Stochastic Resetting and Large Deviations

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Contents

These are the lecture notes from two lectures given at the 2024 summer school on Large Deviations held at Les Houches School of Physics. Reference [13] is a review that is hopefully accessible and pedagogical. It also contains a comprehensive bibliography up to 2020.

1 Motivations

The processes of stochastic resetting can be motivated by a simple everyday experience of searching for misplaced keys. During the search process, we periodically reset back to the place where the keys usually should be and restart the search. It turns out that such resets expedites the completion time of the search process[1, 13]. A few other interesting motivations for studying stochastic resetting are

(i) **Searching for a target:** It is generally observed in search processes that mixing local moves with long range moves improves the search process. This is termed as an "Intermittent search process". In the case of searches with stochastic resetting, the local moves are diffusion and the long range moves are the resettings.

(ii) **Expediting completion times:** It has also been observed that restarting a complex task, for example a complicated chemical reaction [9], such as,

$$\mathrm{E} + \mathrm{S} \underset{k_{\mathrm{off}}}{\overset{k_{\mathrm{on}}}{\rightleftharpoons}} \mathrm{ES} \xrightarrow{\ \tau\ } \mathrm{P}.$$

The above equation represents a reaction where E is the enzyme, and S is the substrate which combines to form ES and then takes a random time τ (which could have long tails in its distribution) to produce the product P. k_{off} can be thought of as the resetting rate, and it turns out that having such a rate enhances the completion time of the process by cutting off the long tails in the distribution of τ.

(iii) **Generating non-equilibrium steady states (NESS):** Resetting the system to its initial state prevents the system from relaxing to its equilibrium state. The resettings generate a probability current back to the initial condition, which results in a simple but non-trivial NESS.

2 Preliminaries: Diffusion

Consider a usual (over-damped) diffusing particle starting at $x(t=0) = x_0$, which can be represented as,

$$x_{t+\Delta t} = x_t + \xi \ , \tag{1}$$

where ξ is a Gaussian white noise given by the distribution

$$P(\xi) = \frac{1}{\sqrt{4\pi D\Delta t}} e^{-\frac{\xi^2}{4D\Delta t}} \ .$$

In the continuum limit (1) leads to diffusion equation

$$\frac{\partial}{\partial t} P(x,t|x_0) = D\frac{\partial^2}{\partial x^2} P(x,t|x_0) \tag{2}$$

with the initial condition

$$P(x, t=0|x_0) = \delta(x - x_0) \ ,$$

whose solution is

$$P(x,t|x_0) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{(x-x_0)^2}{4Dt}} \ . \tag{3}$$

We can also a backward equation, which for diffusion looks the same as the forward equation, where the spatial variable is now x_0

$$\frac{\partial}{\partial t}P(x,t|x_0) = D\frac{\partial^2}{\partial x_0^2}P(x,t|x_0) \tag{4}$$

We'll review later how to derive the forward and backward diffusion equations from (1).

The usual method to derive the solution (3) is through the Fourier transform. Perhaps the simplest method is just to assume a scaling solution $P = t^{-1/2}g(x/t^{1/2})$. However here, we derive the solution by Laplace transform and discuss the subsequent inversion using a Bromwich contour to obtain the diffusion propagator. The reasons for choosing this method will become apparent when we move on to more complicated problems.

We define the Laplace transform with respect to time for the PDF as,

$$\widetilde{P}(x,s|x_0) = \int_0^\infty dt\, e^{-st}P(x,t|x_0)\ .$$

By making use of integration by parts, we can rewrite the Laplace transform of the time derivative of $P(x,t|x_0)$ as

$$\begin{aligned}\int_0^\infty e^{-st}\frac{\partial P}{\partial t}dt &= [e^{-st}P]_0^\infty + s\int_0^\infty e^{-st}Pdt \\ &= -\delta(x-x_0) + s\widetilde{P}\ .\end{aligned}$$

The space derivative goes through unaltered as the Laplace transform only acts on the time variable. Hence we obtain the Laplace transform of the diffusion equation as

$$D\frac{\partial^2}{\partial x^2}\widetilde{P}(x,s|x_0) - s\widetilde{P}(x,s|x_0) = -\delta(x-x_0) \tag{5}$$

The solution of (5) is

$$\widetilde{P}(x,s|x_0) = \frac{1}{\sqrt{4sD}}e^{-\sqrt{\frac{s}{D}}|x-x_0|}\ , \tag{6}$$

which can be verified by making use of the identity

$$\frac{d^2}{dx^2}\left[e^{-a|x-b|}\right] = -2a\delta(x-b) + a^2e^{-a|x-b|}\ .$$

Finally, to obtain the solution back in the time-domain, we have to invert the Laplace transform by making use of the Bromwich contour

$$P(x,t|x_0) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} ds\, e^{st}\widetilde{P}(x,s|x_0)\ , \tag{7}$$

where γ is a real number which lies to the right of all the singularities of the function to be inverted. The inversion integral for (6) has to be evaluated around a branch cut which will finally result in (3).

Exercice: Another method to obtain the same final result is to make use of a known integral (a particular case of identity 3.174.9 from [19])

$$\int_0^\infty t^{-1/2}e^{-\frac{\beta}{t}-st}dt = \frac{\pi^{1/2}}{s^{1/2}}e^{-2(\beta s)^{1/2}}$$

and match $\beta = \frac{(x-x_0)^2}{4D}$. Perform the calculation and verify that this also results in the propagator for diffusion (3).

3 Preliminaries : Diffusion with an absorbing target

Now consider the situation where in addition to a diffusing particle, there is an absorbing target [20, 13]. When the particle touches the target situated at $x_T = 0$, it get absorbed, and the process is considered to be completed.

We are interested in the probability that the particle survive up to a time t. The presence of the absorbing target can be modelled as an additional a boundary condition $P(x_T, t) = 0$. using the backward Fokker-Planck approach, we obtain the following equation,

$$\frac{\partial}{\partial t} P(x, t|x_0) = D \frac{\partial^2}{\partial x_0^2} P(x, t|x_0) \ . \tag{8}$$

The survival probability $q(t|x_0)$ is then obtained by integrating over all final positions at time t

$$q(t, x_0) = \int_0^\infty dx \ P(x, t|x_0) \ . \tag{9}$$

Integrating over x from 0 to ∞, and exchanging the order of differentiation and integration (since integration is with respect to x while the derivatives are with respect to x_0), we end up with an equation for the survival probability as

$$\frac{\partial}{\partial t} q(t|x_0) = D \frac{\partial^2}{\partial x_0} q(t|x_0) \ , \tag{10}$$

with the boundary condition

$$q(t|0) = 0 \ ,$$

and initial condition

$$q(0|x_0) = 1 \ .$$

The solution to (10) with the above mentioned boundary and initial conditions is the given in terms of the error function as

$$q(t, x_0) = \operatorname{erf}\left(\frac{x_0}{\sqrt{4Dt}} \right) \ , \tag{11}$$

where

$$\operatorname{erf}(z) = \frac{2}{\pi} \int_0^z du \ e^{-u^2} \ . \tag{12}$$

Again one can use the method of Laplace transform to obtain this solution. The Laplace transform of (10) is given by

$$D \frac{\partial^2 \widetilde{q}}{\partial x_0^2} - s\widetilde{q} = -1 \ , \tag{13}$$

which can be solved to obtain

$$\widetilde{q}(s|x_0) = \frac{1 - e^{-x_0 \sqrt{\frac{s}{D}}}}{s} \ . \tag{14}$$

Again (14) can be inverted using the Bromwich integral by integrating around the branch cut at $s = 0$ to obtain (11).

We can also obtain the large t behaviour quite easily from the Laplace transform. For this we expand $q(s|x_0)$ for small values of s and invert it using the inverse Laplace transform identity

$$\mathcal{L}^{-1}_{s \to t} \left\{ \frac{1}{s^\alpha} \right\} = \frac{1}{\Gamma(\alpha)} t^{\alpha - 1} \ . \tag{15}$$

Following this we get,

$$\widetilde{q}(s|x_0) \simeq \frac{x_0}{\sqrt{sD}} + C + \mathcal{O}(\sqrt{s}) \;, \tag{16}$$

$$q(t, x_0) \simeq \frac{x_0}{D}^{1/2} \Gamma(1/2) t^{-1/2} + \mathcal{O}(t^{-3/2}) \;. \tag{17}$$

3.1 Mean First Passage Time (MFPT)

The mean first passage time is obtained from the survival probability as follows

$$\langle T(x_0) \rangle = \int_0^\infty dt\; t \left(-\frac{\partial q}{\partial t} \right) \;, \tag{18}$$

but due to the $t^{-1/2}$ tails for the survival probability for diffusive processes as obtained in (17), the mean first passage time diverges as $t^{1/2}$ for the process. That is the mean first passage time for a diffusive process in infinite!

Note: When the survival probability goes to zero faster than t^{-1}, the mean first passage time can be simplified using integration by parts as

$$\begin{aligned} \langle T(x_0) \rangle &= -\left[q(t|x_0) t \right]_0^\infty + \int_0^\infty dt\; q(t|x_0) \\ &= \int_0^\infty dt\; q(t|x_0) \end{aligned} \tag{19}$$

We will use this equation later on.

4 Diffusion with stochastic resetting

Now in addition to the diffusive process mentioned in the previous sections, consider Poissonian resetting [1, 13]. This is defined by a constant resetting rate (per unit time) r This leads to the following process:

$$x_{t+\Delta t} = \begin{cases} x_t + \xi & \text{with proba } 1 - r\Delta t, \\ x_r & \text{with proba } r\Delta t \end{cases} \tag{20}$$

As before, we first write the Forward master equation for the process with resetting. Upon averaging over all possible events in between the time t to $t + \Delta t$, we get

$$\begin{aligned} P_r(x, t + \Delta t) &= r\Delta t \delta(x - x_r) + (1 - r\Delta t) \langle P_r(x - \xi, t) \rangle \\ &= r\Delta t \delta(x - x_r) + (1 - r\Delta t) \left[P_r - P_r^{'} \langle \xi \rangle + \frac{P_r^{''}}{2} \langle \xi^2 \rangle + \cdots \right] \\ &= P_r + \Delta t \left[r\delta(x - x_r) - rP_r + D \frac{\partial^2}{\partial x^2} P_r + \cdots \right] \end{aligned}$$

In the calculations above, we have made use of the following properties of Gaussian white noise

$$\begin{aligned} \langle \xi(t) \rangle &= 0 \;, \\ \langle \xi(t) \xi(t') \rangle &= 2D\delta(t - t') \;. \end{aligned}$$

Then in the limit $\Delta t \to 0$, we obtain the equation for the evolution of distribution with resetting as

$$\frac{\partial}{\partial t} P_r(x, t) = D \frac{\partial^2}{\partial x^2} P_r(x, t) - rP_r(x, t) + r\delta(x - x_r) \;. \tag{21}$$

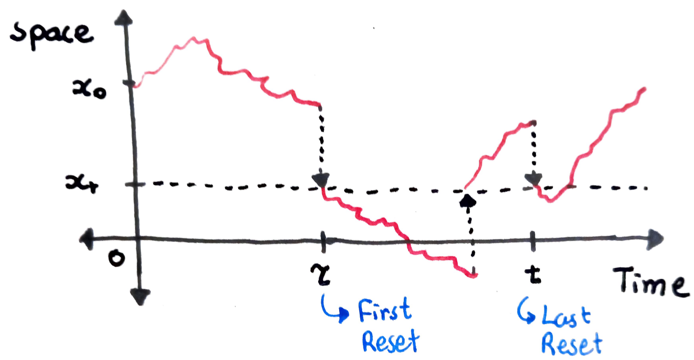

Figure 1: A schematic of trajectory for a particle undergoing stochastic resetting. The particle starts at x_0 and resets to x_r. t' represents the time of last reset and τ represents the time of first reset.

We thus obtain a diffusion equation with two additional terms proportional to r on the r.h.s.. The first represents the loss of probability with rate r from any position x, the second represents the gain of probability at the resetting position x_r with rate r from all other positions x.
Similarly, to obtain the backward master equation, we consider the evolution of the particle in the first time step from 0 to Δt. We then obtain,

$$P_r(x, t + \Delta t|x_0) = r\Delta t P_r(x, t|x_r) + (1 - r\Delta t)\langle P_r(x, t|x_0 + \xi)\rangle \ , \tag{22}$$

which upon expanding and taking the $\Delta t \to 0$ limit leads to

$$\frac{\partial}{\partial t} P_r(x, t|x_0) = D\frac{\partial^2 P_r}{\partial x_0^2}(x, t|x_0) + rP_r(x, t|x_r) - rP_r(x, t|x_0) \ . \tag{23}$$

As $t \to \infty$ we reach a steady state with $\frac{\partial P}{\partial t} = 0$, and initial condition $P(x, 0) = \delta(x - x_0)$ for both the forward (21) and backward equation (23) as

$$P_r^*(x) = \frac{\alpha_0}{2} e^{-\alpha_0|x - x_r|} \ , \tag{24}$$

where

$$\alpha_0 = \sqrt{\frac{r}{D}} \ . \tag{25}$$

Due to the existence of a non-zero probability current in the system with resetting, the resulting steady state is a Non-Equilibrium Steady State (NESS).

4.1 Renewal Approach

Another equivalent, but simpler method to obtain the propagator for the system with resetting is to use the renewal approach [1, 4, 13]. This approach is based on the fact that between resets, the process behaves exactly like a normal diffusion process with the resetting position as the initial condition (See Fig:1).
To show this more concretely, denote the last time of the process as t'. The last renewal equation can then be written as:

$$P_r(x, t|x_0) = \underbrace{e^{-rt} P_0(x, t|x_0)}_{\substack{\text{Probability of no reset} \\ \text{up to time} t}} + \int_0^t \underbrace{dt' r e^{-r(t-t')}}_{\substack{\text{Probability of reset between} \\ t' \text{ and } t' + dt' \text{ and no} \\ \text{resets for the remaining time.}}} \underbrace{P_0(x, t - t'|x_r)}_{\substack{\text{After the last reset,} \\ \text{the system evolves like usual} \\ \text{diffusion starting from } x_r.}} \tag{26}$$

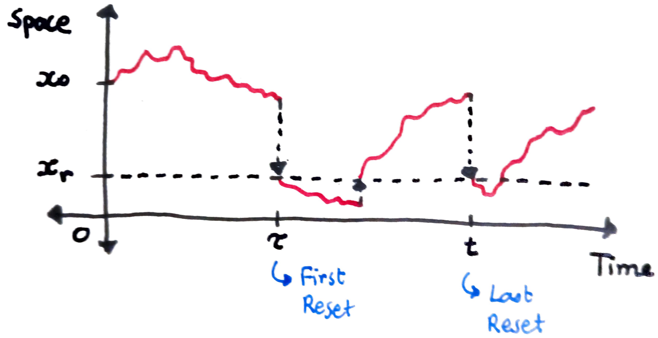

Figure 2: A schematic of a trajectory that does not cross $x = 0$ and hence survives till time t. t' represents the time of last reset and τ represents the time of first reset.

To obtain the steady state solution from the (26), we look at the $t \to \infty$ limit, which makes the first term zero, and the remaining term gives the steady state solution as

$$P^*(x) = r \int_0^{\infty} e^{-r\tau} P_0(x, \tau | x_r) d\tau \ , \tag{27}$$

which is the Laplace transform of a Gaussian, which we derived in Section 2 Eq. (6).

5 Diffusion with Resetting and Absorbing Target

Again we model the target with an absorbing target at $x_T = 0$ [1, 3, 13]. Using a similar (See Fig. 2) last renewal approach we can write the survival probability for the system with resetting in terms of the system without resetting as

$$q_r(t|x_0) = \underbrace{e^{-rt} q_0(t|x_0)}_{\substack{\text{Probability of surviving} \\ \text{with no reset up to time } t}} + \int_0^t \underbrace{dt' r e^{-r(t-t')}}_{\substack{\text{Probability of reset between} \\ t' \text{ and } t'+dt' \text{ and no} \\ \text{resets for the remaining time.}}} \underbrace{q_r(t'|x_0)}_{\substack{\text{Probability of survival} \\ \text{till } t' \text{ with resetting.}}} \underbrace{q_0(t-t'|x_r)}_{\substack{\text{Probability of survival} \\ \text{without resetting for} \\ \text{the remaining time.}}} \ . \tag{28}$$

Recognizing that in (28), the term inside the integral is a convolution, we can make use of the convolution theorem to obtain the Laplace transform of $\widetilde{q}_r(t|x_0)$ as,

$$\widetilde{q}_r(s|x_0) = \widetilde{q}_0(s+r|x_0) + r\widetilde{q}_r(s|x_0)\widetilde{q}_0(s+r|x_r) \ , \tag{29}$$

which upon taking the initial and reset position both to be the same *ie*, $x_r = x_0$, can be rewritten as,

$$\widetilde{q}_r(s|x_0) = \frac{\widetilde{q}_0(s+r|x_0)}{1 - r\widetilde{q}_0(s+r|x_0)} \tag{30}$$

$$= \frac{1 - e^{-\alpha(s)x_0}}{s + re^{-\alpha(s)x_0}} \ , \tag{31}$$

where $\alpha(s) = \sqrt{\frac{r+s}{D}}$.

Note: Instead of a last renewal equation for the survival probability as given in (28), one could also write an equivalent first renewal equation with τ representing the time of first reset as (See Fig. 2),

$$q_r(t|x_0) = \underbrace{e^{-rt} q_0(t|x_0)}_{\substack{\text{Probability of surviving} \\ \text{with no reset up to time } t}} + \int_0^t \underbrace{d\tau \ r e^{-r\tau}}_{\substack{\text{Probability of a single} \\ \text{reset between } \tau \\ \text{and } \tau + d\tau.}} \underbrace{q_0(\tau|x_0)}_{\substack{\text{Probability of survival} \\ \text{till } \tau \text{ without resetting.}}} \underbrace{q_r(t-\tau|x_r)}_{\substack{\text{Probability of survival} \\ \text{with resetting for} \\ \text{the remaining time.}}} \ . \tag{32}$$

Taking the Laplace transform then yields an alternative equation for the Laplace transforms

$$\widetilde{q}_r(s|x_0) = \widetilde{q}_0(s+r|x_0) + r\widetilde{q}_0(r+s|x_0)\widetilde{q}_r(s|x_r) \ , \tag{33}$$

which reduces to (29) when $x_0 = x_r$.

5.1 Mean First Passage Time

Using (19), the MFPT can be calculated as,

$$\langle T_r(x_0)\rangle = \widetilde{q}_r(s=0|x_0) = \frac{e^{\alpha_0 x_0} - 1}{r} \ . \tag{34}$$

(34) has diverging behaviour for both the small r and large r limits,

- For $r \ll 1 : \langle T_r(x_0)\rangle \to \frac{x_0}{\sqrt{Dr}}$.
- For $r \gg 1 : \langle T_r(x_0)\rangle \to \frac{e^{\sqrt{\frac{r}{D}}x_0}}{r}$.

This implies that the MFPT has an optimal value of resetting which minimizes the MFPT, and this can be obtained by evaluating $\frac{d\langle T_r(x_0)\rangle}{dr} = 0$, which results in

$$\frac{y}{2} = 1 - e^{-y} \ , \tag{35}$$

where $y = \frac{x_0}{(D/r)}$. y is dimensionless parameter, which is the ratio of the two length scales in the system, the distance to the target and the distance diffused between resets. (35) has a unique non-zero solution given by $y^* = 1.5936\ldots$, that minimizes the MFPT for searches with resetting.

Note that here the optimisation of the MFPT assumes that we know the distance to the target x_0. Nevertheless, the calculation shows that the MFPT can be optimised by tuning the resetting rate and any amount of resetting is better than the process without resetting. Also, when the target position itself is random, it turns out that Poissonian resetting is again the optimal choice. This is discussed in Section 7.

5.2 Long Time Behaviour of Survival Probability

The asymptotic behaviours of the survival probability (31) can be obtained from the singularities of the Laplace transform [1, 13]. The largest singularity dominates the long term behaviour. The denominator of (31) has two singularities

(i) Branch point at $s = -r$.

(ii) Pole at $s_0 + r\exp\left(-\sqrt{\frac{r+s}{D}}x_0\right)$, with $s_0 > -r$.

Since $s_0 > -r$, the contribution from the pole determines the leading order behaviour,

$$q(t|x_0) \underset{t\gg 1}{\simeq} e^{s_0 t} \ . \tag{36}$$

If $|s_0| \ll 1$ then,

$$s_0 \simeq -re^{-y} \ , \tag{37}$$

where $y = \sqrt{\frac{r}{D}}x_0$.

Now in addition, if $y \gg 1$, then

$$q(t|x_0) \simeq e^{-rte^{-y}} \ , \tag{38}$$

which is the Gumbel distribution.

Aside on Gumbel Distribution

Consider a situation where N iid random variables are drawn from a PDF, $f(x)$. We are interested in finding the probability that the maximum of these N random numbers is less than a given number M. That is,

$$\mathbb{P}(\max_{1\leq i\leq N} X_i \leq M) = [\mathbb{P}(X_1 \leq M)]^N = \left[\int_{-\infty}^{M} f(x)dx\right]^N = \exp N \ln\left[1 - \int_{M}^{\infty} f(x)dx\right] . \tag{39}$$

Now if $f(x)$ has an exponential tail, $f(x) \sim Ae^{-ax}$ for $x \gg 1$, we obtain the Gumbel distribution from (39),

$$\mathbb{P}(\max_{1\leq i\leq N} X_i \leq M) \approx \exp\left[-\frac{NA}{a}e^{-aM}\right] . \tag{40}$$

In the case of survival of the system with stochastic resetting, we have a similar situation. We look at the probability that the maximum excursion to the lest is less than the distance to the target x_0. Averaging over the duration of each excursion, and with an expected number of resets rt in the long time limit, we obtain

$$\begin{aligned} q_r(t|x_0) &\approx \left[\int_0^\infty dtre^{-rt} q_0(t|x_0)\right]^{rt} \qquad \text{(Using (14))} \\ &= \left[1 - e^{-x_0\sqrt{\frac{r}{D}}}\right]^{rt} \\ &\simeq e^{-rte^{-y}} , \end{aligned} \tag{41}$$

with $y = \sqrt{\frac{r}{D}}x_0$.

6 Large Deviations in Stochastic Resetting

6.1 With an additive functional

We are interested in calculating the large deviations of an additive functional for a process with resetting [6, 14, 15]. Consider a path with N resettings. Let $f(x_t)$ be an additive functional of the trajectory, using which we can define

$$\begin{aligned} A_t &= \int_0^t dt\ f(x_t) \\ &= \sum_{i=1}^{N+1} A_{(t_n - t_{n-1})} , \end{aligned} \tag{42}$$

where we have defined $t_0 = 0$ and $t_{N+1} = t$.
We expect a large deviation principle to hold with the form

$$P_r(A_t, t) \sim \exp\left(-tI\left(\frac{A_t}{t}\right)\right) . \tag{43}$$

Using the first renewal formalism, we can write an equation for $P_r(A_t, t)$ as

$$P_r(A_t, t) = e^{-rt}P_0(A_t, t) + \int_0^t d\tau\ e^{-r\tau} r\left(\int dA_\tau P_0(A_\tau, \tau)P_r(A - A_\tau, t - \tau)\right), \tag{44}$$

where we have used the additive property of the functional to write a convolution over A_τ. Define a generating function

$$G(k, t) = \langle e^{kA_t}\rangle = \int dA_t e^{kA_t} P(A_t, t) , \tag{45}$$

using which we can rewrite (44), making use of the convolution theorem as

$$G_r(A_t,t) = e^{-rt}G_0(A_t,t) + \int_0^t d\tau\; re^{-r\tau}G_0(k,\tau)G_r(k,t-\tau)\ . \tag{46}$$

Now define the Laplace transform of $G_r(A_t,t)$ as

$$\widetilde{G}(k,s) = \int_0^\infty e^{-st}G(k,t)dt\ , \tag{47}$$

from which we obtain again, making use of the convolution theorem,

$$\widetilde{G}_r(k,s) = \widetilde{G}_0(k,s+r) + r\widetilde{G}_0(k,s+r)\widetilde{G}_r(k,s)\ , \tag{48}$$

which can be rearranged to obtain,

$$\widetilde{G}_r(k,s) = \frac{\widetilde{G}_0(k,s+r)}{1-r\widetilde{G}_0(k,s+r)}\ . \tag{49}$$

In (49), we have obtained the the generating function for the system with resetting in terms of the generating function for the system without resetting. (49) has a pole at $1-r\widetilde{G}_0(k,s+r)=0$, which we denote by $s_0(k,t)$. So upon inverting, we obtain the leading behaviour of $G_r(k,t)$ as

$$G_r(k,t) \sim e^{s_0(k,r)t}\ . \tag{50}$$

Then upon a second inversion with respect to k, we obtain (ignoring sub-exponential terms),

$$P_r(A_t,t) \sim \int dk e^{t[\frac{A_t}{t}k+s_0(k,r)]}\ , \tag{51}$$

which is dominated by the saddle point,

$$\frac{A_t}{t} + \frac{d}{dk}s_0(k,r) = 0\ . \tag{52}$$

So we obtain the large deviation form for $P_r(A_t,t)$ as

$$P_r(A,t) \sim e^{tI(a)} \tag{53}$$

where $I\left(a\right)$ is the Legendre-Fenchel tranfsorm of s_0

$$I\left(a\right) = -\sup_k\left(ka + s_0(k,r)\right)\ , \tag{54}$$

with $a=\frac{A_t}{t}$.

6.2 Example of Large Deviation: The Cost of Stochastic Resetting

In the model so far, we have assumed that the resets are instantaneous, but this assumption is unrealistic. To rectify this and make the model more applicable, we introduce a cost to each of the resets in the model [16, 17]. We consider additive costs that occur at the resets (See Fig. 3) of the following form,

$$C = \sum_{i=1}^{N} c(|x_i|)\ , \tag{55}$$

where we take x_i as the position just before the reset and the resetting position as the origin. The resetting costs are functionals of resetting jumps. Example costs include:

- Linear cost per reset: $c(x)=\frac{|x|}{V}$. This can be interpreted as the time to return to the origin at a constant speed V.

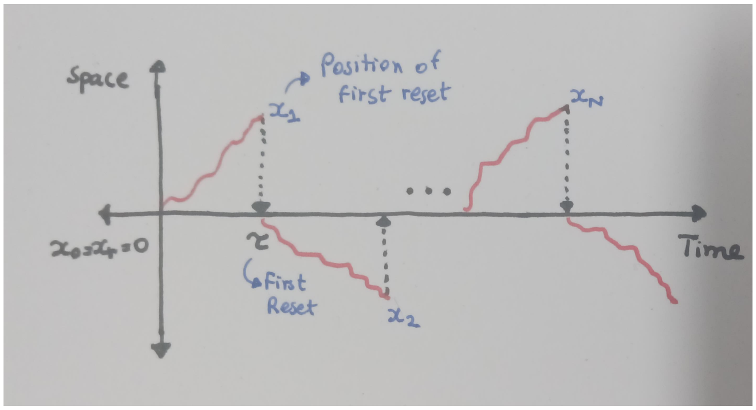

Figure 3: For simplicity, we the the particle to start and reset to $x = 0$. Cost is incurred at the jumps denoted by x_i. τ represents the time of first reset.

- Quadratic cost per reset $c(x) = x^2$. This is related to "thermodynamics of resetting".
- Exponential cost per reset: $c(x) = e^{|x|}$. This could be used to model situation where resets are highly unfavourable.

We are interested in calculating the mean total cost $\langle C \rangle$ for the process up to a fixed time t. To begin with the calculation for the mean total cost, we first write down a first renewal equation for $\boldsymbol{P}_r(C,t)$, which is the joint PDF of having incurred a total cost C at time t. We obtain the first renewal equation as

$$\boldsymbol{P}_r(C,t) = \delta(C)e^{-rt} + \int_0^t d\tau\ re^{-r\tau} \int_{-\infty}^{\infty} dx\ P_0(x,\tau)\boldsymbol{P}_r(C - c(x), t-\tau)\Theta(C - c(|x_i|)) \ , \tag{56}$$

where $\Theta(C - c(|x_i|))$ ensures that the total cost is kept positive.

We then define a generating function

$$G_r(k,t) = \int_0^{\infty} e^{-kC}\boldsymbol{P}_r(C,t)dC \tag{57}$$

Using the definition of the generating function on (56), we obtain after a few steps of calculation,

$$G_r(k,t) = e^{-rt} + \int_0^t d\tau\ re^{-r\tau} \int_{-\infty}^{\infty} dx\ P_0(x,\tau)e^{-kc(|x|)}G_r(k, t-\tau) \ . \tag{58}$$

Finally taking a Laplace transform of (58), we get

$$\widetilde{G}_r(k,s) = \frac{1}{s+r} + r\int_{-\infty}^{\infty} dx\ \widetilde{P}_0(x, r+s)e^{-kc(|x|)}G_r(k,s) \ , \tag{59}$$

which can be rewritten as

$$\widetilde{G}_r(k,s) = \frac{1}{s+r}\frac{1}{\left[1 - r\int_{-\infty}^{\infty} dx\ \widetilde{P}_0(x, r+s)e^{-kc(|x|)}\right]} \ . \tag{60}$$

The mean total cost can then be obtained as

$$\langle C \rangle = \mathcal{L}^{-1}_{s\to t}\left[-\partial_k \widetilde{G}_r(k,s)\Big|_{k\to 0^+}\right] \ . \tag{61}$$

Performing these calculations for the linear and quadratic cost per resetting leads to the following mean total costs, which is written in terms of the dimensionless resetting rate $R = rt$.

- Linear cost per resetting: $c(x) = |x|$

$$\langle C \rangle_{\text{lin}} = \frac{e^{-R}}{\sqrt{\pi}} + \frac{(2R-1)\text{erf}(R)}{2\sqrt{R}} \ . \tag{62}$$

For large values of R, diverges as, $\langle C \rangle_{\text{lin}} \to R^{1/2}$.

- Quadratic cost per resetting: $c(x) = |x|$

$$\langle C \rangle_{\text{quad}} = \frac{2(R + e^{-R} - 1)}{R} \ . \tag{63}$$

Unlike the case for linear cost, for large values of R, the mean total quadratic cost obtains a finite value, $\langle C \rangle_{\text{quad}} \to 2$.

7 Non-Poissonian Resetting

So far we have assumed that that the resetting is Poissonian, ie, we have assumed that the waiting time between resets is distributed exponentially. We now relax this assumption as assume a general waiting time distribution $\psi(t)$ [7, 8]. We also define the probability of no resets up to time t as

$$\Psi(t) = \int_t^\infty d\tau \ \psi(\tau) \ . \tag{64}$$

Note that we recover the all the results for Poissonian resetting if we set $\psi(t) = re^{-rt}$. Another situation that is also of interest is the case of deterministic reset (or sharp restart) which is given by $\psi(t) = \delta(t - t_r)$. Similar to before, in the absence of an absorbing target and with $x_0 = x_r$, we can write the propagator for system with resetting using a first renewal equation as

$$P_r(x,t|x_0) = \Psi(t)P_0(x,t|x_0) + \int_0^t d\tau \ \psi(\tau)P_r(x,t-\tau|x_0) \ . \tag{65}$$

Then using Laplace transform we get,

$$\begin{aligned} \widetilde{P}_r(x,s|x_0) &= \frac{\int_0^\infty dt \ e^{-st}\Psi(t)P_0(x,t|x_0)}{1 - \int_0^\infty dt \ e^{-st}\psi(t)} \\ &= \frac{\int_0^\infty dt \ e^{-st}\Psi(t)P_0(x,t|x_0)}{s\int_0^\infty dt \ e^{-st}\Psi(t)} \ , \end{aligned} \tag{66}$$

where the last equality is obtained by performing integration by parts on the denominator. The stationary state will then be given by the coefficient of the $\frac{1}{s}$ term in the small s expansion. So we obtain the stationary state to be

$$P^*(x) = \lim_{s\to 0} \frac{\int_0^\infty dt \ e^{-st}\Psi(t)P_0(x,t|x_0)}{\int_0^\infty dt \ e^{-st}\Psi(t)} \ . \tag{67}$$

Thus, for the NESS to exist, this limit should not vanish. Therefore we require

$$\int_0^\infty \Psi(t)dt = \int_0^\infty \tau\psi(\tau)d\tau = \mathbb{E}(\tau) < \infty \ . \tag{68}$$

Similar to the cases discussed in previous sections, we could consider the MFPT for the system with non-poissonian resetting and optimise it [2, 11].

8 Optimal waiting time distribution

Another interesting question to ask is, in the presence of an absorbing target, what is the best choice of waiting time distribution to minimize the mean time for first passage? It turns out that the answer to this question is that a deterministic resetting is the best [8, 10, 11], if the resetting period is chosen appropriately. But choosing the period for the sharp restart appropriately, assumes that we know the distance to the target.

A more realistic problem is when we know where the target ought to be (at the origin say) but the actual distance from this position is a random variable drawn from a target distribution, thus we do not know the position of the target exactly. We then would choose our resetting position to be the origin and wish to select an optimal waiting time distribution for resetting. The optimal waiting time distribution will depend on the target distribution. For the case of an exponential target distribution it turns out that Poissonian resetting, that is, an exponential waiting time distribution with appropriately chosen mean, is the waiting time distribution that is optimal i.e. it minimises the averaged MFPT to the target . Thus stochastic resetting prevails over sharp restart when the target distribution is sufficiently broad [18].

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