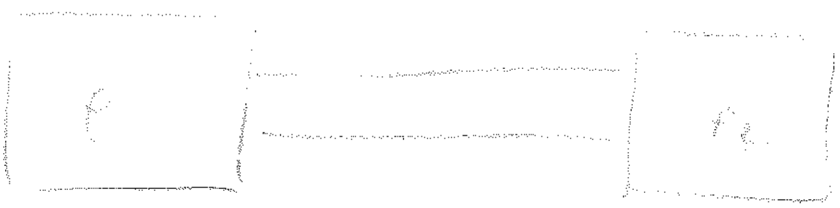
Les Houches C1

I Last time

Diffusive system



$\rho_1 - \rho_2$ small Fick's law $\frac{Q_t}{t} = \frac{D(\rho)}{L}(\rho_1 - \rho_2)$

$\rho_1 = \rho_2$ $\frac{\langle Q_t^2 \rangle}{t} = \frac{\sigma(\rho)}{L}$

<u>SSEP</u> $D = 1$ $\sigma = 2\rho(1-\rho)$

<u>non interacting</u> $D = 1$ $\sigma = 2\rho$ (low density limit of the SSEP)

Fluctuating Hydrodynamics

$$j = -D(\rho)\rho' + \sigma(\rho)E + \sqrt{\epsilon\,\sigma(\rho)}\,\eta(x,t)$$

$$\frac{d\rho}{dt} = -\frac{dj}{dx}$$

$$\Psi_t = \langle e^{\sum_i \lambda_i n_i} \rangle \sim e^{L\, G_t(\Lambda(x))}$$

$$x = \frac{i}{L}$$

$$\lambda_i = \Lambda\left(\frac{i}{L}\right)$$

II . Steady state profile (E)

$\boxed{\rho_a}$ ——— $\boxed{\rho_b}$

$$j = \langle j \rangle = -D(\rho)\,\rho'$$

$$j \int_0^x 1\, dx = -\int_0^x D(\rho)\,\rho'\, dx = -\int_{\rho_a}^{\rho(x)} D(\rho)\, d\rho$$

$\Rightarrow$ parametric form of the profile

$$\boxed{\begin{aligned} x &= \frac{1}{j}\int_{\rho(x)}^{\rho_a} D(\rho)\, d\rho \\ 1 &= \frac{1}{j}\int_{\rho_b}^{\rho_a} D(\rho)\, d\rho \end{aligned}}$$

SSEP $\boxed{D = 1}$

$$\boxed{\begin{aligned} \rho(x) &= \rho_a(1-x) + \rho_b\, x \\ j &= \rho_a - \rho_b \end{aligned}}$$

SSEP

$$\frac{d\langle n_i\rangle}{dt} = \langle n_{i+1}(1-n_i) + n_{i-1}(1-n_i) - n_i(1-n_{i+1}) - n_i(1-n_{i-1})\rangle$$

$$= \langle n_{i+1}\rangle - 2\langle n_i\rangle + \langle n_{i-1}\rangle \qquad i \neq 1, L$$

$\Rightarrow$ Steady state profile

$$\frac{d\langle n_1\rangle}{dt} = \alpha\,(1-\langle n_1\rangle) + \langle n_2\rangle - \langle n_1\rangle - \gamma\langle n_1\rangle$$

$$\langle n_i\rangle = \frac{\rho_a\,(L+b-i) + \rho_b\,(i+a-1)}{L+a+b-1}$$

$$\rho_a = \frac{\alpha}{\gamma+\alpha} \qquad \rho_b = \frac{\delta}{\beta+\delta}$$ densities of the reservoirs

$$a = \frac{1}{\gamma+\alpha} \qquad b = \frac{1}{\beta+\delta}$$ $\longleftarrow$ inverse strength of the contacts

$a \to \infty$ weak contact with contact a

a and b $= O(1)$ good contacts

III Steady State correlations

$$\frac{d}{dt} \langle n_i n_j \rangle = \langle (n_{i+1} - 2n_i + n_{i-1}) n_j \rangle + \langle n_i (n_{j+1} - 2n_j + n_{j-1}) \rangle$$

i and j not neighbours

$$\frac{d \langle n_i n_{i+1} \rangle}{dt} = \langle n_{i-1} n_{i+1} \rangle + \langle n_i n_{i+2} \rangle - \langle n_{i-1} n_i \rangle - \langle n_i n_{i+1} \rangle$$

+ boundaries

Solved by Spohn 1983

Result steady state

$$\langle n_i n_j \rangle - \langle n_i \rangle \langle n_j \rangle = -\frac{(\rho_a - \rho_b)^2 (i+a-1)(L+b-j)}{(L+a+b-1)^2 (L+a+b-2)}$$

$i < j$

Large L, good contacts

$$\boxed{x = \frac{i}{L} \; ; \; y = \frac{j}{L}} \qquad \simeq -\frac{x(1-y)(\rho_a - \rho_b)^2}{L}$$

$x < y$

- $L \to \infty$ No correlation !!

Not really so :

N = total number of particles

$$\langle N \rangle = \sum_{i=1}^{L} \langle n_i \rangle$$

$$\langle N^2 \rangle - \langle N \rangle^2 = \underbrace{\sum_{i=1}^{L} \langle n_i \rangle - \langle n_i \rangle^2}_{L} + \underbrace{2 \sum_{i<j} \underbrace{\langle n_i n_j \rangle_c}_{\frac{1}{L}}}_{L}$$

$$\langle N \rangle = \frac{L\left(\frac{1}{2}(\rho_a^2 - \rho_b^2) - \frac{1}{3}(\rho_a^3 - \rho_b^3)\right)}{\rho_a - \rho_b}$$

$$+ \frac{L}{12}(\rho_a - \rho_b)^2$$

long range correlation

- $\rho_a - \rho_b = 0$

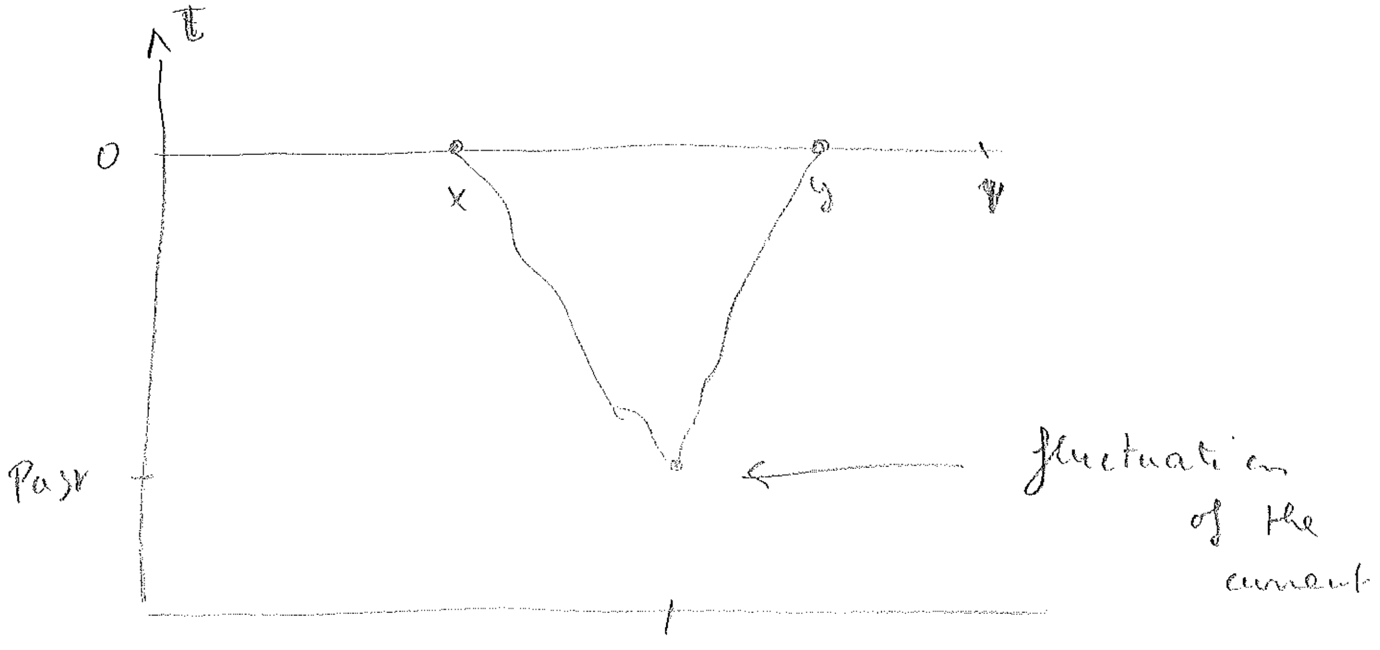
No long range correlation

IV Steady state correlations using fluctuating Hydrodynamics

The calculation of $\langle n_i \, n_j \rangle$ can be done explicitly only for few models.

But the long range correlations are generic except for very particular models

- Origin of these correlations



Expliquer le calcul

$\bar{\rho}(x)$ = steady state

$$\rho(x,\tau) = \bar{\rho}(x) + \underbrace{r(x,\tau)}_{\text{small fluctuation}}$$

$$j(x,\tau) = -D(\rho)\,\frac{d\rho}{dx} + \underbrace{\sqrt{\epsilon}\,\sqrt{\sigma(\bar{\rho})}\;\eta(x,\tau)}_{\varphi(x,\tau)}$$

$$\frac{dr}{d\tau} = \frac{d^2}{dx^2}\Big(D(\bar{\rho})\, r(x,\tau)\Big) - \frac{d}{dx}\varphi$$

$$r(x,\tau) = \int_{-\infty}^{\tau} d\tau' \int_0^1 dz \; G_{\tau-\tau'}(x;z)\left(-\frac{d\varphi(z,\tau')}{dz}\right)$$

$$= \int_{-\infty}^{\tau} d\tau' \int_0^1 dz \left(\frac{d}{dz}\, G_{\tau-\tau'}(x,z)\right)\varphi(z,\tau')$$

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where $$\frac{d}{d\tau}\, G_\tau(x;z) = \frac{d^2}{dx^2}\Big(D(\bar{\rho}(x))\cdot G_\tau(x;z)\Big)$$

$$= D(\bar{\rho}(z))\,\frac{d^2}{dz^2}\, G(x;z)$$

$x \neq y$ $\qquad \dfrac{\sigma(\bar\rho(x))}{2D(\bar\rho(x))}\,\delta(x-y)$ (8)

$$\boxed{\langle \rho(x)\,\rho(y)\rangle_c = -\,J \int_0^1 dz\, \frac{d}{dz}\left(\frac{\sigma'(\bar\rho(z))}{2\,D(\bar\rho(x))}\right) \int_0^\infty d\xi\; G_s(x;\xi)\, G_s(y;\xi)}$$

~~for general σ and ρ difficult to be more explicit~~

J stationary macroscopic current

$$J = \frac{1}{L}\int_{\rho_b}^{\rho_a} D(\rho)\,d\rho$$

Correlations $= O\left(\frac{1}{L}\right)$

$= 0$ if $J = 0$ equilibrium

$= 0$ for some models

- non interacting

$\sigma = 2\rho$, $D = 1$

$\Rightarrow \dfrac{\sigma'}{D} = \text{cste}$

- ZRP zero range process

$\sigma'(\rho) \propto D(\rho)$

- For general σ and D, it is difficult to be more explicit except for

  - $\rho_a - \rho_b$ small

(C8b)

<u>Exercise</u> . derive these correlations

$\boxed{\text{Ingredients}}$:

- $\frac{d}{d\tau_1} G_{\tau_1+\tau_2}(x;z) = \frac{d}{d\tau_2} G_{\tau_1+\tau_2}(x;z)$

$$= \int dy \; G_{\tau_1}(x;y) \frac{d}{d\tau_2} G_{\tau_2}(y;z)$$

$$= \int dy \; \frac{dG_{\tau_1}}{d\tau_1}(x;y) \; G_{\tau_2}(y;z)$$

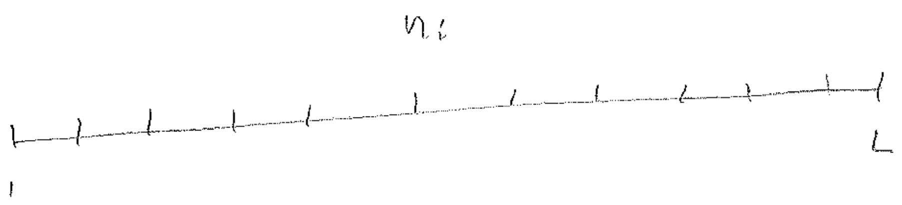
- integrations by parts

- $\langle \varphi(z,\tau)\, \varphi(z',\tau') \rangle = \sigma(\bar\rho(z))\, \delta(\tau-\tau')\, \delta(z-z')$

- $-\bar\rho'\, D(\bar\rho) = j = \frac{J}{L}$ stationary current

V Large deviations of the density at equilibrium

$d=1$

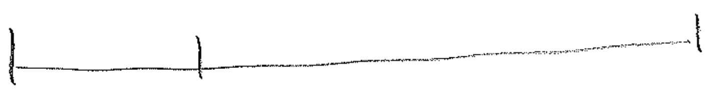


$$P(n_1, \ldots n_L) \longrightarrow P_L(\{\rho(x)\})$$

$$P_L(\{\rho(x)\}) \sim e^{-L\,\mathcal{F}(\{\rho(x)\})}$$

Quasipotential

$\mathcal{F}$ = functional



k Boxes of length $\ell = \frac{L}{k}$

L large

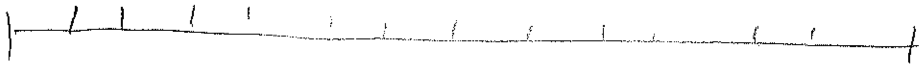
$$\sum_{\substack{i \in \\ \text{box } 1}} n_i = \ell\, r_1$$

$$\sum_{\text{box } 2} = \ell\, r_2$$

$$P(r_1, \ldots r_k) \sim e^{-\frac{L}{k} F(r_1, \ldots r_k)}$$

$\mathcal{F}(\{\rho(x)\})$ obtained by taking the limit

<u>Equilibrium</u> $\quad \ell = \frac{L}{k}$



$$P(r_1, \ldots r_k) \simeq \frac{Z_\ell(r_1) \ldots Z_\ell(r_k)}{Z_L} e^{O(L)} \longleftarrow \text{normalisation } e^{L F_0}$$

$$\sim e^{-\ell f(\rho_1) - \ell f(\rho_2) \ldots - \ell f(\rho_k)} \qquad (kT = 1)$$

$\Rightarrow$

$$P(\{\rho(x)\}) \sim e^{-L \int f(\rho(x))\, dx}$$

$\Rightarrow$

$$\mathcal{F}(\{\rho(x)\}) = \int f(\rho(x))\, dx - F_0$$

$\mathcal{F}(\{\rho(x)\})$ = quasipotential = free energy

C11

$$\boxed{\mathcal{F} = \text{local}}$$

$$\langle e^{\sum_i \alpha_i n_i} \rangle \simeq \langle e^{L \int_0^1 \Lambda(x) \rho(x)\, dx} \rangle \simeq e^{L\, G(\Lambda(x))}$$

$$\alpha_i = \Lambda\left(\frac{i}{L}\right)$$

$$\begin{cases} G(\{\Lambda(x)\}) = \max_{\rho(x)} \left[\int \Lambda(x)\rho(x)\,dx - \mathcal{F}(\{\rho(x)\}) \right] \\ \mathcal{F}(\{\rho(x)\}) = \min_{\{\Lambda(x)\}} \left[G(\{\Lambda(x)\}) - \int \Lambda(x)\rho(x)\,dx \right] \end{cases}$$

Legendre transforms

$\Rightarrow$ $\boxed{\mathcal{F} \text{ local} \Leftrightarrow G \text{ local}}$ at equilibrium

~~Non equilibrium~~ —

VI Large deviation functional of the density out of equilibrium

(C12)

$$\Psi = \langle e^{\sum_{i=1}^{L} \lambda_i n_i} \rangle \leftarrow \text{steady state}$$

$$= \sum_{n_1} \cdots \sum_{n_L} e^{\sum \lambda_i n_i} P_{\text{steady state}}(n_1, \ldots n_L)$$

$$\simeq e^{L\, G(\{\Lambda(x)\})} \qquad \lambda_i = \Lambda\left(\frac{i}{L}\right)$$

$$G(\{\Lambda(x)\}) = \max_{\{\rho(x)\}} \left[\int \Lambda(x)\rho(x)dx - \mathcal{F}(\{\rho(x)\}) \right]$$

$$\frac{\log \Psi}{L} = \sum_i \lambda_i \langle n_i \rangle + \sum_i \frac{\lambda_i^2}{2} \left(\langle n_i^2 \rangle - \langle n_i \rangle^2 \right)$$

$$+ \sum_{i<j} \lambda_i \lambda_j \left(\langle n_i n_j \rangle - \langle n_i \rangle \langle n_j \rangle \right)$$

$$+ O(\lambda^3)$$

$$\langle n_i n_j \rangle_c = \frac{c_2(x,y)}{L} \qquad \langle n_i \rangle = \bar{\rho}(x)$$

$$\langle n_i^2 \rangle - \langle n_i \rangle^2 = v(x)$$

$$G = \int_0^1 \rho(x)\, \Lambda(x)\, dx + \int_0^1 \frac{v(x)\, \Lambda(x)^2}{2} dx$$

$$+ \int_0^1 dx \int_x^1 dy \quad c_2(x,y)\, \Lambda(x)\, \Lambda(y)$$

$$+ O(\Lambda(x))$$

G is <u>non local</u>

- If $c_2(x,y) \neq 0 \Rightarrow G$ non local

- Expanding in powers of $\Lambda(x)$
  (in other words if G can be expanded
  in powers of $\Lambda(x)$ and this is not
  garanteed) $\Rightarrow$

C14

$$\bar{\rho}(x) = \frac{\delta G}{\delta \Lambda(x)}\Big|_{\Lambda=0}$$

$$\langle \rho(x)\rho(y)\rangle_c = \frac{1}{L}\frac{\delta^2 G}{\delta\Lambda(x)\delta\Lambda(y)}\Big|_{\Lambda=0}$$

$$\langle \rho(x)\rho(y)\rho(z)\rangle_c = \frac{1}{L^2}\frac{\delta^3}{\delta\Lambda(x)\delta\Lambda(y)\delta\Lambda(y)}\Big|_{\Lambda=0}$$

$$\underbrace{\cdots\cdots}_{p\text{ points}} = \frac{1}{L^{p-1}}\cdots$$

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order 1 in Λ $\Rightarrow$ LLN

2 in Λ $\rightarrow$ CLT

$\rho(x)$ gaussian fluctuations around $\bar{\rho}(x)$

$\Lambda = O(1)$ $\Rightarrow$ large deviations

$\mathcal{F}(\{\rho(x)\})$ has been computed for

few models

- non interacting
- Zero range process

} no long range correlations

- SSEP

Microscopic calculation
D. Lebowitz Speer 2001

The Italians 2001 →

Macroscopic Fluctuation Theory

$$\mathcal{F}(\{\rho(x)\}) = \max_{F(x)} \left[\rho(x) \log \frac{\rho(x)}{F(x)} + (1-\rho(x)) \log \frac{1-\rho(x)}{1-F(x)} + \log \frac{F'(x)}{\rho_b - \rho_a} \right]$$

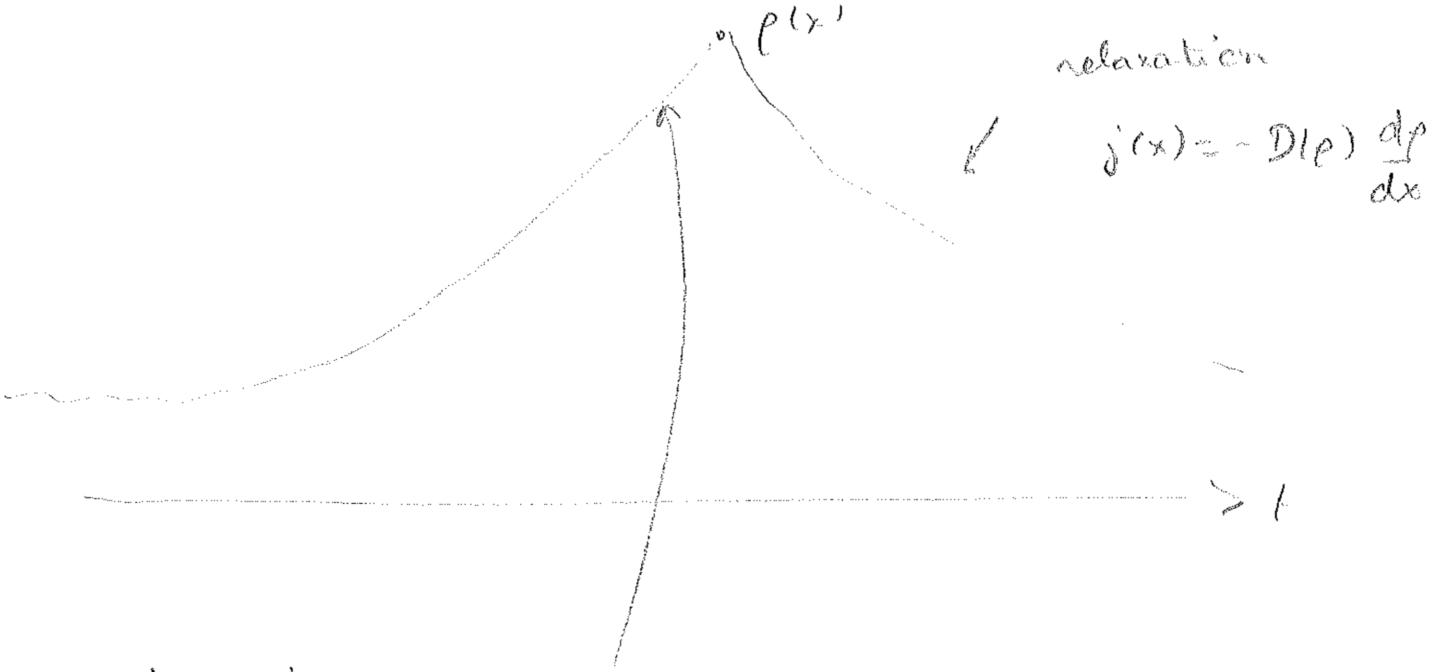
$F(0) = \rho_a$
$F(1) = \rho_b$
F monotone

$\boxed{\underline{\max} \text{ : surprising}}$

$$\rho(x) = F(x) + \frac{F(1-F)F''}{F'^2}$$

non locality

Last remark



Last time $\quad j = -D(\rho)\dfrac{d\rho}{dx} + \sigma(\rho)\dfrac{d\Lambda(x)}{dx}$

$$G(\{\Lambda(x)\}) = \max_{\rho(x)}\left\{-\mathcal{F}(\{\rho(x)\}) + \int \Lambda(x)\rho(x)\,dx\right\}$$

$$\Rightarrow \Lambda(x) = \frac{\delta \mathcal{F}}{\delta \rho(x)}$$

Eq.

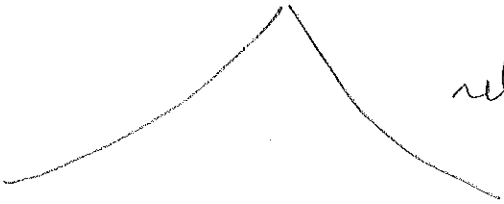
$$\mathcal{F} = \int f(\rho(x))\,dx$$

$$\frac{\delta \mathcal{F}}{\delta \rho(x)} = f'(\rho(x))$$

$$\frac{d\Lambda}{dx} = f''(\rho(x))\,\rho'$$

$$f'' = \frac{2D}{\sigma}$$

$$j = D(\rho)\,\frac{d\rho}{dx}$$



relaxation path ≡ symmetric of the excitation path.