mathcal{F}$$

minimizer ρ_P

Toda

BUT $\partial_P (P\, \rho_P) = \rho^*$

7. Quantum approach

to be explored

Calogero $\log|w|^2 \Longrightarrow \log\left(1+\frac{1}{w^2}\right) \geq 0$

in general: $\Phi : \quad \Phi' > 0\,, \quad \Phi(0) = 0$ $\quad \Phi$ phase, Φ' scattering shift

- classical

mean field

input: $u_1 < \dots < u_N\,, \quad d^N u$

output: $\{\lambda_1, \dots, \lambda_N\}$ solution of $u_j = \nu N \lambda_j + \sum_{i=1}^{N} \Phi(\lambda_j - \lambda_i)$

GGE $\frac{1}{Z} e^{-\sum_{j=1}^{N} V(\lambda_j)} d^N u$ DOS $\rho(w) = \frac{1}{N} \sum_{j=1}^{N} \delta(w - \lambda_j)$

$T(\lambda, \lambda') = \Phi'(\lambda - \lambda')$

$\rho_s = \nu + T\rho$ large deviations $\mathcal{F}(\rho) = \langle V, \rho \rangle + S(\rho | \rho_s)$

$\rho \geq 0\,, \quad \langle \rho \rangle = 1$ Jacobian $\left\{\frac{\partial u_i}{\partial \lambda_j}\right\}_{i,j=1,\dots,N}$

■ quantum $\qquad$ δ-Bose gas $\qquad \Phi'(w) = \frac{2c}{c^2+w^2}$

input: $\qquad I_1 < \dots < I_N \qquad \| I_j \in \mathbb{Z} \| \qquad$ counting measure

output: $\{k_1, \dots, k_N\} \qquad I_j = \nu N k_j + \sum_{i=1}^{N} \Phi(\lambda_j - \lambda_i)$

GGE, DOS as before

⇒ large deviations $\qquad$ Yang-Yang free energy functional $\qquad (V(w) = w^2)$

1969

$\rho + \rho_h = \rho_s, \qquad \rho_s = \nu + T\rho$

$$\mathcal{F}_{YY}(\rho) = \langle V, \rho \rangle + \int dw \left(\rho \log \rho + \rho_h \log \rho_h - \rho_s \log \rho_s \right)$$