

The theoretical capacity of the Parity Source Coder

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Received 24 June 2005

Accepted 21 September 2005

Published 6 October 2005

Online at stacks.iop.org/JSTAT/2005/P10003

[doi:10.1088/1742-5468/2005/10/P10003](https://doi.org/10.1088/1742-5468/2005/10/P10003)

Abstract. The Parity Source Coder is a protocol for data compression which is based on a set of parity checks organized in a sparse random network. We consider here the case of memoryless unbiased binary sources. We show that the theoretical capacity saturates the Shannon limit at large K . We also find that the first corrections to the leading behaviour are exponentially small, with the result that the behaviour at finite K is very close to the optimal one.

Keywords: random graphs, networks, source and channel coding

ArXiv ePrint: [cond-mat/0506652](https://arxiv.org/abs/cond-mat/0506652)

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1. Introduction

The Parity Source Coder (PSC) is a new scheme for lossy data compression, which uses a kind of dual approach [1] to the LDPC codes used in channel coding [2]. It has been introduced in [3], and discussed recently in [4] and [5]. We discuss here its theoretical performances.

The idea of the PSC is to use the M bits $\mathbf{x}_M \equiv \{x_1, \dots, x_M\}$ that we want to compress to build M parity checks on a low-density graph involving $N (< M)$ Boolean variables $\mathbf{y}_N \equiv \{y_1, \dots, y_N\}$. From the theoretical point of view we will be interested in the ‘thermodynamic’ limit where N and M go to infinity while the *rate* $R \equiv N/M$ is kept fixed. The topology is defined as follows: each constraint is connected to exactly K variables chosen at random. This implies that the probability distribution of the variable connectivity is Poissonian (as in Erdős–Renyi random graphs) with mean $K\alpha$. This is the general setting for a number of constraint satisfaction problems [6]. In our case such a graph (cf figure 1) defines a set of M linear equations for the N variables:

$$y_{i_1^a} + y_{i_2^a} + \dots + y_{i_K^a} = x_a \pmod{2}, \quad a = 1, \dots, M, \quad (1)$$

where $x_i, y_i \in \{0, 1\}$, and the indices $i_1^a, i_2^a, \dots, i_K^a$ are chosen in $\{1, \dots, N\}$ with uniform distribution (the repetition of two indices in the same constraint can be forbidden, but this is irrelevant in the large N limit which interests us here). This problem is called K -XORSAT [7] and it has been recently studied in [8] and [9]. It is also a diluted version of the p -spin model used in spin glass theory [10]. Here we use it to set up a data compressor, following [4]. The encoded word corresponds to the solution of the linear system (1) which minimizes the number of errors. In the thermodynamic limit, it has been shown that the critical value α_c that signals the K -XORSAT problem has a phase transition at a critical value α_c of the ratio $\alpha = M/N$. For $\alpha < \alpha_c$ a random instance is satisfiable (in the sense that there exists an assignment of the N variables satisfying all M equations) with probability one. This is the SAT phase. For $\alpha > \alpha_c$ a random instance is unsatisfiable with probability one: there is no assignment satisfying all constraints. The critical density of constraints α_c increases with K and goes exponentially fast to 1 as K increases (figure 2), as can be computed using the formalism introduced in [8, 9]. The K -XORSAT can be

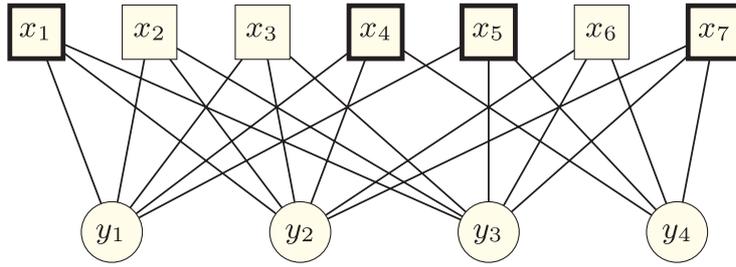


Figure 1. A Tanner graph for a PSC with $M = 7$ checks and $N = 4$ variables. In this example the string to be compressed is $\{x_1, x_2, \dots, x_7\} \equiv 1001101$. The constraints x_1, x_4, x_5, x_7 impose the sum of the variables y_i involved in each constraint to be 1 mod 2, while x_2, x_3, x_6 require that the variables add up to 0 mod 2.

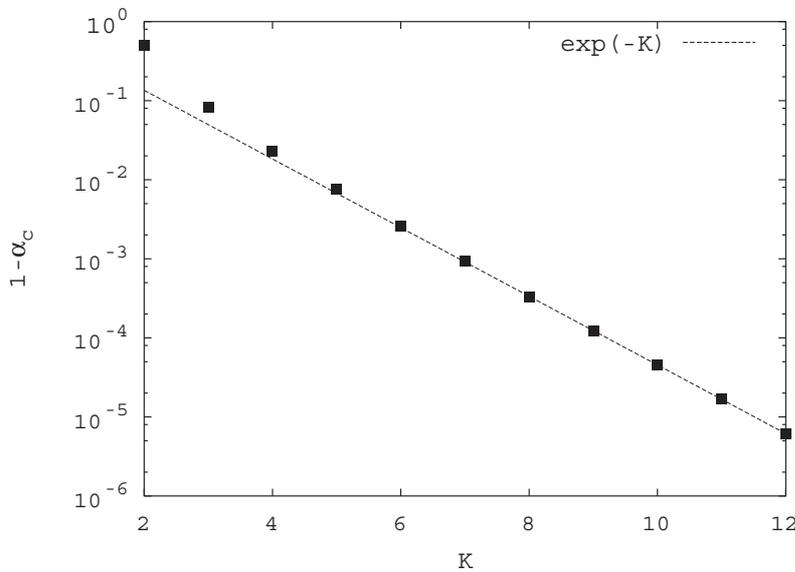


Figure 2. The critical value of the control parameter marking the transition between SAT/UNSAT is plotted versus K . Following [8], one can show that the leading behaviour at large K is $\alpha_c(K) = 1 - e^{-K} - (K^2 - K/2)e^{-2K} + \mathcal{O}(K^5 e^{-3K})$.

used for data compression by working in the UNSAT phase with $\alpha > 1$. As the encoding step $\mathbf{x}_M \rightarrow \mathbf{y}_N$ consists in finding the string \mathbf{y}_N which violates the smallest number of constraints in (1), the compression rate is $R = 1/\alpha$. Once we have the encoded word, the decompression step $\mathbf{y}_N \rightarrow \mathbf{x}_M^*$ is done by setting $x_a^* = 0$ or 1 according to equation (1). The *distortion* is defined as the number of bits which are not properly recovered, divided by the total number of bits M . We can look at the problem in terms of a ‘cost’ function $\varepsilon_a(y_{i_1^a} \dots y_{i_K^a} | x_a)$ which is 0 if equation (1) is verified and 2 otherwise. The total cost E of the compression process is then twice the total number of unsatisfied equations in the linear system (1). The distortion is related to it by

$$D = \frac{E}{2M} = \frac{E}{2N\alpha}. \tag{2}$$

We consider here the simplest version of the lossy compression problem: we deal with uncorrelated unbiased binary sources, i.e. $\text{prob}(x_1, \dots, x_M) = \prod_{a=1, M} \text{prob}(x_a)$ and $\text{prob}(x_a = 0) = \text{prob}(x_a = 1) = 1/2$. The rate distortion theorem [11] states that a distortion D can be achieved if and only if the rate is large enough, $R \geq R^*$, where the Shannon bound R^* is given by

$$R^* = 1 - H_2(D),$$

and $H_2(x) = -x \log x - (1 - x) \log(1 - x)$ is the binary entropy. Basically the proof of achievability in this theorem relies on a choice of codewords (the set of all possible encoded words) which is a random set. This is intimately related to the random energy model (REM) [12]. On the other hand, our PSC can be argued to become a random energy model in the large K limit, in the same way as the p -spin model becomes a REM in the large p limit¹ [12]. Seen from this point of view, it is not surprising that the performances of the PSC converge to the Shannon bound in the limit of large K , as we shall prove here. In fact the same optimal performance has been found in a recent work [14] using a non-monotonic perceptron. Again in such a device each bit of the decoded word is chosen to be a function of the complete encoded word, which is the same as letting $K = N$, i.e. infinity in the thermodynamic limit, in our language.

However all these ‘optimal’ source coding devices, based either on a random codebook like in the REM, on a fully connected perceptron, or on the PSC at $K \rightarrow \infty$, have a serious drawback: there is no known fast algorithm for performing the encoding. Physically, the encoding step is a search of the ground state, the one which minimizes the number of violated constraints. This has to take place in the UNSAT phase $\alpha > 1$ where these systems are frustrated. Finding the exact ground state is an NP-complete problem, but it turns out that we do not even have good heuristics for finding approximate ground states. Such a heuristic of course cannot exist for the REM, but one could hope to find one for the PSC with finite K . For instance in the related problem of K -satisfiability [15], or source coding devices based on random nodes [4], there exist good heuristics based on the message passing ‘survey propagation’ (SP) algorithm which can be seen as a generalization of the celebrated ‘belief propagation’ algorithm [16]–[18]. While this algorithm, as such, does not work for the PSC, it seems possible that one could develop powerful algorithms for the finite K PSC in the future. Actually, a very recent work [5] proposes a message passing algorithm, inspired by SP, which seems to show very good performance. This motivates the present study of the theoretical capacity of the PSC at finite K .

In this work we compute explicitly the distortion of the PSC in the limit where the clause connectivity K becomes large. We first show that for $K \rightarrow \infty$ the distortion becomes optimal (it saturates the Shannon bound). As for the finite K corrections, we find that, for a given value of the rate $R = 1/\alpha$, the distortion is

$$D = D_{\text{Sh}} + a\sqrt{K}e^{-K\Delta}(1 + \mathcal{O}(1/K)), \quad (3)$$

where D_{Sh} satisfies $1 - H_2(D_{\text{Sh}}) = 1/\alpha$ and the coefficients a and Δ depend on α . In particular, the actual Δ lies in $[\log 2, 1]$, and goes to $\log 2$ in the large α limit. The fact that the first finite K corrections are exponentially small must be stressed: this means also that a PSC with $K = 5$ or 6 is in practice nearly optimal. A good encoding algorithm

¹ This point, along with the ‘hardening’ of the constraints at large K , is discussed in some detail in [13].

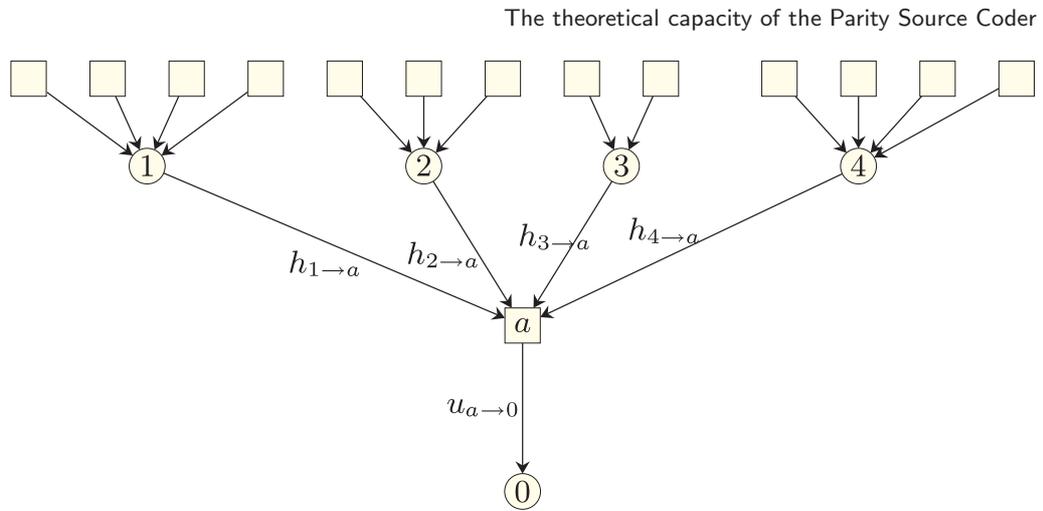


Figure 3. The iterative idea behind the cavity equations is illustrated here for $K = 5$.

for this case could thus turn this PSC into a very good compressor. We stress that the range of validity of the result of this paper is limited to the case of uncorrelated sources. This is confirmed by the statistical description of a family of code ensembles presented in [19]. On the other hand, the hypothesis of a non-biased input message does not seem to play a role.

As we mentioned previously, a protocol very similar to this PSC (the only difference being the underlying graph topology) has been introduced in [3], and Murayama [20] has shown that some belief propagation based algorithm can be used for encoding in the $K = 2$ case. Our result shows that the optimal capacity (i.e. Shannon's bound) can be obtained only in the limit of large K , at variance with some of the statements in [3]. It gives the analogue, for source coding, to the result of Kabashima and Saad [21] on channel capacity of error-correcting codes at large K .

2. Cavity equations

In order to deal with the K -XORSAT problem we take advantage of the cavity method as explained in [15]. This method is heuristic (the main assumptions can be checked self-consistently) but it is believed to be exact. As for the K -XORSAT problem, its range of validity has been rigorously established in [8] and [22]. In particular, the cavity result for the critical threshold α_c is exact. For $\alpha > \alpha_c$ (the regime where we use it) this method finds the correct ground state energy up to a threshold value α_G , which is $\simeq 3.07$ for $K = 3$ [22] and increases with K as one can see from numerics.

For the sake of simplicity, we pass from Boolean variables to Ising spins, thus taking values in $\{-1, +1\}$. The general idea behind the cavity approach is summarized in figure 3. Since the local structure of the random graph is tree-like, we focus on a single clause and look at the variables connected to it. We introduce two types of message, cavity biases $u_{a \rightarrow i}$ going from clause a to variable i , and cavity fields $h_{i \rightarrow a}$ going from variable i to clause a . A cavity bias can be 0 (which means that, as for the clause a , variable i is free to assume any value), or ± 1 (meaning that this is the value that i should take in order

to satisfy clause a). The message sent from clause a must take into account all the other variables connected to it; each of these sends to a a cavity field which is nothing but the sum of all the other incoming cavity biases: $h_{j \rightarrow a} = \sum_{b \in j-a} u_{b \rightarrow j}$. In the most general case, the space of low-energy configurations is broken into many disconnected components (clustering). The general object we need to deal with this is then a functional distribution $\mathcal{Q}[\mathbf{q}(u)]$ giving the probability that, if one link $a \rightarrow i$ is chosen at random, the probability (with respect to the choice of the cluster) of observing a bias $u_{a \rightarrow i}$ is $\mathbf{q}_{a \rightarrow i}(u_{a \rightarrow i})$. The same holds for the distribution of cavity fields, $\mathcal{P}[\mathbf{p}(h)]$. We thus suppose we have a population of $\mathbf{q}(u)$ s and $\mathbf{p}(h)$ s. In order to simplify the notation, we shall simply call u_0 the bias on variable 0, with no regard for the clause it is coming from. According to [8], we iterate the following self-consistent equations:

$$\mathbf{q}_0(u_0) = \sum_{h_1, \dots, h_{(K-1)}} \mathbf{p}_1^{(p_1)}(h_1) \cdots \mathbf{p}_{(K-1)}^{(p_{K-1})}(h_{(K-1)}) \delta(u, S(Jh_1 \cdots h_{(K-1)})),$$

$$\text{with prob. } \prod_{i=1}^{K-1} f_{K\alpha}(p_i) \quad (4)$$

$$\mathbf{p}^{(p)}(h) = \frac{1}{A^{(p)}(y)} \sum_{u_1, \dots, u_p} \mathbf{q}_1(u_1) \cdots \mathbf{q}_p(u_p) \delta\left(h, \sum_{a=1}^p u_a\right) \exp\left\{y \left| \sum_{a=1}^p u_a \right| - y \sum_{a=1}^p |u_a|\right\}, \quad (5)$$

$$A^{(p)}(y) = \sum_{u_1, \dots, u_p} \mathbf{q}_1(u_1) \cdots \mathbf{q}_p(u_p) \exp\left\{y \left| \sum_{a=1}^p u_a \right| - y \sum_{a=1}^p |u_a|\right\}. \quad (6)$$

Here $S(x) \equiv \text{sgn}(x)$ for $x \neq 0$, $S(0) \equiv 0$, and $f_{K\alpha}(\cdot)$ is the Poisson distribution with mean $K\alpha$. The first of these equations is the direct implementation of the recursion illustrated in figure 3: the delta function ensures that clause a sends the proper value to variable 0. In the second equation, a reweighting term is present [15]. This is due to the fact that if we add one variable and want to compute the new probability distributions at a given value of the energy E , then we need all the contributions from the states at energy $E - \Delta E$, where ΔE is the energy shift caused by the addition of one variable. If the number of clusters at energy E is $\exp(N\Sigma(E/N))$, then the expansion $\Sigma(E) \simeq \Sigma(E - \Delta E) - y\Delta E$ leads to a reweighting $\exp(-y\Delta E)$, with $y = \partial\Sigma/\partial E$. The knowledge of these distributions allows one to compute the free energy $\Phi(y)$:

$$\Phi(y) = \Phi_1(y) - (K - 1)\alpha\Phi_2(y), \quad (7)$$

$$\Phi_1(y) = -\frac{1}{y} \overline{\log A^{(p)}(y)}, \quad (8)$$

$$\Phi_2(y) = -\frac{1}{y} \overline{\log \sum_u \mathbf{q}(u; \{p_i\}) \sum_h \mathbf{p}^{(p)}(h) e^{y(|u+h| - |u| - |h|)}}, \quad (9)$$

where the average is taken over the random graph ensemble and over the population of the distributions $\mathbf{q}(u)$ s and $\mathbf{p}(h)$ s. The free energy in (7) is obtained by adding one variable (and a certain number of clauses) to a system with N variables and computing the contribution arising from the corresponding shift in energy, $\exp(-y\Phi_1) = \langle \exp(-y\Delta E) \rangle$. The correction term is due to the fact that in the $(N + 1)$ -variable system the probability of generating the clauses is slightly lower; thus we have to cancel a fraction of them at

random (see [15] for a detailed derivation). The ground state energy is then evaluated as the $\min_y \Phi(y)$.

Actually, the nature of messages allows for a simplification of the cavity equations: we write

$$\mathbf{q}(u) = \eta \delta_{u,0} + \frac{1-\eta}{2} [\delta_{u,-1} + \delta_{u,+1}]. \quad (10)$$

Also, it should be clear that, as for the $\mathbf{p}(h)$, what matters is only the sign of the field h ; then,

$$\mathbf{p}^{(p)}(h) = \frac{1}{A^{(p)}} (w_0^{(p)} \delta_{S(h),0} + w_+^{(p)} \delta_{S(h),+1} + w_-^{(p)} \delta_{S(h),-1}), \quad (11)$$

with $A = w_0 + w_+ + w_-$ and $w_+ = w_-$ because of the up-down symmetry of the problem. In practice, one needs to work with a single population of real numbers η_i , that leads to a stationary distribution $\rho(\eta)$. For any fixed value of y , the self-consistent equations (4)–(6) are solved as follows.

- (1) Consider a population of η_i randomly distributed in $[0, 1]$.
- (2) Do $K - 1$ times:
 - Pick a random integer p with probability $f_{K\alpha}(p)$.
 - Choose p values η_1, \dots, η_p and compute a probability distribution $\mathbf{p}(h)$ according to (5). Given (11), this amounts to computing two real numbers: w_0 and the normalization A .
 - Compute Φ_1 as in (8) through this A .
- (1) Using these $K - 1$ distributions $\mathbf{p}(h)$ s, compute a new $\mathbf{q}(u)$ according to (4). Given (10) this is the same as computing a new value η_0 .
- (2) Use this new $\mathbf{q}(u)$ and a new extracted $\mathbf{p}(h)$ to compute Φ_2 as in (9). The total free energy can now be evaluated via (7).
- (3) Replace an η value randomly chosen in the distribution with the new value η_0 .
- (4) Go to step 2 until a stationary distribution $\rho(\eta)$ is reached. (The free energy then attains a stationary value.)

We are now going to discuss the cavity equations for large K and we will use the algorithm that we have just described to check our asymptotic results numerically.

3. The Shannon bound

The cavity equations (4)–(6) have been discussed in [8] mainly as regards the value of $\alpha_c(K)$ and the behaviour of the ground state energy $E_0(K)$ close to $\alpha_c(K)$. We want to compute $E_0(K)$ at any α in the large K limit.

For large K , there is a self-consistent solution of the cavity equations such that all the w_0 are very small, in fact exponentially small. We just need to assume that the typical value of a w_0 is much smaller than $1/K$. This condition on w_0 s shows that η is zero to

leading order, because from equation (4) one finds that

$$\eta = 1 - \prod_{i=1}^{K-1} (1 - w_0^{(p_i)}). \quad (12)$$

We shall be more precise below as we verify self-consistently the assumption on w_0 and will be able to compute the first non-zero term. Here we work directly with $\eta = 0$. We need to compute the new value of w_0 and w_+ using equation (5). If K is large, p is generically large (it is Poisson distributed with mean $K\alpha$). If p is even (the case of p odd is an immediate generalization) one finds

$$w_0^{(p)} = \binom{p}{p/2} \frac{e^{-py}}{2^p} \simeq \frac{2e^{-py}}{\sqrt{2\pi p}}, \quad (13)$$

$$w_+^{(p)} = \frac{1}{2^p} \sum_{q=0}^{(p/2)-1} \binom{p}{q} e^{-qy} \simeq \frac{p}{2^p} \int_0^{1/2} \frac{dx}{\sqrt{2\pi px(1-x)}} \times \exp\{p(-x \log x - (1-x) \log(1-x) - xy)\}, \quad (14)$$

$$w_-^{(p)} = w_+^{(p)}. \quad (15)$$

The integral can be evaluated for p large by the saddle point method (the saddle point being $x^* = -y + \log(1 + e^y)$) and we have

$$w_+^{(p)} = \left(\frac{1 + e^{-2y}}{2} \right)^p.$$

Since for any finite y this is exponentially larger than w_0 , the leading term in the normalization constant is just

$$A^{(p)}(y) = 2w_+^{(p)} = 2 \left(\frac{1 + e^{-2y}}{2} \right)^p. \quad (16)$$

Now, it is not difficult to show that equation (9) can be rewritten as

$$\Phi_2(y) = -\frac{1}{y} \log \left(\frac{A^{(p+1)}(y)}{A^{(p)}(y)} \right), \quad (17)$$

and thus the free energy can be computed from the normalization (16) alone. We find that

$$\begin{aligned} \Phi(y) &= -\frac{1}{y} \left(\log A^{(p)}(y) - (K-1)\alpha \log \left(\frac{A^{(p+1)}(y)}{A^{(p)}(y)} \right) \right) \\ &= -\frac{1}{y} \left[\log 2 + \alpha \log \left(\frac{1 + e^{-2y}}{2} \right) \right] \equiv \Phi_\infty(y). \end{aligned} \quad (18)$$

The ground state energy is the maximum of $\Phi(y)$ [15] and, according to equation (2), this gives a distortion D for the PSC at large K :

$$D = \frac{1}{2\alpha} \max_y \Phi_\infty(y). \quad (19)$$

The Shannon bound says that the minimum distortion satisfies $1 - H_2(D_{\text{Sh}}) = 1/\alpha$. A few lines of computation show that the distortion in (19) actually saturates the Shannon

bound. Let us call z the value of y where $\Phi_\infty(y)$ is maximal. It satisfies

$$\log 2 = \alpha(z \tanh z - \log \cosh z). \quad (20)$$

Then one gets

$$\frac{\Phi_\infty(z)}{2\alpha} = \frac{1}{e^{2z} + 1} \Rightarrow H_2\left(\frac{\Phi_\infty(z)}{2\alpha}\right) = \frac{1}{\log 2} \left(-\log \frac{1}{e^{2z} + 1} - 2z \frac{e^{2z}}{e^{2z} + 1} \right). \quad (21)$$

After some algebra one can derive from this the result sought:

$$H_2\left(\frac{\Phi_\infty(z)}{2\alpha}\right) = 1 - \frac{1}{\alpha}. \quad (22)$$

This shows that at very large K the XORSAT problem gives exactly the Shannon limit. We now look at finite K corrections in order to see how this asymptotic performance is reached.

4. Corrections

In order to compute the first-order corrections to the leading behaviour we compute the normalization constant in (6) under the hypothesis of small (but finite) η :

$$\begin{aligned} A^{(p)}(y) &= \prod_{a=1}^p \left(\frac{1 - \eta_a}{2} \right) e^{-py} \sum_{q=0}^p \binom{p}{q} e^{y|p-2q|} + p\eta_1 \prod_{a=2}^p \left(\frac{1 - \eta_a}{2} \right) \\ &\quad \times e^{-(p-1)y} \sum_{q=0}^{p-1} \binom{p-1}{q} e^{y|p-1-2q|} + \dots \\ &= \frac{e^{-py}}{2^p} g_p(y) \left(1 - \sum_{a=1}^p \eta_a + \mathcal{O}(p\eta)^2 \right) + p\eta \frac{e^{-(p-1)y}}{2^{p-1}} g_{p-1}(y) (1 + \mathcal{O}(p\eta)) \\ &\quad + \frac{e^{-(p-2)y}}{2^{p-2}} g_{p-2}(y) \mathcal{O}(p\eta)^2 + \dots, \end{aligned} \quad (23)$$

$$g_p(y) \equiv \sum_{q=0}^p \binom{p}{q} e^{y|p-2q|}. \quad (24)$$

As we have shown above, the whole free energy can be computed from the knowledge of $A^{(p)}(y)$. In order to calculate it, we compute the function $g_p(y)$ in the large p limit. We first notice that it can be written as

$$g_p(y) = \sum_{\sigma_1, \dots, \sigma_p} \exp \left[y \left| \sum_{i=1}^p \sigma_i \right| \right], \quad (25)$$

where σ_i are Ising spins. Thus,

$$g_p(y) + g_p(-y) = \sum_{\{\sigma_i\}} [e^{y|\sum_i \sigma_i|} + e^{-y|\sum_i \sigma_i|}] = \sum_{\{\sigma_i\}} [e^{y\sum_i \sigma_i} + e^{-y\sum_i \sigma_i}] = 2(2 \cosh y)^p. \quad (26)$$

We use a Fourier transformation to express $g_p(-y)$:

$$\begin{aligned} g_p(-y) &= \frac{y}{\pi} \int \frac{dk}{k^2 + y^2} e^{ikp} \sum_{q=0}^p \binom{p}{q} e^{-i2qk} = \frac{y}{\pi} 2^p \int \frac{dk (\cos k)^p}{k^2 + y^2} \\ &= \frac{y}{\pi} 2^p \sqrt{\frac{2\pi}{p}} \left(1 - \frac{1}{4p} + \mathcal{O}\left(\frac{1}{p^2}\right) \right) \sum_{n=-\infty}^{+\infty} \frac{(-1)^{np}}{(\pi n)^2 + y^2}. \end{aligned}$$

The sum can be done exactly and we have

$$\begin{aligned} g_{p\text{even}}(-y) &= \frac{2^{p+1}}{\sqrt{2\pi p}} \frac{1}{\tanh y} (1 - 1/4p + \mathcal{O}(1/p^2)), \\ g_{p\text{odd}}(-y) &= \frac{2^{p+1}}{\sqrt{2\pi p}} \frac{1}{\sinh(y)} (1 - 1/4p + \mathcal{O}(1/p^2)). \end{aligned} \tag{27}$$

Using (26), we get for p even

$$g_p(y) = 2^{p+1} (\cosh y)^p \left[1 - \frac{(\cosh y)^{-p}}{\sqrt{2\pi p} (\tanh y)} \left(1 - \frac{1}{4p} + \mathcal{O}\left(\frac{1}{p^2}\right) \right) \right], \tag{28}$$

with the replacement $\tanh y \rightarrow \sinh y$ if p is odd. To the leading order we have thus

$$A^{(p)}(y) = 2 \left(\frac{1 + e^{-2y}}{2} \right)^p (1 + \mathcal{O}(p^\gamma e^{-p})), \tag{29}$$

with some exponent γ which depends on the actual order of magnitude of η . To compute it we first need to know the weight for $h = 0$. If p is even we use equation (5) and we note that the main contribution (in the same hypothesis of η small) is given by

$$\begin{aligned} w_0^{(p\text{even})} &= \frac{1}{A^{(p)}(y)} e^{-py} \binom{p}{p/2} \frac{1}{2^p} \prod_{a=1}^p (1 - \eta_a) + \mathcal{O}(p\eta) \\ &= \frac{1}{2} \left(\frac{1 + e^{-2y}}{2} \right)^{-p} (1 + \mathcal{O}(p^\gamma e^{-p})) e^{-py} \frac{2^{p+1}}{\sqrt{2\pi p}} \left(1 - \frac{1}{4p} + \mathcal{O}\left(\frac{1}{p^2}\right) \right) \frac{1}{2^p} (1 + \mathcal{O}(p\eta)) \\ &= \frac{(\cosh y)^{-p}}{\sqrt{2\pi p}} (1 - 1/4p + \mathcal{O}(1/p^2) + \mathcal{O}(p\eta)). \end{aligned} \tag{30}$$

(Here we have also assumed that $p > 0$, since $w_0 = 1$ if $p = 0$.) On the other hand, if p is odd we have

$$\begin{aligned} w_0^{(p\text{odd})} &= \eta \frac{1}{A^{(p)}(y)} e^{-(p-1)y} p \binom{p-1}{(p-1)/2} \frac{1}{2^{p-1}} \prod_{a=1}^{p-1} (1 - \eta_a) + \mathcal{O}(p\eta)^2 \\ &= \sqrt{\frac{p}{2\pi}} \eta e^y (\cosh y)^{-p} (1 + \mathcal{O}(1/p)). \end{aligned}$$

To the leading order, η does not fluctuate and takes the value

$$\begin{aligned} \eta &\simeq -\log(1 - \eta) \simeq \sum_{i=1}^{K-1} w_0^{(p_i)} = (K-1) \left[e^{-K\alpha} + e^{-K\alpha} (\cosh(K\alpha) - 1) \overline{w_0^{(p\text{even})}} \Big|_{p>0} \right. \\ &\quad \left. + e^{-K\alpha} \sinh(K\alpha) \overline{w_0^{(p\text{odd})}} \right] \\ &\simeq \frac{K-1}{2} \left[\frac{(\cosh y)^{-p}}{\sqrt{2\pi p}} \left(1 - \frac{1}{4p} + \mathcal{O}\left(\frac{1}{p^2}\right) \right) \right]_{p\text{even}>0} + \mathcal{O}(K e^{-K\alpha}) + \mathcal{O}(\eta^2), \end{aligned}$$

since the two other terms ($p = 0$ and p odd) are exponentially subleading. In order to perform this average we use

$$\frac{1}{p^z} = \frac{1}{\Gamma(z)} \int dt t^{z-1} e^{-pt} \quad (31)$$

to express the denominator. This allows us to perform the average over p even. We then have

$$\begin{aligned} \eta &= \frac{Ke^{-K\alpha}}{4\pi\sqrt{2}} \left(1 - \frac{1}{K}\right) \left\{ 2 \int dt t^{-1/2} (\cosh(\beta e^{-t}) - 1) \right. \\ &\quad \left. - \int dt t^{1/2} (\cosh(\beta e^{-t}) - 1) + \mathcal{O}(\beta^{-5/2}) \right\} + \mathcal{O}(Ke^{-K\alpha}) \\ &= \frac{Ke^{-K\alpha}}{4\pi\sqrt{2}} 4\beta \int dt t^{1/2} e^{-t} \sinh(\beta e^{-t}) \left(1 - \frac{t}{6} + \mathcal{O}(t^2)\right) \left(1 - \frac{1}{K}\right) + \mathcal{O}(Ke^{-K\alpha}) \end{aligned}$$

where $\beta \equiv K\alpha / \cosh y$. We then set $t = \tau/\beta$ and expand in $1/\beta$. This gives

$$\eta = \frac{\sqrt{\cosh y}}{2\sqrt{2\pi\alpha}} K^{1/2} e^{-K\alpha(1-1/\cosh y)} \left[1 + \frac{1}{K} \left(\frac{\cosh y}{8\alpha} - 1 \right) + \mathcal{O}\left(\frac{\cosh y}{K\alpha}\right)^2 \right] + \mathcal{O}(Ke^{-K\alpha}), \quad (32)$$

which shows *a posteriori* that the small η hypothesis is consistent. We now go back to (23) and get

$$\begin{aligned} A^{(p)}(y) &= 2 \left(\frac{1 + e^{-2y}}{2} \right)^p [1 + p\eta \tanh y + \mathcal{O}(p\eta)^2] \\ &= 2 \left(\frac{1 + e^{-2y}}{2} \right)^p \left[1 + \frac{p \tanh y \sqrt{\cosh y}}{2\sqrt{2\pi\alpha}} K^{1/2} e^{-K\alpha(1-1/\cosh y)} \right. \\ &\quad \left. \times \left(1 + \frac{1}{K} \left(\frac{\cosh y}{8\alpha} - 1 \right) + \mathcal{O}\left(\frac{\cosh y}{K\alpha}\right)^2 \right) + \mathcal{O}(pKe^{-K\alpha}) \right]. \end{aligned} \quad (33)$$

From this result and from equation (9) one finds that

$$\Phi_2(y) = -\frac{1}{y} \left(\log \left(\frac{1 + e^{-2y}}{2} \right) - \eta \tanh y + \mathcal{O}(K\eta)^2 \right). \quad (34)$$

Moreover,

$$\Phi_1(y) = -\frac{1}{y} \left(\log 2 + K\alpha \log \left(\frac{1 + e^{-2y}}{2} \right) + K\alpha \eta \tanh y + \mathcal{O}(K\eta)^2 \right). \quad (35)$$

We can now compute the total free energy (7). One can check directly that the leading corrections to the infinite K limit, of order $\mathcal{O}[K^{3/2} \exp(-K\alpha(1-1/\cosh y))]$, vanish. We are then left with

$$\begin{aligned} \Phi(y) &= -\frac{1}{y} \left[\log 2 + \alpha \log \left(\frac{1 + e^{-2y}}{2} \right) \right] + \frac{\sqrt{\cosh y} \tanh y}{2y\sqrt{2\pi}} (\alpha K)^{1/2} e^{-K\alpha(1-1/\cosh y)} \\ &\quad \times \left(1 + \frac{1}{K} \left(\frac{\cosh y}{8\alpha} - 1 \right) + \mathcal{O}\left(\frac{1}{K^2}\right) \right) \\ &= \Phi_\infty(y) + \Delta\Phi_K(y), \end{aligned} \quad (36)$$

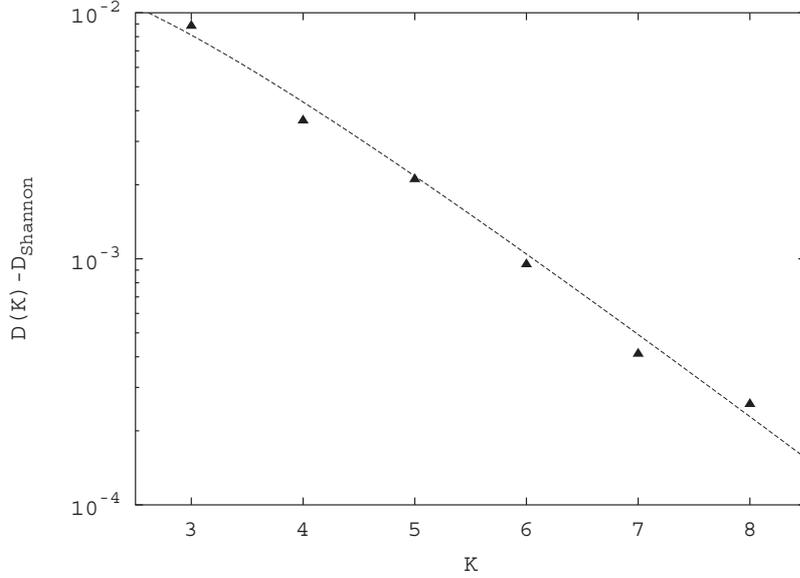


Figure 4. Theoretical capacity of the PSC, $\alpha = 1.3$.

where $\lim_{K \rightarrow \infty} \Delta\Phi_K = 0$. We assume that the maximum of the $\Phi(y)$ in (36) is at $y = z + \varepsilon$, where ε is exponentially small at large K (we shall verify this hypothesis self-consistently) and z is the solution of equation (20). The condition $\Phi'(y) = 0$ then results in

$$\varepsilon = -\frac{\Delta\Phi'_K(z)}{\Phi''_\infty(z)} = \mathcal{O}(K^{-1/2}e^{-K\alpha(1-1/\cosh z)}), \quad (37)$$

where the dependence of z on α is extracted from (20). One finds that z is a monotonic decreasing function. In particular, $z \sim \sqrt{2\log 2/\alpha}$ at large α while z diverges as $(-1/2)\log(\alpha - 1)$ as $\alpha \rightarrow 1$: it follows that ε is exponentially small in any case. Coming back to equation (19), it is then easy to see that to the leading order

$$D = \frac{1}{2\alpha}(\Phi_\infty(z) + \Delta\Phi_K(z)) = D_{\text{Sh}} + C_K(\alpha), \quad (38)$$

where the corrections $C_K(\alpha)$ are finally

$$C_K(\alpha) = \frac{\sqrt{\cosh z} \tanh z}{4z\sqrt{2\pi\alpha}} K^{1/2} e^{-K\alpha(1-1/\cosh z)} \left(1 + \frac{1}{K} \left(\frac{\cosh z}{8\alpha} - 1 \right) + \mathcal{O}\left(\frac{1}{K^2}\right) \right) (1 + \mathcal{O}(K^{1/2} e^{-K\alpha})), \quad (39)$$

z being the solution of equation (20).

We now look at numerical data in order to verify our analytical prediction. In figure 4 we plot the difference between the actual distortion of the PSC as obtained from the numerical solution of the cavity equations at $\alpha = 1.3$ and the corresponding Shannon value. The curve is the theoretical prediction in (39), where we neglected the $1/K^2$ corrections. The same plot but for $\alpha = 2$ is shown in figure 5. In both cases there is a very good agreement with the analytical prediction.

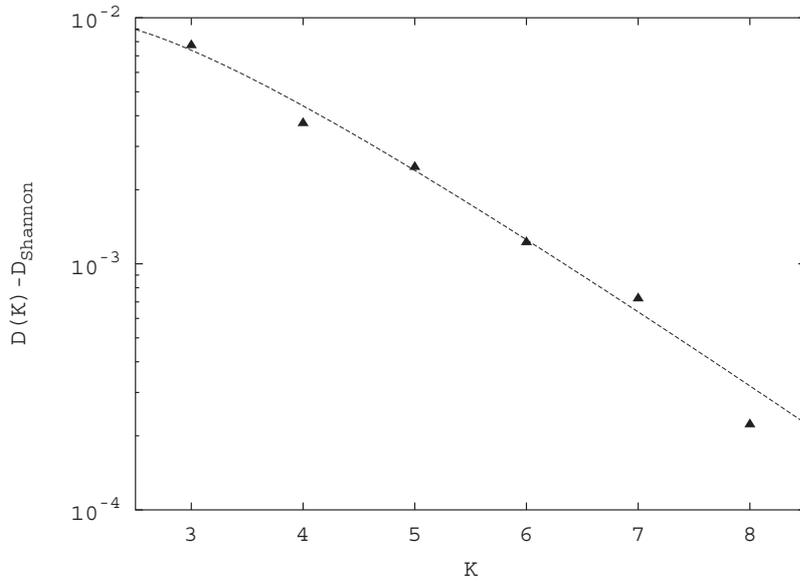


Figure 5. Theoretical capacity of the PSC, $\alpha = 2.0$.

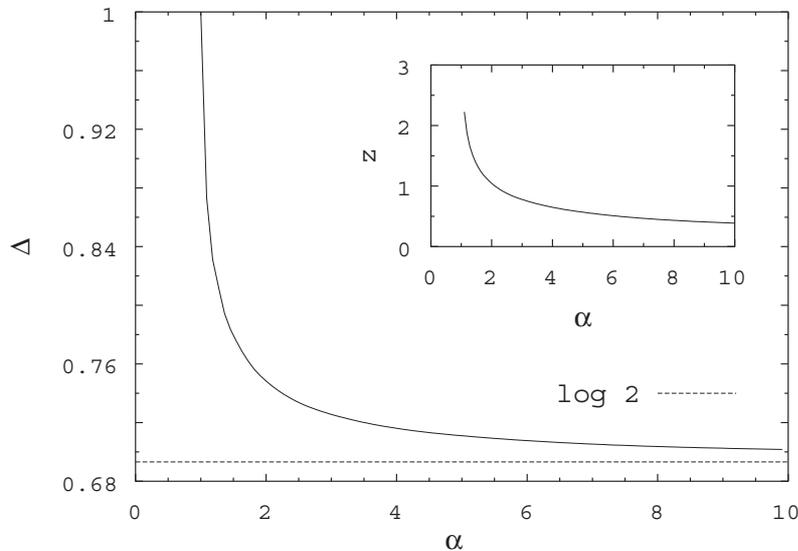


Figure 6. The value of $\Delta \equiv \alpha(1 - 1/\cosh z)$, z being the solution of equation (20), is plotted versus α . One finds that $\Delta = \log 2 + \mathcal{O}(1/\alpha)$ at large α , while $\Delta = 1 - 2\sqrt{\alpha - 1} + \mathcal{O}(\alpha - 1)$ as $\alpha \rightarrow 1^+$. Inset: the actual value of z as a function of α , as given by equation (20). It diverges as $-\log(\alpha - 1)$ as $\alpha \rightarrow 1^+$.

5. Conclusions

We have shown that the theoretical capacity of the Parity Source Coder is optimal at large K and that the corrections to the leading behaviour are exponentially small. Nevertheless, due to the smallness of Δ (cf figure 6), the exponential decreases quite slowly, and $1/K$ corrections are needed to take into account the deviations from the leading behaviour at relatively small values of K .

Acknowledgments

We thank O Rivoire, M Wainwright, J S Yedidia and R Zecchina for important discussions, and an anonymous referee for useful suggestions and references. SC is supported by EC through the network MTR 2002-00307, DYGLAGEMEM. This work has been supported in part by the EC through the network MTR 2002-00319 STIPCO and the FP6 IST consortium EVERGROW.

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