Les Houches

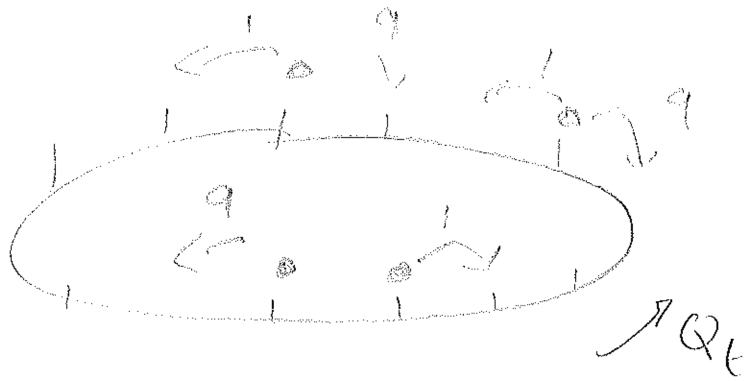
I Diffusive versus driven systems

Examples:

SSEP WASEP — diffusive

ASEP TASEP — driven

N particles on a ring of L sites



$q = 1$ SSEP symmetric simple exclusion process

$q = 1 - \frac{\nu}{L}$ WASEP ($\uparrow$ weakly δ)

$q < 1$ ASEP

$q = 0$ TASEP

Steady state of all these systems :

all configurations are equally likely

N particles on L sites

(Note that for $q \neq 1$, they don't satisfy detailed balance)

~~for~~ But :

$$\frac{dP(\mathcal{C})}{dt} = \sum_{\mathcal{C}'} M(\mathcal{C},\mathcal{C}')\, P(\mathcal{C}') - M(\mathcal{C}',\mathcal{C})\, P(\mathcal{C})$$

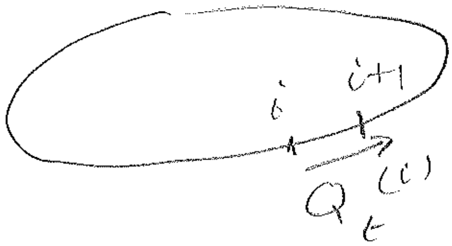
if all $P(\mathcal{C})$ are equal $= P$

$$\sum_{\mathcal{C}'} M(\mathcal{C},\mathcal{C}') = (1+q)\; n(\mathcal{C})$$

number of clusters in $\mathcal{C}$

$$\sum_{\mathcal{C}'} M(\mathcal{C},\mathcal{C}') = (1+q)\; n(\mathcal{C})$$

So: $P(\mathcal{C})$ indep of $\mathcal{C}$ is the steady state



$$J = \frac{dQ_t^{(i)}}{dt} = \langle n_i(1-n_{i+1})\rangle - q\langle n_{i+1}(1-n_i)\rangle$$

$$= \frac{N(L-N)(1-q)}{L(L-1)}$$

Steady state

$$\simeq \rho(1-\rho)(1-q)$$

For $q = 1 - \frac{d}{L} \Rightarrow J = \rho(1-\rho)\frac{d}{L}$

diffusive

TASEP $q = 0$ $J = \rho(1-\rho)$

driven

Hydrodynamic equation

$$\cancel{\frac{d\rho}{dt} = -\frac{dJ}{dx}}$$

q indep of L $J = \rho(1-\rho)(1-q)$

II Hydrodynamic equation

Suppose a situation close to equilibrium where the density varies on a large scale L

$$\langle n_i \rangle = \rho\left(\frac{i}{L}\right) \qquad \frac{i}{L} = x$$

Diffusive system	Driven system
microscopic current $J_i = \frac{1}{L} j(x)$	$J = j(x)$
evolution $\frac{d\rho}{dt} = J_{i-1} - J_{i} = -\frac{1}{L^2}\frac{dj}{dx}$	$\frac{d\rho}{dt} = -\frac{1}{L}\frac{dj}{dx}$
time $\sim L^2$	$\sim L$
	$J = \rho(1-\rho)$
	TASEP

$$\frac{d\rho}{d\tau} = -\frac{d}{dx}\left[\rho(1-\rho)\right]$$

Burgers equation

deterministic equation

$$\rho(x,0) = \rho_0(x)$$

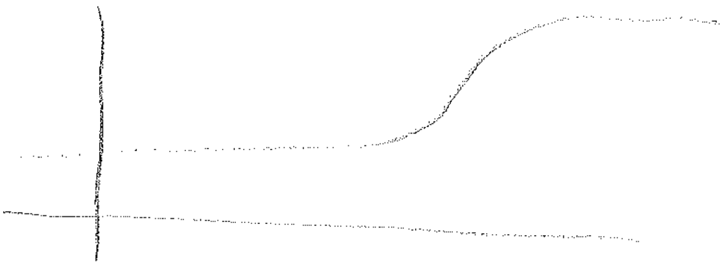
can be solved in a parametric form : one varies y

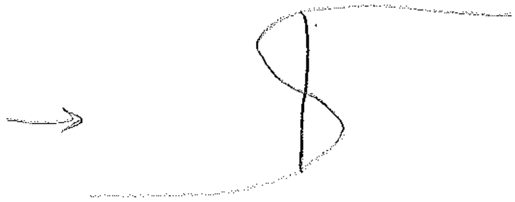
$$\rho(x,t) = \rho_0(y)$$

$$x = y + \left(1 - 2\rho_0(y)\right) t$$

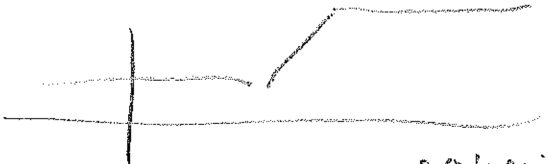
Exercice : show that this is a solution of Burgers equation

Generalise it to $\frac{d\rho}{dt} = -\frac{d}{dx} f(\rho)$





shocks

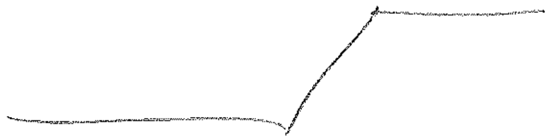


easier

E6

$$\frac{d\rho}{dt} = -\frac{d}{dx}\left(\rho(1-\rho)\right)$$

$$\rho = A(t) + B(t)x$$



$$\frac{dB}{dt} = 2B^2$$

$$\frac{dA}{dt} = 2AB - B$$

$$\boxed{B(t) = \frac{B(0)}{1-2B(0)t}}$$

$B(0) < 0$

fan

$B(t) \to 0$

$B(0) > 0$

shock

$B(t) = \infty$ at

$t = \frac{1}{2B(0)}$

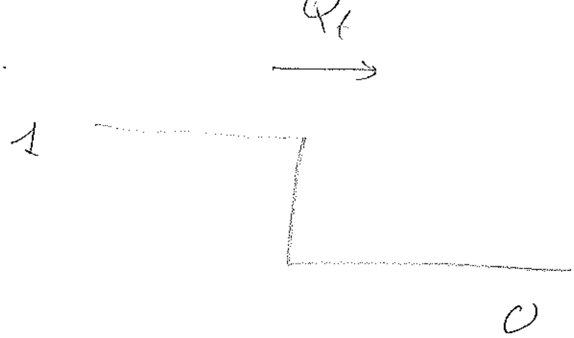
III Fluctuations of the current

Burgers + Noise ($\Leftarrow$)

Kardar - Parisi Zhang 1986

KPZ

Ex.



TASEP

$$Q_t = \frac{t}{4} + \text{fluctuation}$$

(diffusive $\sqrt{t}$)

$\downarrow$

$t^{1/3} \times$ random variable

$\downarrow$

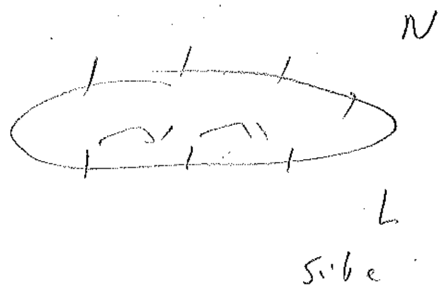
Tracy Widom distribution

(random matrix)

~ 2008

J. Lebowitz E7a

On the ring



exact

$P_{N,L}(Q = qt) \sim \exp - t \underbrace{F_{N,L}(q)}_{\text{explicitly}}$ for all N and L

Universal scaling function

$\frac{N}{L} = \rho$

$q > q_{typ}$ F much large than

$q < q_{typ}$: F_{slow}

slow down

F slower much less F faster

Bethe ansatz calculation q large

$\boxed{q=0}$ e^{-t}

$q \to \infty$ $e^{-t(N q \log q)}$

E7b

TASEP

D. Lebowitz 98

D. Appert 99

$$\langle e^{\lambda Q_t} \rangle \sim e^{\mu(\lambda) t}$$

$$\mu(\lambda) = \rho(1-\rho)\lambda + \sqrt{\frac{\rho(1-\rho)}{2\pi N^3}}\; G\left(\lambda \sqrt{N(\rho(1-\rho))}\right)$$

G universal function

$$G(\beta) = \sum_{n \geq 1} (-)^{n+1} \frac{z^n}{n^{5/2}}$$

$$\beta = \sum_{n \geq 1} (-)^{n+1} \frac{z^n}{n^{3/2}}$$

G universal

IV. Matrix method for the

Open systems } in the steady state

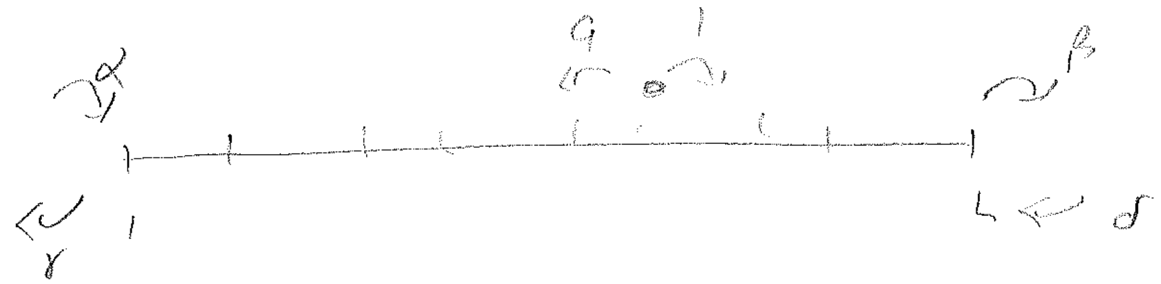
goal $\longrightarrow$ $P_{ss}(\{\rho(x)\}) \sim e^{-L\,\mathcal{F}(\{\rho(x)\})}$

Matrix method (initially used for Quantum Systems)

$\boxed{\text{Exact description of the steady state}}$

for the SSEP, ASEP --

Evans, Hakim, Pasquier 1992



$\boxed{P_{\text{steady state}}(n_1, \cdots n_L) \text{ can be determined exactly}}$

$\langle W| = (W_1, W_2 \cdots W_K)$ ← vectors

$$|V\rangle = \begin{pmatrix} V_1 \\ \vdots \\ V_K \\ \vdots \end{pmatrix}$$

D, E matrices or operators

$$P(\tau_1, \dots \tau_L) = \frac{1}{Z_L} \langle W | \prod_{i=1}^{L} (n_i D + (1-n_i) E) | V \rangle$$

$$\left.\begin{aligned} &DE - qED = D + E \\ &\langle W | (\alpha E - \gamma D) = \langle W | \\ &(\beta D - \delta E) | V \rangle = | V \rangle \end{aligned}\right\}$$

decreases the degree

(Note that any product

$$D^{n_1} E^{m_1} D^{n_2} E^{m_2} \dots$$

can be reduced to sums of

$$E^m D^n \quad)$$

(2^L numbers reduce to L^2 numbers

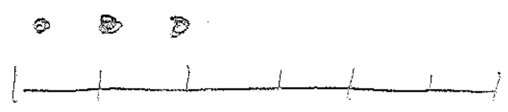
E9a

Example TASEP $\gamma = \delta = 0$

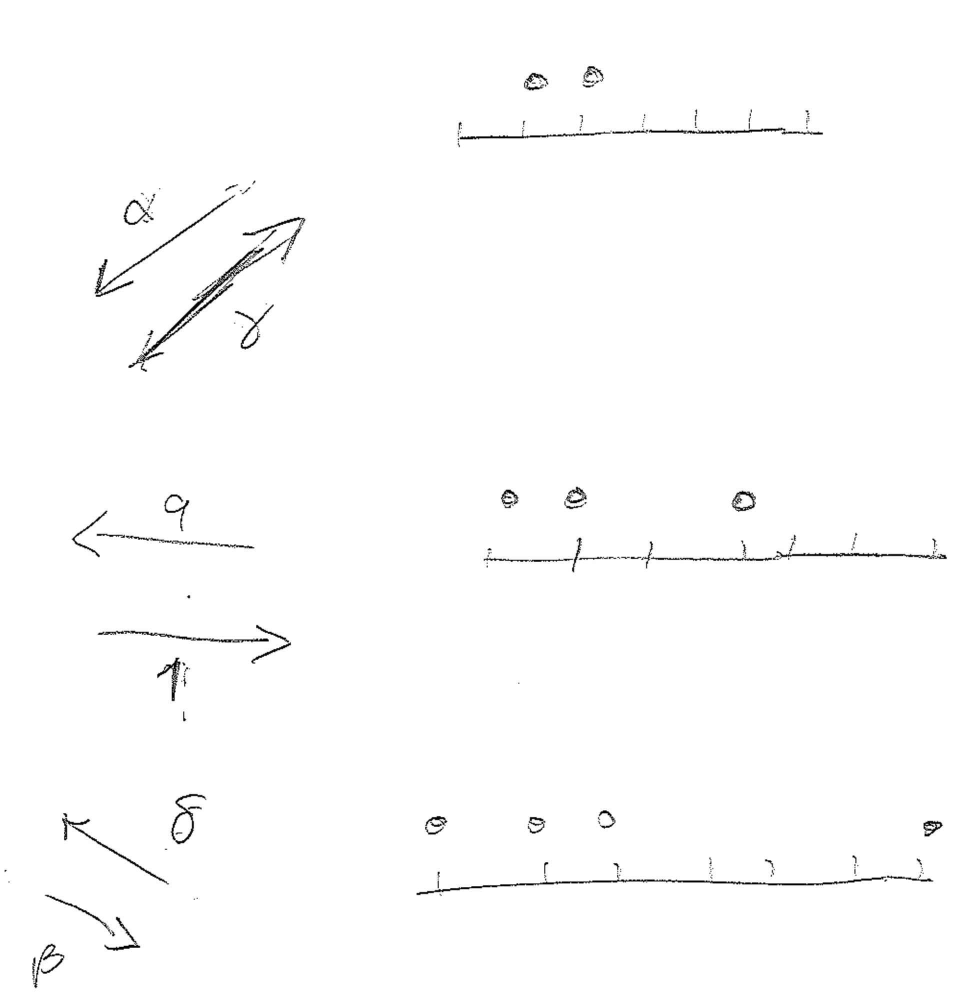
$$\langle W | E^n D^m | V \rangle = \frac{1}{\alpha^n} \frac{1}{\beta^m}$$

E.10

Check



$\langle W | D^3 E^4 \rangle$



- gain   Loss

$$\alpha \langle W | E D^2 E^4 | V \rangle - \gamma \langle W | D^3 E^4 | V \rangle = \langle W | D^2 E^4 | V \rangle$$

$$+ q \langle W | D^2 E D E^3 | V \rangle - \langle W | D^3 E^4 | V \rangle = - \langle W | D^2 (D+E) E^3 | V \rangle$$

$$+ \beta \langle W | D^3 E^3 D | V \rangle - \delta \langle W | D^3 E^4 | V \rangle = \langle W | D^3 E^3 | V \rangle$$

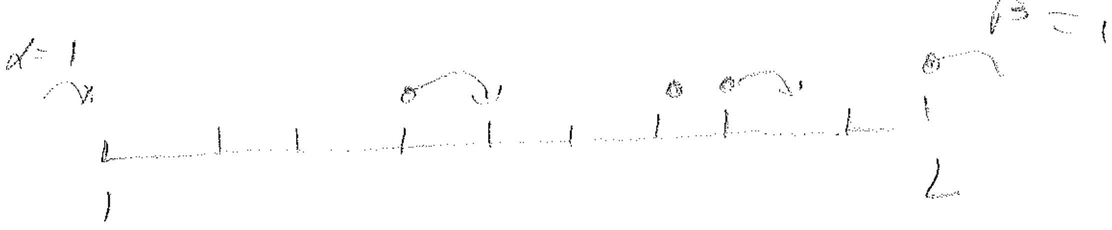
E11

Can one find such matrices

Yes

Example $\alpha = 1$ $\beta = 1$ TASEP

$\gamma = 0$ $\delta = 0$



$$D = \begin{pmatrix} 1 & 1 & & & 0 \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ 0 & & & \ddots & \ddots \end{pmatrix} \qquad E = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ & \ddots & \ddots & \\ & & & \ddots \end{pmatrix}$$

$$\langle W| = (1, 0 \ldots\ldots)$$

$$|V\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}$$

(E 112)

Is it necessary : no

the algebric rules allow to calculate all the weights

V Steady state properties

(1) Profile

$$\langle n_i \rangle = \frac{\langle W | (D+E)^{i-1} D (D+E)^{L-i} | V \rangle}{\langle W | (D+E)^L | V \rangle}$$

(2) correlations

$$\langle n_i n_j \rangle = \frac{\langle W | (D+E)^{i-1} D (D+E)^{j-i-1} D (D+E)^{L-j} | V \rangle}{\langle W | (D+E)^L | V \rangle}$$

(3) current in the steady state

$$\langle n_i (1-n_{i+1}) \rangle - q \langle n_{i+1} (1-n_i) \rangle$$
$$= \frac{\langle W | (D+E)^{i-1} (DE - qED) D^{L-i-1} | V \rangle}{\langle W | (D+E)^L | V \rangle}$$

The current is

$$\frac{\langle W | (D+E)^{L-1} | V \rangle}{\langle W | (D+E)^{L} | V \rangle}$$

VI Non gaussian fluctuations

Prob on large scale of a density profile

TASEP $\alpha = 1$, $\beta = 1$

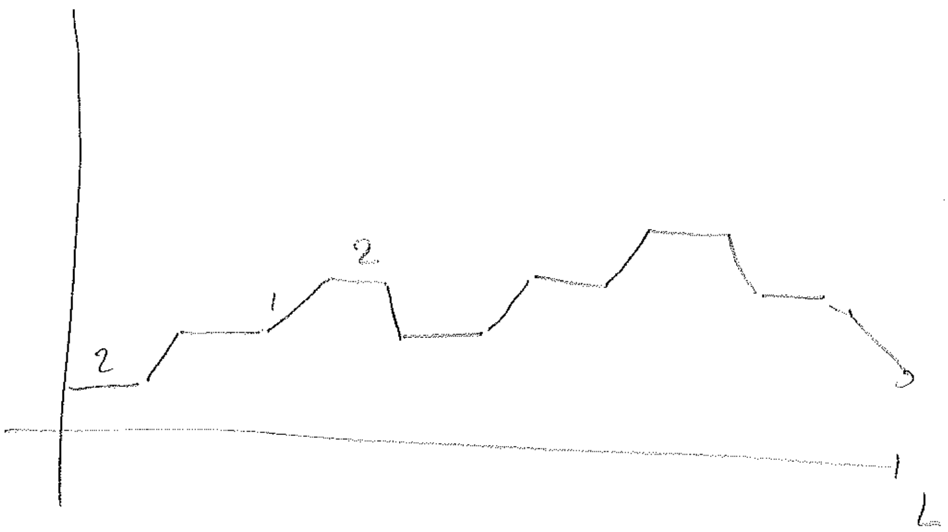
$$D+E = \begin{pmatrix} 2 & 1 & & \\ 1 & 2 & 1 & \\ & 1 & 2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

$\langle W | = (1, 0 \ldots$

$$|V\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

E 14

$\langle W | (D+E)^L | V \rangle$



Paths which remain positive

$$\langle W | (D+E)^L | V \rangle = \frac{(2L+2)!}{(L+2)!\,(L+1)!} \qquad L \geq 2$$

Catalan number

$\Rightarrow$

$$\rho(x) - \frac{1}{2} \simeq \frac{1}{\sqrt{L}} \left(\dot{B}(x) + \dot{Y}(x) \right)$$

$x = \frac{i}{L}$

Brownian ~~exc~~ motion

Brownian excursion

Non gaussian fluctuations

VII How to compute large deviations

$$P(\{\rho(x)\}) \sim e^{-L\,F(\{\rho(x)\})}$$

1 way

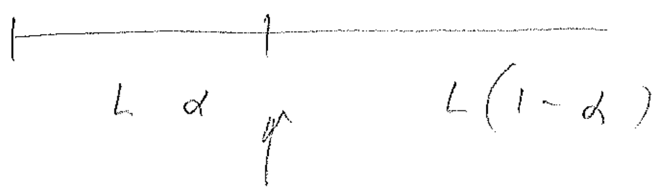
~~Exp~~ $P(n_1 \cdots n_L) = \langle W| \prod \left(n_i D + (1-n_i) E\right) |V\rangle$

$$\langle e^{\sum_i \lambda_i n_i} \rangle = \langle W| \prod \left(D e^{\lambda_i} + E\right) |V\rangle \sim e^{L\, G(\lambda(x))}$$

$$\lambda_i = \lambda\left(\frac{i}{L}\right)$$

in general G is non local.

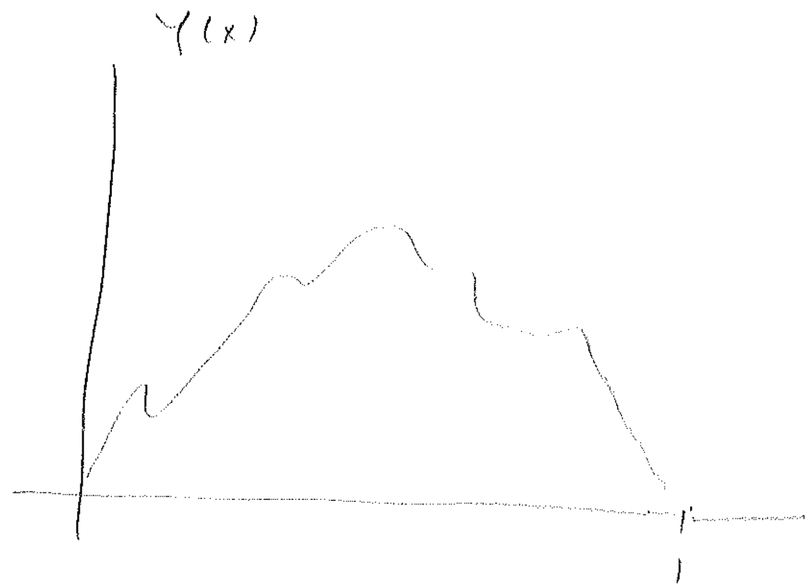
One ...



insert a basis of vectors

(2) Another way

E16



$Y(x)$ Brownian excursion

$L\,\psi'(x)\,\delta x$

$$\Rightarrow \left(\rho(x) - \tfrac{1}{2}\right) = \frac{1}{\sqrt{L}}\left(\dot{B}(X) + \dot{Y}(x)\right)$$

↑ Brownian motion

↑ Brownian excursion

↑ Non gaussian

Evans - Lebowitz
2004