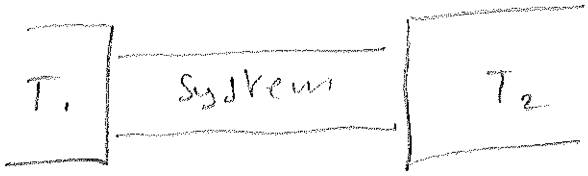
Les Houches

I Last time

Einstein relations

Heat



$T_1 - T_2$ small

$$\frac{\langle Q_t \rangle}{t} = \tilde{D} \, (T_1 - T_2)$$

$T_1 - T_2$ $\quad \dfrac{\langle Q_t^2 \rangle}{t} \quad \tilde{\sigma}$. $\quad \boxed{\tilde{\sigma} = 2kT^2 \tilde{D}}$

Particles

$\mu_1 - \mu_2$ small

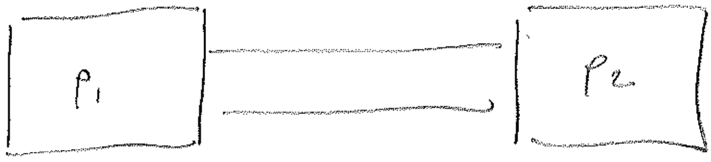
$$\frac{\langle Q_t \rangle}{t} = \tilde{D} \, \Delta\mu \qquad \boxed{\tilde{\sigma} = 2kT\tilde{D}}$$

$\mu_1 = \mu_2$

$$\frac{\langle Q_t^2 \rangle}{t} \sim \tilde{\sigma}$$

B2

For $\rho_1 - \rho_2$ small



$$\langle Q_t \rangle = \tilde{\tilde{D}} \, (\rho_1 - \rho_2)$$

$$\tilde{\tilde{D}} = \frac{1}{2kT} f''(\rho) \, \tilde{\sigma}$$

$f(\rho)$ = free energy per unit volume

$$F = V f\left(\frac{N}{V}\right) \quad \leftarrow \text{ total free energy}$$

$$\mu = \frac{dF}{dN} = f'(\rho)$$

$$\frac{d\mu}{d\rho} = f''(\rho)$$

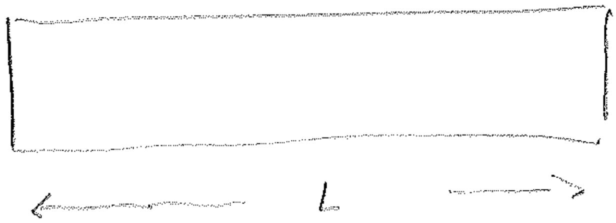
~~susceptibility~~ compressibility

Here $\quad 2D = f''(\rho) \, \sigma \quad$ f free energy per unit volume

mathematicians $\quad \frac{1}{f''(\rho)} =$ susceptibility

physicists $\quad \frac{1}{\rho^2 f''(\rho)} = \frac{1}{\rho} \frac{d\rho}{dp}$ compressibility

II Fourier's - Fick's Law



large L

$\tilde{D} = \frac{D}{L}$ $\qquad$ $\tilde{\sigma} = \frac{\sigma}{L}$ $\qquad$ Heat Fourier law

$\tilde{\tilde{D}} = \frac{D}{L}$ $\qquad$ $\tilde{\sigma} = \frac{\sigma}{L}$ $\qquad$ Particles Fick's law

$\vdots$ $\qquad$ $\vdots$

(Ohm's law for electric current

Here we will only discuss particles for $kT = 1$

$\Rightarrow$ $\sigma(\rho)$

$D(\rho)$

$$\boxed{\frac{2D}{\sigma} = f''(\rho)}$$

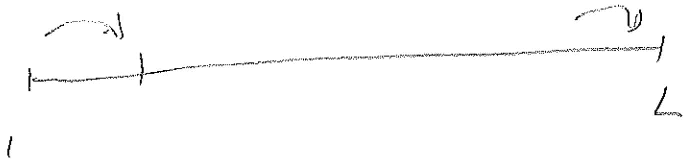
$f(\rho)$ known from the statics

($D(\rho)$? depend on the dynamics

$\sigma(\rho)$

III How to determine $\mathcal{D}$ and σ

(i) SSEP



$$Z = \frac{L!}{N!\,(L-N)!}$$ N particles in volume L

$$\Rightarrow F = -kT \log Z$$

$$F = L\left(\rho \log \rho + (1-\rho)\log(1-\rho)\right) = L\,f(\rho)$$

$$\Rightarrow f''(\rho) = \frac{1}{\rho(1-\rho)}$$

So $$\boxed{\sigma(\rho) = 2\rho(1-\rho)\,\mathcal{D}(\rho)}$$

n_i = occupation of site i

$$\frac{d\langle n_i\rangle}{dt} = \langle n_{i+1}(1-n_i)\rangle + \langle n_{i-1}(1-n_i)\rangle - \langle n_i(1-n_{i+1})\rangle - \langle n_i(1-n_{i-1})\rangle$$

$$= \langle n_{i+1}\rangle + \langle n_{i-1}\rangle - 2\langle n_i\rangle$$

Steady state $\frac{d\langle n_i\rangle}{dt} = 0$ $\quad\langle n_i\rangle$ linear

$$\langle n_i \rangle = A + iB$$

$$\langle n_1 \rangle \simeq \rho_a \qquad \Rightarrow \qquad B \simeq \frac{\rho_b - \rho_a}{L}$$

$$\langle n_L \rangle \simeq \rho_b$$

$$\frac{\langle Q_i \rangle}{t} = \langle n_i (1-n_{i+1}) \rangle - \langle n_{i+1} (1-n_i) \rangle = B$$

$$\frac{\langle Q \rangle}{t} \simeq \frac{\rho_a - \rho_b}{L} \qquad \Rightarrow \qquad \boxed{D = 1}$$

$$D = 1 \qquad \sigma = 2\rho(1-\rho)$$

Other models

gradient models

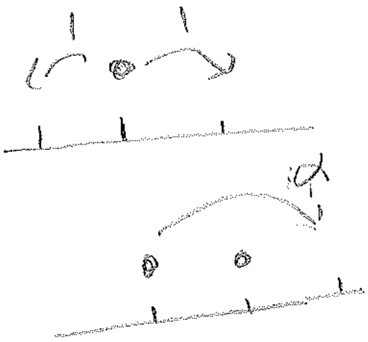
$$\frac{\langle Q_i \rangle}{t} = \langle \phi(n_{i-2}, n_{i-1}, n_i, n_{i+1}) \rangle - \langle \phi(n_{i-1}, n_i, n_{i+1}, n_{i+2}) \rangle$$

$$\frac{\sum \langle Q_i \rangle}{t} = \text{telescopic sum}$$

$$\frac{\langle Q \rangle}{t} \simeq \frac{1}{L} \underbrace{\langle \phi(n_1, n_2, n_3, n_4) \rangle}_{\text{eq at } \rho_a} - \underbrace{\langle \phi(n_{L-3}, n_{L-2}, n_{L-1}, n_L) \rangle}_{\text{eq } (\rho_b)}$$

$$= \frac{1}{L} \left(\Psi(\rho_a) - \Psi(\rho_b) \right)$$

Exercices



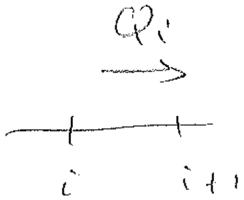
Exclusion

jump either to 1st or second neighbor if 1st neighbor occupied.

$f(\rho)$?

$D(\rho)$?

$\sigma(\rho)$



$$
\begin{aligned}
\frac{d\langle Q_i\rangle}{dt} &= \alpha\langle n_{i-1}\, n_i\,(1-n_{i+1})\rangle + \langle n_i\,(1-n_{i+1})\rangle \\
&\quad + \alpha\langle n_i\, n_{i+1}\,(1-n_{i+2})\rangle \\
&\quad - \alpha\langle n_{i+2}\, n_{i+1}\,(1-n_i)\rangle - \langle n_{i+1}(1-n_i)\rangle \\
&\quad - \alpha\langle n_{i+1}\, n_i\,(1-n_{i-1})\rangle \\
&= \langle n_i\rangle + \left[\langle n_{i-1}\, n_i\rangle + \langle n_i\, n_{i+1}\rangle\right]\alpha \\
&\quad - \langle n_{i+1}\rangle - \left[\langle n_i\, n_{i+1}\rangle + \langle n_{i+1}\, n_{i+2}\rangle\right]\alpha
\end{aligned}
$$

Ans

$$
\begin{pmatrix} \sigma(\rho) = 2\rho(1-\rho)\, D \\ D = 1 + 4\rho\alpha \end{pmatrix}
$$

CAN ONE SIMPLIFY?

Other exercice more difficult

$i \rightarrow i+2$ only if $i+1$ occupied

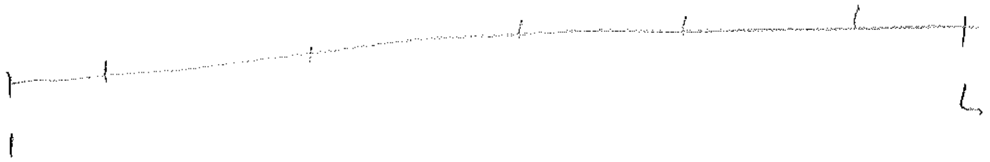
$\longleftarrow$

IV Fluctuating Hydrodynamics

~~and large deviations~~

Microscopic model

$n_i(t)$ nb of particles on site i



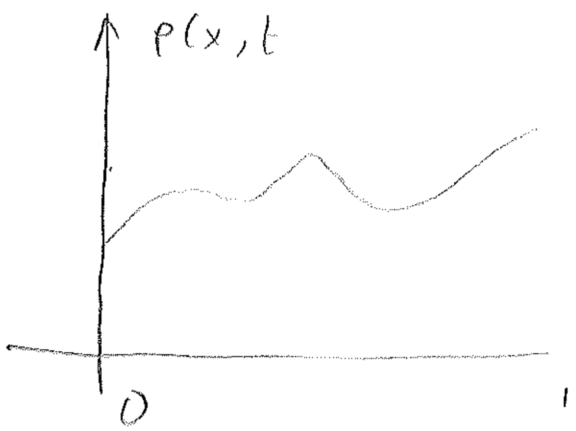
Large scale

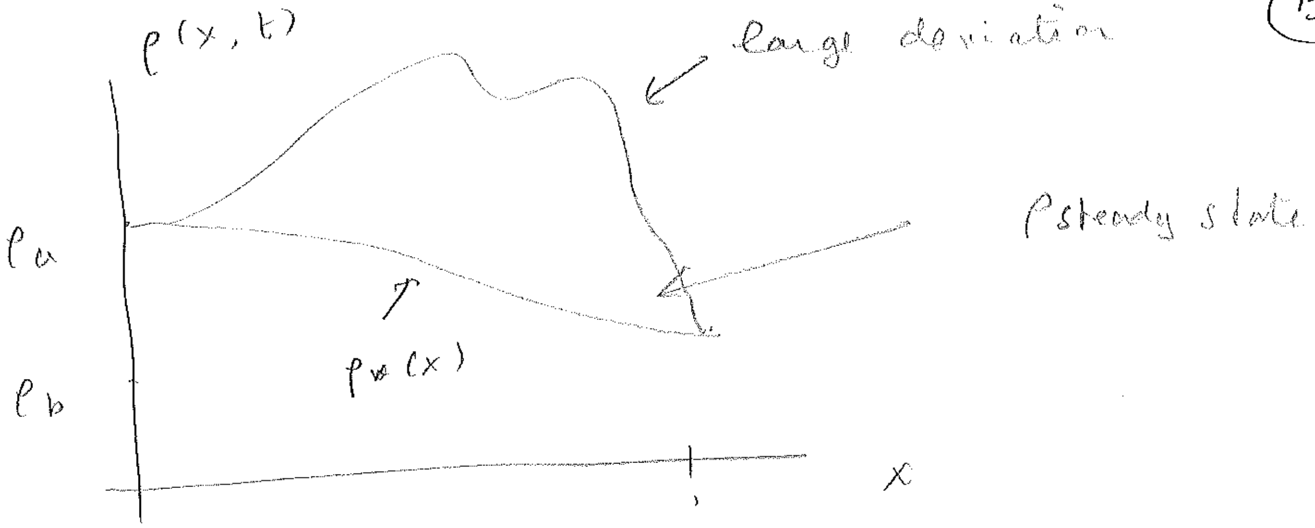
$i = L\,x \qquad 0 < x < 1$

$\ell = L\,\Delta x \qquad 1 \ll \ell \ll L$

$$\boxed{\rho(x,t)\, L\, \Delta x = \sum_{Lx < i \leq L(x+\Delta x)} n_i(t)}$$

Main idea





globally $\quad \rho(x,t)$ far from $\rho^*(x)$

Locally $\quad \rho(x,t) - \rho(x+\Delta x, t)$ small

$\Rightarrow$ Einstein Relation locally

Fluctuating Hydrodynamics

Observer at large scale

$0 < x < 1$ rescaled space

τ rescaled time

$j(x,\tau)$ rescale current

$\rho(x,\tau)$ density

$Lx \qquad L(x+\Delta x)$

large

but locally

close to equilibrium

$$\boxed{\frac{d\rho}{d\tau} = -\frac{dj}{dx}}$$

conservation law

$\longleftarrow$ No noise

$$\Delta x\left[\rho(x,t+\Delta t) - \rho(x,t)\right] = \Delta t\left[j(x,t) - j(x+\Delta x,t)\right]$$

$$\Delta t\,\Delta x\,\frac{d\rho}{dt} = -\Delta t\,\Delta x\,\frac{dj}{dx}$$

$$\boxed{j = \underbrace{-D(\rho)\,\rho'}_{\text{Fick's law}} + \underbrace{\sqrt{\epsilon}\,\sqrt{\sigma(\rho)}\,\eta(x,t)}_{\text{fluctuation}}}$$

white noise

$$\langle \eta(x,t)\,\eta(x',t')\rangle = \delta(t-t')\,\delta(x-x')$$

(Is this description O.K. ?

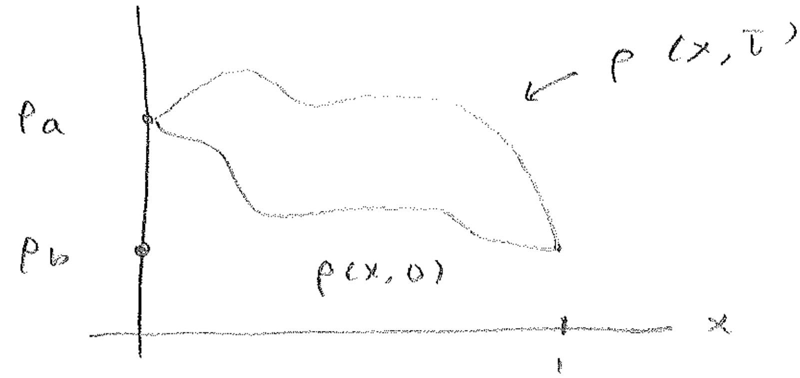
What is ϵ, are σ and D the same as in the microscopic calculation ?

⚠ Multiplicative noise

but here ϵ very small

V Large deviations : Macroscopic fluctuation theory

[Bertini - De Sole, Gabrielli
Jona Lasinio
Landim
2000]



$$P(\rho(x,\tau)\,|\,\rho(x,0)) = \int \underbrace{\mathcal{D}j\,\mathcal{D}\rho}_{\text{constraint}} \exp\left[-\frac{1}{\epsilon}\int_0^1 dx \int_0^\tau d\tau' \,\frac{(j + D(\rho)\rho')^2}{2\,\sigma(\rho)}\right]$$

constraint

$$\frac{d\rho(x,\tau')}{d\tau'} = -\frac{d}{dx}\, j(x,\tau')$$

$\boxed{\epsilon \to 0 \quad \text{a single path dominates}}$

Let us try to determine for $\epsilon \to 0$

$$\left\langle \exp\left(\frac{1}{\epsilon}\int_0^1 \rho(x,\tau)\,\Lambda(x)\,dx\right)\right\rangle$$

$$\simeq e^{\frac{1}{\epsilon} G_\tau(\{\Lambda(x)\})}$$

Ring growth (B11)

$$e^{\frac{1}{\epsilon} G_{\tau+\Delta\tau}\{\Lambda(x)\}} = \left\langle_{j} \left[\frac{1}{\epsilon} \int \Lambda(x)\, \rho(x,\tau+\Delta\tau)\,dx - \Delta\tau \frac{(j + D(\rho)\rho')^2}{2\sigma(\rho)} \right] \right\rangle$$

$\cancel{d\rho}$ $\cancel{\Delta\rho}$

$$\rho(x,\tau+\Delta\tau) = \rho(x;\tau) - \frac{dj}{dx}\Delta\tau$$

$$\boxed{G_{\tau+\Delta\tau}(\Lambda) - G_{\tau}(\Lambda) = \Delta\tau \max_{j(x)} \left[\int dx \left(\underbrace{\Lambda(x)\left(-\frac{dj}{dx}\right)}_{+\int\left(\frac{d\Lambda(x)}{dx}\right) j\,dx} - \frac{(j + D(\rho)\rho')^2}{\sigma(\rho)} \right) \right]}$$

$$\Rightarrow \boxed{j(x) = -D(\rho)\rho' + \underbrace{\sigma(\rho)\frac{d\Lambda(x)}{dx}}_{\uparrow \text{ forcing}}}$$

TO REMEMBER TOMORROW

$$\boxed{\frac{dG_\tau(\Lambda(x))}{d\tau} = \int dx \left[\frac{\sigma(\rho)}{2}\left(\frac{d\Lambda}{dx}\right)^2 - D(\rho)\rho'\frac{d\Lambda}{dx} \right]}$$

$$\rho(x,\tau) = \frac{\delta G_\tau(\{\Lambda(x)\})}{\delta\Lambda(x)}$$

GARDER AU TABLEAU

Hamilton Jacobi

σ and D arbitrary

(1) These equations (or equivalent equations)

are HARD to solve.

If you can solve them for arbitrary boundary conditions, σ, D and initial condition, many unsolved problems could be solved.

(2) We are going to see the link with a microscopic model and how time has to be rescaled.

(3)

Fluctuating Hydro $\Rightarrow$ long range correlation out of equilibrium

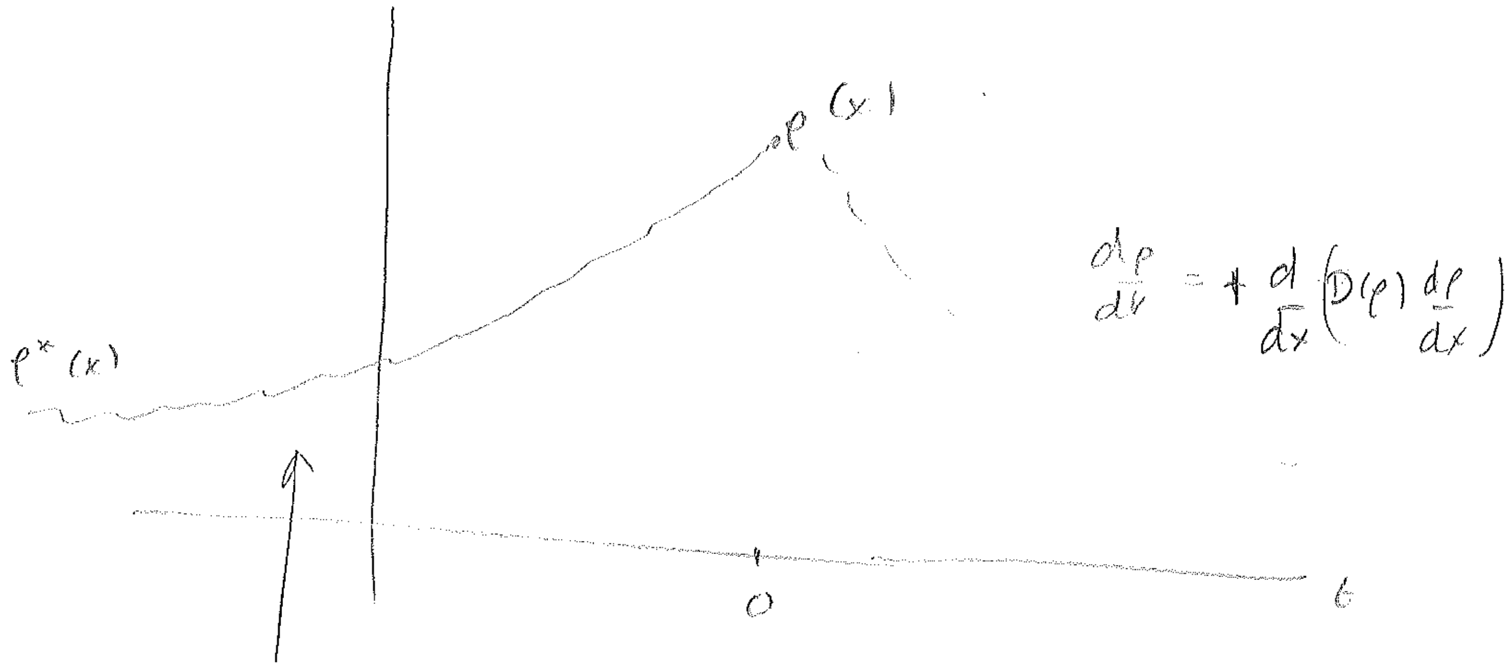
(4) Quasi Potential

Steady state $\mathcal{P}(\{\rho(x)\}) \sim e^{-\frac{1}{\epsilon}\mathcal{F}(\rho(x))}$

$\mathcal{F}(\{\rho(x)\})$?

(B13)

idea



Path to generate the large deviation $\rho(x)$

Easy at eq. $\frac{d\rho}{dt} = - \frac{d}{dx}\left(D(\rho)\frac{d\rho}{dx}\right)$

Out of equilibrium HARD in general

Some cases one knows the solution

SSEP

See next lecture

VI Microscopic model

SSEP ring

$$\psi(\lambda_1, \dots \lambda_L) = \langle e^{\lambda_1 n_1 + \dots + \lambda_L n_L} \rangle$$

$n_i = 0, 1$

$$\frac{d\psi}{dt} = \sum_i \left(e^{\lambda_{i+1} - \lambda_i} - 1\right) \langle n_i (1 - n_{i+1})\, e^{\lambda_1 n_1 + \dots \lambda_L n_L} \rangle$$

$$+ \left(e^{\lambda_i - \lambda_{i+1}} - 1\right) \langle n_{i+1} (1 - n_i)\, e^{\lambda_1 n_1 \dots \lambda_L n_L} \rangle$$

$$= \sum_i \left(e^{\lambda_{i+1} - \lambda_i} - 1\right) \left(\frac{d}{d\lambda_i} - \frac{d^2}{d\lambda_i \, d\lambda_{i+1}}\right) \psi$$

$$+ \sum_i \left(e^{\lambda_i - \lambda_{i+1}} - 1\right) \left(\frac{d}{d\lambda_{i+1}} - \frac{d^2}{d\lambda_i \, d\lambda_{i+1}}\right) \psi$$

$$\frac{d\psi}{dt} = \sum_i \left(e^{\lambda_{i+1} - \lambda_i} - 1 + e^{\lambda_{i-1} - \lambda_i} - 1\right) \frac{d\psi}{d\lambda_i} \quad (\text{ring})$$

$$- \sum_i \left(e^{\lambda_{i+1} - \lambda_i} - 1 + e^{\lambda_i - \lambda_{i+1}} - 1\right) \frac{d^2\psi}{d\lambda_i \, d\lambda_{i+1}}$$

$\boxed{\text{EXACT}}$

---

large scale

$$\boxed{\lambda_i = \Lambda\left(\frac{i}{L}\right)} \qquad \frac{i}{L} = x$$

B15

$$\psi \simeq e^{L\, G_t(\{\rho(x)\})} \qquad x = \frac{i}{L}$$

$$\rho_{i+1} - \rho_i = \frac{1}{L}\rho'(x) + \frac{1}{2L^2}\rho''(x)$$

$$\frac{d\psi}{d\rho_i} \simeq \langle n_i \rangle_{\rho_1 \cdots \rho_L} = \frac{\delta G}{\delta \rho(x)}$$

(Footnote: if $\psi = e^{\sum_i g(\rho_i)} \simeq e^{L\int_0^1 g(\rho(x))\,dx}$

$$\frac{d\psi}{d\rho_i} = g'(\rho_i) = g'(\rho(x)) = \frac{\delta G}{\delta \rho(x)}$$

$$\frac{d^2\psi}{d\rho_i\, d\rho_{i+1}} = g'(\rho_i)\, g'(\rho_{i+1})\,\psi = \left(\frac{\delta G}{\delta \rho(x)}\right)^2$$

⚠ this remain true also if at eq. no correlations between neighbors i.e. équilibrium is Bernoulli)

$$\frac{d\psi}{dt} = L\,\frac{dG_t}{dt} = L\int_0^1 \frac{dx}{L^2}\left([\rho'(x)^2 + \rho''(x)]\,\frac{\delta G}{\delta \rho(x)} - \rho'(x)^2\left(\frac{\delta G}{\delta \rho(x)}\right)^2\right)$$

B 16

An integration by parts

$$\Lambda'' \frac{\delta G}{\delta \Lambda(x)} \to -\Lambda' \frac{d}{dx}\left(\frac{\delta G}{\delta \Lambda(x)}\right)$$

$\Rightarrow \quad \epsilon = \frac{1}{L} \qquad D(\rho) = 1$

$t = L^2 \tau \qquad \sigma(\rho) = 2\rho(1-\rho)$

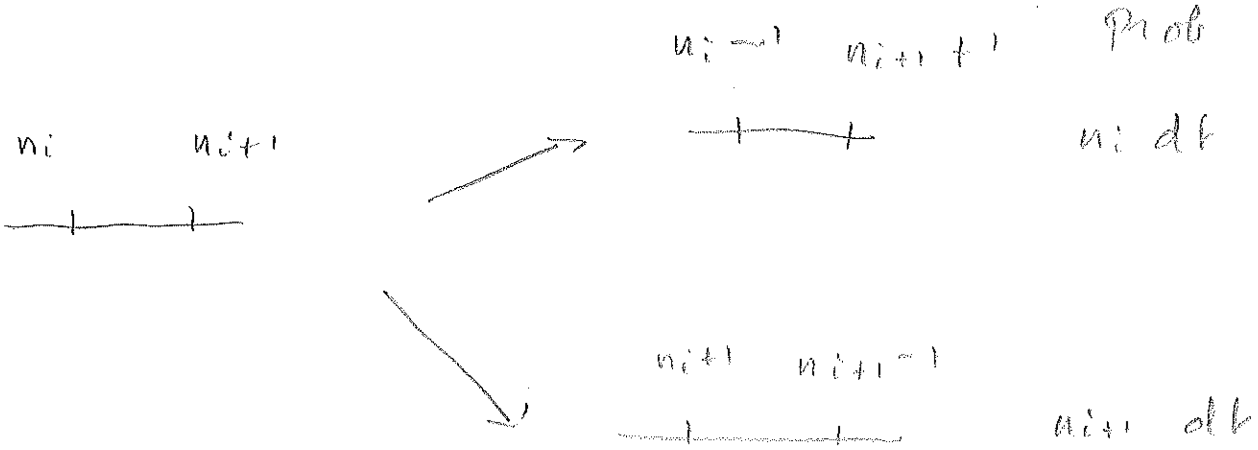
Diffusion

$J = \frac{j}{L} \qquad \left(\frac{\langle Q \rangle}{t} = \frac{J}{L} \right)$

τ	macroscopic time	$= O(1)$
j	" current	$= O(1)$
ρ		$= O(1)$

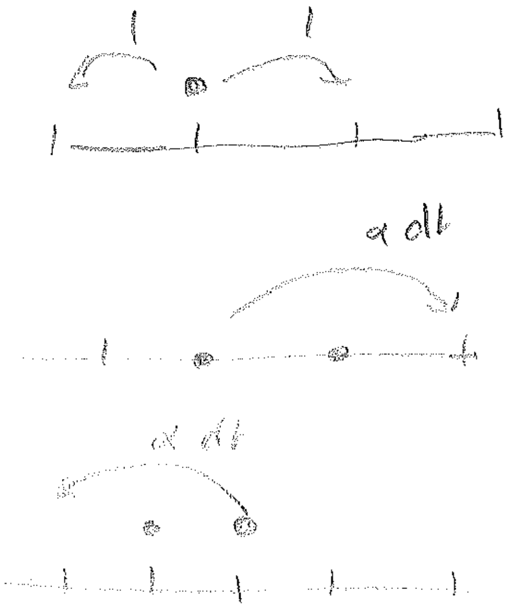
Exercices

(1) Non interacting particles



$\boxed{D = 1 \quad \sigma = 2\rho}$

(2) Jumps to second neighbours

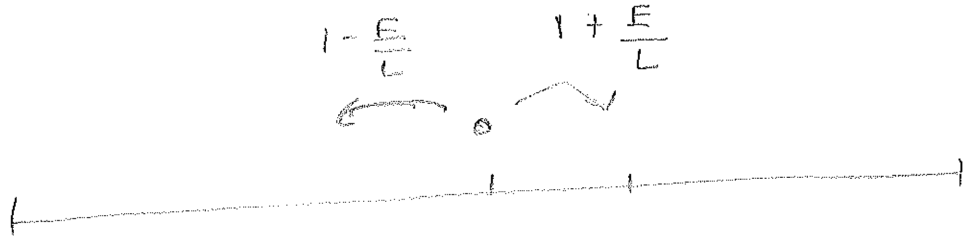


$\sigma = 2\rho(1-\rho) \; D$

$D =$

(All these cases : eq. is Bernoulli)

(3) SSEP + small field = WASEP



show that

$$j = -D(\rho)\,\rho' + \sigma(\rho)\,E + \sqrt{\epsilon\,\sigma(\rho)}\;\eta$$

x SSEP $\sigma = 2\rho(1-\rho)$