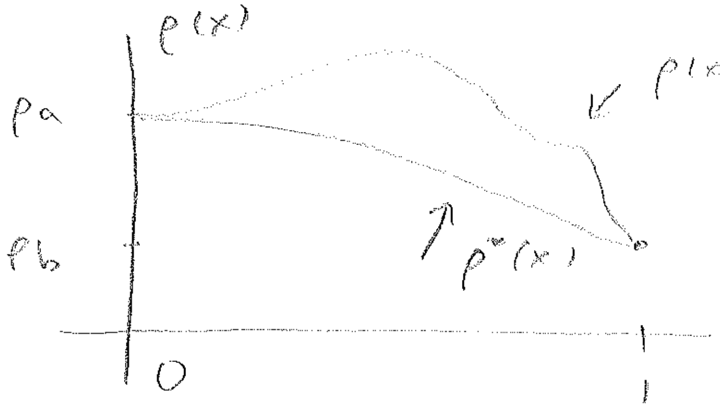
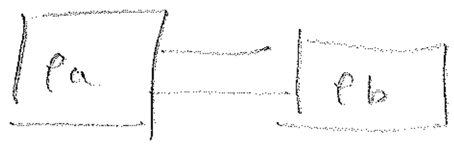
Les Houches

I Last time

Large deviations of the density in the steady state





$$P_{\text{steady state}}(\{\rho(x)\}) \sim e^{-L\,\mathcal{F}_{st.st.}(\{\rho(x)\})}$$

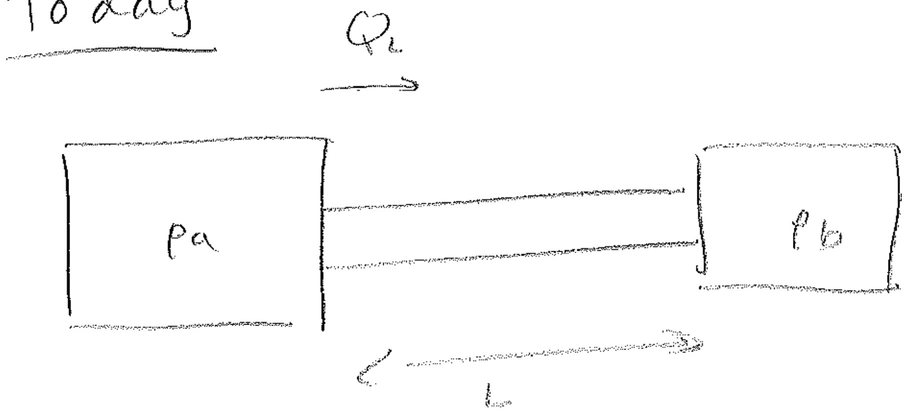
$$\langle e^{\sum \lambda_i n_i} \rangle_{\text{steady state}} \simeq e^{L\,G_{st.st.}(\{\Lambda(x)\})}$$

$$\lambda_i = \Lambda\left(\frac{i}{L}\right)$$

G or $\mathcal{F}$ only known for few models

very few models — SSEP, KMP, WASEP, non interacting

Today



large deviation function of the current

$$P\left(\frac{Q_t}{t} = J\right) \sim e^{-t\, I(J)}$$

can be computed for "general" $\sigma(\rho)$ and $D(\rho)$

II Convexity of $I(J)$

$$P_t(J) \geq P_{t\alpha}(J_1)\; P_{t(1-\alpha)}(J_2)$$

$$J = \alpha J_1 + (1-\alpha) J_2$$

$$\Rightarrow \quad I(J) = I(\alpha J_1 + (1-\alpha) J_2)$$

$$\leq \alpha I(J_1) + (1-\alpha) I(J_2)$$

$I(J)$ convex

Markov process:

$$P(\mathcal{C}_t, Q_t \mid \mathcal{C}_0)$$

$$= \sum_{\mathcal{C}_{t_1}} \sum_{Q_{t_1}} P(\mathcal{C}_t, Q_t - Q_{t_1} \mid \mathcal{C}_{t_1}) P(\mathcal{C}_{t_1}, Q_{t_1} \mid \mathcal{C}_0)$$

For large t

$$P(\mathcal{C}_t, Q_t \mid \mathcal{C}_0) \simeq A(\mathcal{C}_t)\, e^{-t I\left(\frac{Q_t}{t}\right)} B(\mathcal{C}_0)$$

(same as largest eigenvalue of the tilted matrix)

$$\Rightarrow \boxed{I(J) = \max_{J_1}\left[\alpha I(J_1) + (1-\alpha) I\left(\frac{J - \alpha J_1}{1-\alpha}\right)\right]}$$

III Diffusive systems

On large scale

	macroscopic		microscopic
	τ	$=$	$\frac{t}{L^2}$
	x	$=$	$\frac{i}{L}$

$$j = -D(\rho)\,\rho' + \sqrt{\epsilon\,\sigma(\rho)}\;\eta(x,\tau)$$

$$\frac{d\rho}{d\tau} = -\frac{dj}{dx}$$

$$\epsilon = \frac{1}{L}$$

$$\text{Prob}\left(\{\rho(x,\tau'),\,j(x,\tau')\}_{0\leq\tau'\leq\tau}\right)$$

$$\sim e^{-L\int_0^{\tau}d\tau'\int_0^1 dx\,\frac{(j+D(\rho)\rho')^2}{2\sigma(\rho)}}$$

DJ

Bodineau 2004

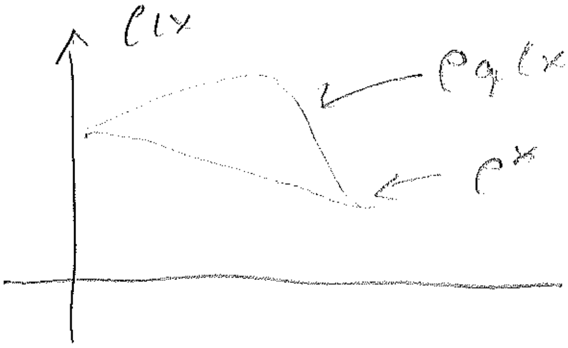
$$q = \frac{1}{\tau} \int_0^{\tau} j(x,\tau')\, d\tau'$$ $\boxed{\text{indep of } x}$

$$\boxed{P(q) \sim e^{-\tau I(q)}}$$?

<u>Guess</u>:

optimal $j(x,\tau') = q$ indep. of time

$\Rightarrow$ $\rho_q(x,\tau') = \rho_q(x)$ ind. of time



$$I(q) = \min_{\rho(x)} \underbrace{\int_0^1 \frac{(q + D(\rho)\rho')^2}{2\sigma(\rho)}\, d\rho}$$

looks like a Lagrangian $\mathcal{L}(\rho, \rho')$

$\mathcal{L}$ $\longrightarrow$ Hamiltonian $\rightarrow$ Conserved quantity

Lagrangian

Remark usually it is time, here it is x

$\mathcal{L}(q, \dot{q})$ here $\mathcal{L}(\rho, \rho')$

Solution for $\rho_{opt}(x, q)$

$$D(\rho)\left(\frac{d\rho}{dx}\right)^2 = q^2\left(1 + 2k\sigma(\rho)\right)$$

$\Rightarrow$ given q $\rightarrow$ ρ_{opt}

$$\Rightarrow \quad q = \int_{\rho_b}^{\rho_a} d\rho \, \frac{D(\rho)}{\sqrt{1+2k\sigma(\rho)}}$$

$$I(q) \quad q \int_{\rho_b}^{\rho_a} \frac{D(\rho)}{\sigma(\rho)}\left[-1 + \frac{1+k\sigma(\rho)}{\sqrt{1+2k\sigma(\rho)}}\right]$$

$k=0 \quad \Rightarrow \quad I(q) = 0$

$$q = \int_{\rho_b}^{\rho_a} D(\rho)\, d\rho$$

$$Q = \int_0^{\tau} j(x, \tau')\, d\tau'$$

$$\frac{\langle Q \rangle}{\tau} = I_1 \qquad \text{large } \tau$$

$$\frac{\langle Q^2 \rangle - \langle Q \rangle^2}{\tau} \quad \frac{I_2}{I_1}$$

$$\frac{\langle Q^3 \rangle_c}{t} = \frac{\langle Q^3 \rangle - 3\langle Q^2 \rangle \langle Q \rangle + 2\langle Q \rangle^3}{t} = \frac{3(I_3 I_1 - I_2^2)}{I_1^5}$$

$\vdots$ where $I_n = \int_{\rho_b}^{\rho_a} D(\rho)\, \sigma(\rho)^{n-1}\, d\rho$

$\rho_a - \rho_b$ small

$$I_1 \simeq D(\rho)(\rho_a - \rho_b)$$

$$I_2 = D(\rho)\, \sigma(\rho)\, (\rho_a - \rho_b)$$

O.K.

Remark: Fluctuation theorem

$$\boxed{I(q) - I(-q) = -2q \int_{\rho_b}^{\rho_a} \frac{D(\rho)}{\sigma(\rho)}\, d\rho}$$

$$\sqrt{1 + 2K\sigma} \longrightarrow -\sqrt{1 + 2K\sigma}$$

$\rho_{opt} = \rho_{opt}$ for q and $-q$

So $I(q) - I(-q)$ is

linear in q

Soon after our paper in 2005

Rome group

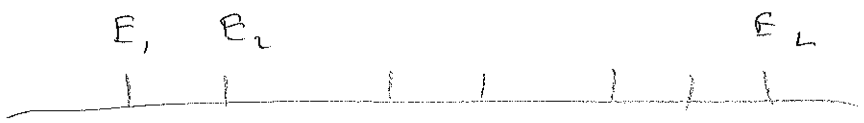
O.K. but one needs $\boxed{\sigma(\rho) \text{ to be concave}}$

SSEP $\quad \sigma = 2\rho(1-\rho)$ ✓

RW $\quad \sigma = 2\rho$ ✓

KMP model $\quad \sigma = 2\rho^2$

<u>KMP model</u> Kipnis, Marchioro, Presutti 1982



$$E_i \quad E_{i+1} \xrightarrow[dt]{\text{Prob}} E'_i \quad E'_{i+1}$$

$$E'_i = (E_i + E_{i+1})\, y \qquad 0 < y < 1$$

$$E'_{i+1} = (E_i + E_{i+1})(1-y)$$

$$\langle J_i \rangle = \left(\langle (E_{i+1} + E_i)(1-y) \rangle - \langle E_{i+1} \rangle \right)$$

$$= \frac{\langle E_{i+1} \rangle - \langle E_i \rangle}{2}$$

$$\boxed{D = \frac{1}{2}} \qquad \langle E \rangle = T$$

We have seen : transport of energy

$$\sigma = 2 k T^2 D$$

So $\left(\begin{matrix} \sigma \sim \rho^2 & \langle E \rangle = \rho \\ D = \frac{1}{2} & \end{matrix} \right)$

D10

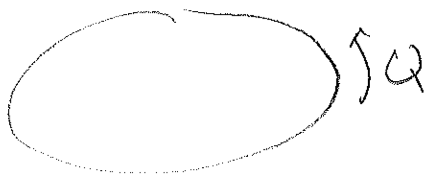
Bodineau 2005

IV Phase transitions

$$I(q) = \min_{\substack{\rho(x,\tau') \\ j(x,\tau')}} \frac{1}{\tau}\int_0^\tau \left(\frac{\left(j(x,\tau') + D(\rho)\rho'\right)^2}{2\sigma(\rho)} \right)$$

$$q = \frac{1}{\tau}\int_0^\tau d\tau' \, j(x,\tau')$$

ii) On a ring   flat profile becomes unstable



$$\rho(x,t) = \rho_0 + \epsilon \sin 2\pi(x - vt)$$

$$j = q + \epsilon v \sin\left(2\pi(x - vt)\right)$$

unstable if

$$\boxed{8\pi^2 D(\rho_0)\sigma(\rho_0) - q^2 \sigma''(\rho_0) < 0}$$

D 11

Open system ρ_a and ρ_b :

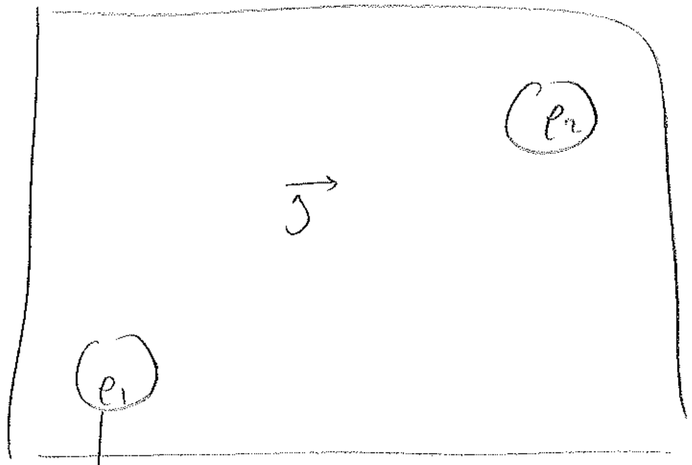
- more difficult because

  ρ_{opt} depends on x

- first order transitions ?

- transition $\rightarrow$ periodic $\rightarrow$ more complicated.

V Higher dimension



contact with
Heat Bath.

Isotropic case

$$\vec{j} = -D(\rho)\,\vec{\nabla}(\rho) + \sqrt{\epsilon\,\sigma(\rho)}\;\vec{\eta}(x,t)$$

Fluctuating hydrodynamics

Large deviation

$\boxed{1\ d}$

$$j_1 = -D(\rho_1)\,\rho_1' + \sigma(\rho)\,\frac{d}{dx}H_1(x)$$

noise to produce deviation

time independent profile

$\underline{d > 1}$

D13

Coste $\int_{space} \int_{time} \frac{\left(\vec{j} + D(\rho)\vec{\nabla}\rho\right)^2}{2\sigma(\rho)}$

$v(\vec{r})$ = solution of

$\Delta v = 0$ with

$v(\partial A) = \rho_1$

$v(\partial B) = \rho_2$ $\Rightarrow v(\vec{r})$

$\rho_d(\vec{r}) = \rho_1(v(r))$

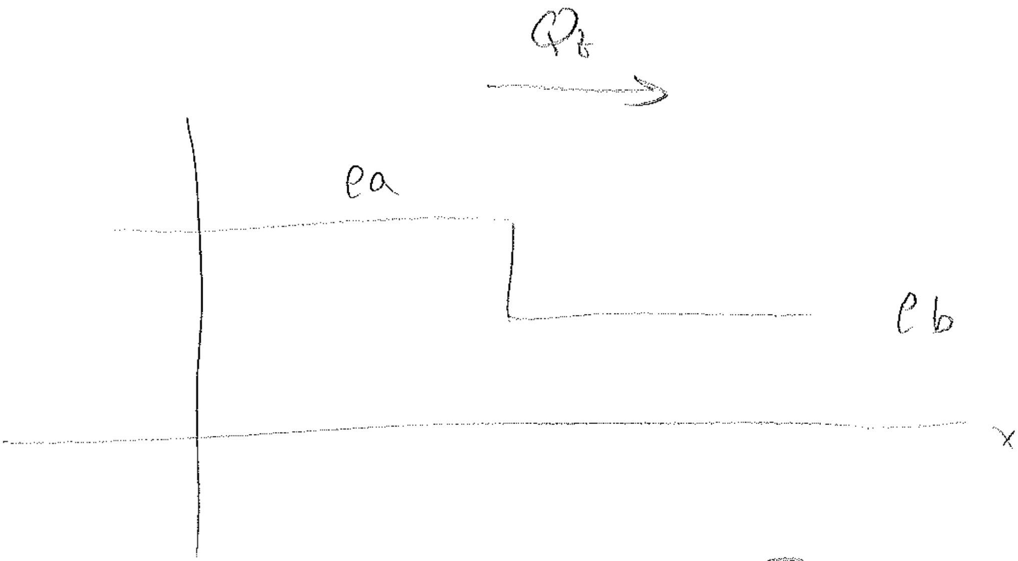
$$\boxed{\vec{j} = -D(\rho_1)\nabla\rho_1 + \sigma(\rho_1)\vec{\nabla} H_1(v(r))}$$

So far

All we discussed was

the steady state

VII Infinite line



$$P(Q_t = q\sqrt{t}) \sim e^{-\sqrt{t}\, I(q)}$$

Different scaling :

not surprising

Diffusive $\qquad L \sim \sqrt{t}$

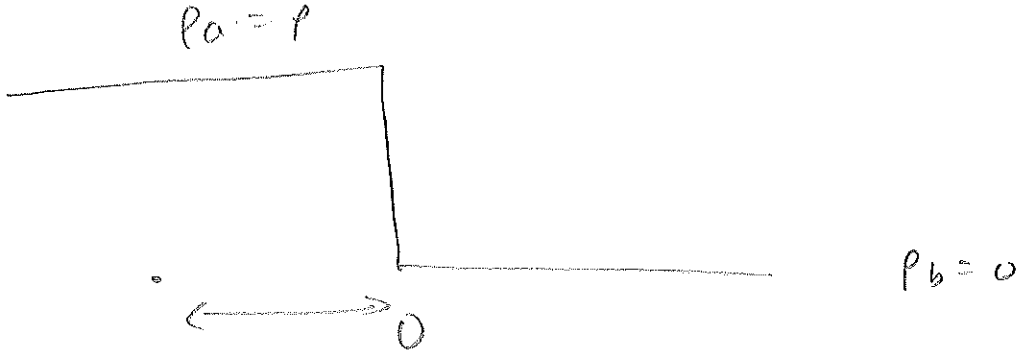
$I(q)$ depends on the initial condition

giving ρ_a and ρ_b is not sufficient

Much harder because the profile becomes time dependent

Soluble case: independent (random walker) Brownian motion

Q_t $\longrightarrow$



$p(x)$ = prob that a particle starting at distance x from the origin will be on the right at time t

$$\boxed{p(x) = \int_x^\infty \frac{e^{-\frac{y^2}{t}}}{\sqrt{\pi t}} = f\left(\frac{x}{\sqrt{t}}\right)}$$

D 16

Annealed case

Prob that a particle in $dx^{(x)}$ is $\rho\, dx$

Average over initial conditions

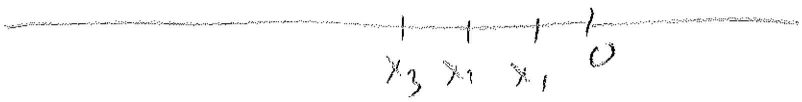
$$\langle e^{\lambda Q} \rangle = \prod_{\substack{dx \\ x>0}} \left[1 - \rho\, dx + \rho\, dx \left(1 - f\left(\tfrac{x}{\sqrt{t}}\right) + f\left(\tfrac{x}{\sqrt{t}}\right) e^{\lambda} \right) \right]$$

$$= \exp \int_0^\infty dx \;\; \rho\, f\left(\tfrac{x}{\sqrt{t}}\right) \left(e^{\lambda} - 1\right)$$

$$\boxed{\begin{aligned} \langle e^{\lambda Q} \rangle &= \exp\left[(e^{\lambda} - 1) \sqrt{t}\, \rho \int_0^\infty du\, f(u) \right] \\ &= \exp\left(\sqrt{t}\, \rho\, (e^{\lambda} - 1)\, \frac{1}{2\sqrt{\pi}} \right) \end{aligned}}$$

We have allowed fluctuations in the initial density

Quenched case



$$\boxed{x_i = \frac{i}{\rho}}$$

D17

$$\langle \exp \lambda Q \rangle = \prod_i \left(1 - p(x_i) + p(x_i)\, e^{\lambda}\right)$$

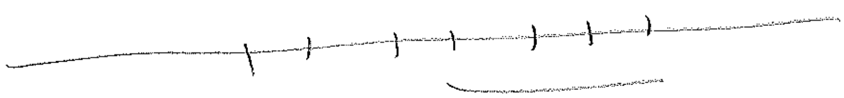
$$= \exp \sum_i \log\left[1 - p(x_i) + p(x_i) e^{\lambda}\right]$$

$$= \exp \rho\sqrt{t} \int_0^{\infty} \log\left[1 + (e^{\lambda} - 1)\, p(u)\, du\right]$$

Large $\lambda > 0$

annealed $\qquad P\left(\frac{Q}{\sqrt{t}} \simeq q\right) \sim e^{-\sqrt{t}\, A q \log q}$

quenched $\qquad P\left(\frac{Q}{\sqrt{t}} \simeq q\right) \sim e^{-B q^3 \sqrt{t}}$



$$e^{-\frac{x^2}{t}}$$

$$\prod_i \exp -\frac{x_i^2}{t} \simeq \exp -\frac{x^3}{t}$$

$$\boxed{X = Q\sqrt{t}}$$

D18

SSEP

~~2000 - 2010~~

~~Gerschenfeld~~

~~Other works recent say the profiles~~

2009 Gerschenfeld

$$\langle e^{\lambda Q} \rangle \sim e^{\sqrt{t} \int_{-\infty}^{+\infty} \frac{dk}{\pi} \log\left(1 + \rho(e^{\lambda}-1)\, e^{-k^2}\right)}$$

SSEP annealed

quenched not known!