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The microstructure of ultrametricity

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Résumé. — L'organisation hiérarchique des états purs d'un verre de spin S.K. (l'ultramétricité) est analysée en termes de la distribution des aimantations locales. Nous montrons que chaque état pur α définit une distance ultramétrique $D_\alpha(i, j)$ entre les N *sites*. Etant donnés deux états α, β dont le recouvrement est q, il y a une distance minimale d_m telle que, pour chaque paire de sites i, j vérifiant $D_\alpha(i, j) \geqslant d_m$, les deux distances D_α et D_β coincident. Il en résulte qu'on peut faire une partition des sites en cellules disjointes à l'intérieur desquelles l'aimantation totale est la même pour tous les sites ayant un overlap mutuel q. Pour cette même famille d'états nous définissons un « ancêtre » qui a, à l'intérieur de chaque cellule, une aimantation locale constante et égale à l'aimantation moyenne des descendants. Les ancêtres vérifient les équations de type T.A.P. La dépendance fonctionnelle de l'aimantation locale en termes du champ local est donnée par la solution de l'équation de diffusion dans l'espace des x qui reçoit une interprétation purement statique.

Abstract. — The hierarchical organization of the pure states of a S.K. spin glass (ultrametricity) is analysed in terms of self-averaging distributions of local magnetizations. We show that every pure state α defines an ultrametric distance $D_\alpha(i, j)$ among the N *sites*. Given two states α, β with overlap q there is a minimum distance d_m such that for two sites i, j with $D_\alpha(i, j) \geqslant d_m$ the two distances D_α and D_β coincide. It follows that the sites can be partitioned in disjoint cells inside which the total magnetization is the same for all the states with mutual overlap q. For this same family of states we then define an « ancestor » that has, inside each cell, constant local magnetization equal to the average magnetization of the descendants. The ancestors satisfy mean field like equations. The functional dependence of the local magnetization in terms of the local field is given by the solution of the diffusion equation in x space which is given a purely static interpretation.

1. Introduction.

Recent progress in the study of the mean field theory of spin glasses is related to the physical interpretation of replica symmetry breaking (R.S.B.) as describing the breaking of ergodicity and the existence of many pure equilibrium states in the spin glass phase. It was shown by Parisi [1] that the order parameter function $q(x)$ which he introduced for describing R.S.B. is related to the distribution of the overlaps (which measure the distances in phase space) between these equilibrium states.

The nature of the spin glass phase is best understood by looking at the geometry of the space of equilibrium states. It was found that this space has a special hierarchical topology characterized by ultrametricity [2], and that the detailed structure of the space, for instance the order parameter function, is not self-averaging, which means that it depends on the special realization of the couplings in the sample, even in the thermodynamic limit [2, 3].

The ultrametric topology of the space of equilibrium states of a spin glass deserves special attention. Such an organization might exist in other systems with frustration and disorder, and it should have consequences in such fields as optimization problems [4] or neural networks [5]. In spin glasses it was found by a direct inspection of the triangles in the space of equilibrium states (all triangles turn out to be either equilateral or isoceles with a shorter third side, which is characteristic of ultrametric spaces), but its physical interpretation (how do the local magnetizations at site i in state α, m_i^α, manage to build up such a strange space ?) was not exhibited. This paper is devoted to the understanding of this microstructure.

A simple representation of an ultrametric space is a genealogical tree (Fig. 1). The different states, α, β, ... of the system are the extremities of the branches of the

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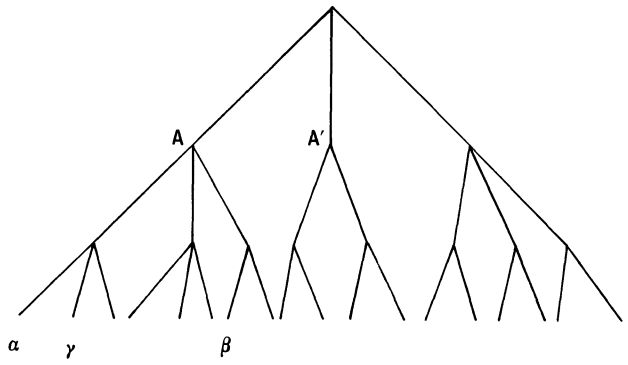

Fig. 1. — The tree of the states. The different states α, β, γ... are the extremities of the branches of the tree. The distance between two states is a monotonic function of the number of steps one has to climb along the tree to find a common ancestor.

tree. The overlap between two states depends only on their closest common ancestor : the higher one must go in the tree to find this ancestor, the lower the overlap q, the larger the distance between the states.

The microstructure deduced in this paper can be schematically summarized as follows :

1) All the pure states α, β, γ, ... which have a given common ancestor (point A in Fig. 1) share a common property : there exists a partition of the N sites into disjoint macroscopic cells $\mathcal{C}_1, \ldots \mathcal{C}_k$ such that the average magnetization of each of these states α, β, γ, ... in every cell $\mathcal{C}_l$ is the same, $\mathcal{M}_l$, within a given resolution $\Delta\mathcal{M}$:

$$\mathcal{M}_l \leqslant \frac{1}{|\mathcal{C}_l|} \sum_{i \in \mathcal{C}_l} m_i^\alpha = \frac{1}{|\mathcal{C}_l|} \sum_{i \in \mathcal{C}_l} m_i^\beta = \cdots \leqslant \mathcal{M}_l + \Delta\mathcal{M} \tag{1}$$

(here $|\mathcal{C}_l|$ is the volume, i.e. the number of sites, of the cell $\mathcal{C}_l$, $\sum_l |\mathcal{C}_l| = N$).

2) In every two states α, β, ..., of which A is the *closest* common ancestor ($q^{\alpha\beta} = q$), the local magnetizations inside each cell $\mathcal{C}_l$ are completely uncorrelated, and have the same distribution. Hence the overlap $q^{\alpha\beta}$ is :

$$q^{\alpha\beta} = \frac{1}{N} \sum_i m_i^\alpha m_i^\beta = \sum_l \frac{|\mathcal{C}_l|}{N} (\mathcal{M}_l)^2 = q \,. \tag{2}$$

3) Considering two states α, γ which link below A ($q^{\alpha\gamma} = q' > q$, see Fig. 1), their magnetizations in each cell $\mathcal{C}_l$ are correlated in a simple way : there exists a subpartition of $\mathcal{C}_l$ into subcells $\mathcal{C}_{l,1}, \ldots \mathcal{C}_{l,l'}$ such that α and γ have the same average magnetization $\mathcal{M}_{l,l'}$ into each cell $\mathcal{C}_{l,l'}$ (always within a given resolution $\Delta\mathcal{M}$), with the obvious properties :

$$\sum_{l'} |\mathcal{C}_{l,l'}| = |\mathcal{C}_l| \,; \quad \sum_{l'} \frac{|\mathcal{C}_{l,l'}|}{|\mathcal{C}_l|} \mathcal{M}_{l,l'} = \mathcal{M}_l \tag{3}$$

and the local magnetizations of α and γ inside each $\mathcal{C}_{l,l'}$ are uncorrelated and have the same distribution, which implies that :

$$q^{\alpha\gamma} = \sum_l \frac{|\mathcal{C}_l|}{N} \sum_{l'} \frac{|\mathcal{C}_{l,l'}|}{|\mathcal{C}_l|} (\mathcal{M}_{l,l'})^2 = q' > q \,. \tag{4}$$

The details of this structure (volumes of the cells and of the subcells, values of the magnetizations $\mathcal{M}_l$ and $\mathcal{M}_{l,l'}$, distribution of local magnetizations inside each cell, ...) are explained in section 3. Their evolutions as functions of the overlap q of the ancestor A are described by differential equations similar to those studied by Parisi and others to obtain the free energy of the system, which receive here a clear interpretation in purely static terms.

On our way to derive the previous properties we have computed the distribution of local magnetizations in each state α :

$$\mathcal{P}_\alpha(m) = 1/N \left[\sum_i \delta(m_i^\alpha - m) \right] \tag{5}$$

and the correlations of local magnetizations in different states α_1, α_2, ..., α_k, which read for $k = 2$:

$$\mathcal{P}_{\alpha,\beta}(m, m') = 1/N \left[\sum_i \delta(m_i^\alpha - m)\, \delta(m_i^\beta - m') \right]. \tag{6}$$

It may be surprising that such quantities can be computed since there is no known way of computing the thermal average of a given observable $\mathcal{O}$ in one give pure state α : $\langle \mathcal{O} \rangle_\alpha$. In fact in the replica method one can compute only :

$$\overline{\langle \mathcal{O} \rangle} = \overline{\sum_\alpha P_\alpha \langle \mathcal{O} \rangle_\alpha} \tag{7}$$

(where $\langle \mathcal{O} \rangle$ is the Gibbs average, decomposed as a sum over pure states [1], and $\overline{(\ )}$ denotes the average over the random couplings), and many quantities, among which the weights P_α, depend on the state α, and on the sample [2, 6]. However we will look for quantities which are both sample independent (self averaging) and state independent (we shall call them reproducible). It turns out that many observables are reproducible (a general sufficient criterion for reproducibility will be given in (14)), among which :

— The distribution of local magnetizations $\mathcal{P}_\alpha(m)$ is independent of α :

$$\mathcal{P}_\alpha(m) = \mathcal{P}(m) \,. \tag{8}$$

— The correlation of local magnetizations in two different states α, β depends only on the overlap $q^{\alpha\beta} = (1/N) \sum_i m_i^\alpha m_i^\beta$ between them :

$$\mathcal{P}_{\alpha,\beta}(m, m') = \mathcal{P}_{q^{\alpha\beta}}(m, m') \,. \tag{9}$$

The reproducible quantities such as $\mathcal{P}(m)$ and $\mathcal{P}_q(m, m')$ can then be computed by the replica method, along lines similar to those introduced by Parisi [1].

The organization of the paper is as follows :

In section 2 we explain the general formalism which enables us to prove the reproducibility and to compute local averages like the distribution of local magnetization. This section is rather lengthy and technical, since it is intended to provide the reader all the basic computational techniques used in the replica solution of the mean field theory of spin glasses. Some of the results have already appeared in other works [7-9]. The reader only interested in the results should skip this section.

In section 3 we analyse the correlation of the distribution of local magnetizations between different pure states, and we deduce from this analysis the structure in cells of the space, together with the properties of the cells at different scales.

Section 4 is devoted to the direct computation of the average couplings between the cells. We deduce from it a set of T.A.P. like equations which are shown to be compatible with the usual T.A.P. equations together with the assumption of the existence of the cells.

Conclusions and perspectives are summarized in section 5.

2. Distribution of local magnetizations-reproducibility.

In this section we explain the basic techniques used to prove the reproducibility of certain observables and to compute their averages. First let us introduce our notations and normalizations : we work in the mean field theory of spin glasses defined by the Sherrington-Kirkpatrick (S.K.) Hamiltonian

$$H_{\text{S.K.}}(J, \sigma) = - \sum_{i<j} J_{ij} \sigma_i \sigma_j - h \sum_i \sigma_i \qquad (10)$$

$\sigma_i = \pm 1$ are N Ising spin variables, and the infinite range couplings J_{ij} are independent random variables with a Gaussian distribution of mean zero and variance $\overline{J_{ij}^2} = 1/N$.

The replica method enables one to perform the quenched average over the random couplings, by introducing at each site i n replicas of the spin variable : σ_i^a, $a = 1, ..., n$. The quenched free energy at temperature $1/\beta$ is then :

$$\frac{\overline{\ln Z}}{N} = \lim_{n \to 0} \left(\frac{\beta^2}{4} - \frac{\beta^2}{4} \frac{1}{n} \sum_{a \neq b} Q_{ab}^2 + \right.$$

$$\left. + \frac{1}{n} \ln \operatorname{Tr}_{\{\sigma\}} e^{-\beta H(Q, \sigma)} \right) \qquad (11)$$

where Q_{ab} is a $n \times n$ symmetric matrix determined by a saddle point condition. In the following we shall always use the form of Q_{ab} found by Parisi [10], which describes replica symmetry breaking in the sense that the matrix elements Q_{ab} are not all equal. This R.S.B. scheme has been shown to be stable [11], and gives results in agreement with all the numerical computations on the S.K. model today [12, 13]. Our precise notations for Q_{ab} are the standard ones, as described for instance in [2].

In (11), $H(Q, \sigma)$ is a one site Hamiltonian coupling the n replicas. given by :

$$H(Q, \sigma) = - \frac{\beta^2}{2} \sum_{a \neq b} Q_{ab} \sigma_a \sigma_b - \beta h \sum_a \sigma_a . \qquad (12)$$

The techniques we use enable one to compute products of averages in different states at fixed distances, of the generic form :

$$A = \overline{\sum_{\alpha_1, ..., \alpha_k} P_{\alpha_1} ... P_{\alpha_k} \langle \mathcal{O}(\sigma_i^{\alpha_1}, ..., \sigma_i^{\alpha_k}) \rangle_{\alpha_1, ..., \alpha_k}} \times$$

$$\times \overline{\prod_{1 \leq r < s \leq k} \delta(q^{\alpha_r \alpha_s} - q^{rs})} \qquad (13)$$

where $\mathcal{O}$ is any observable which is an extensive function of the local spin variables $\sigma_i^{\alpha_s}$ in any of the k-states. A class of reproducible observables is given by the ones which we call $1/N$ dominant. They are arbitrary functions of the following invariants :

$$\lim_{N \to \infty} \frac{1}{N} \sum_i (\sigma_i^{\alpha_{j_1}}) (\sigma_i^{\alpha_{j_2}}) ... (\sigma_i^{\alpha_{j_p}}) ,$$

$$j_1, ..., j_p \in \{ 1, ..., k \} . \qquad (14)$$

We are leaving out those functions which in the limit $N \to \infty$ are not scaled as in (14), such as for instance the susceptibility which is a double sum over sites but is divided by only one power of N. We can generalize this class by including invariants which depend on the matrix J_{ij}. In this case and for the purpose of classifying them as $1/N$ dominant or not, J_{ij} may be replaced by $1/N \sigma_i^{\alpha_0} \sigma_j^{\alpha_0}$ (with α_0 an arbitrary singled out state) thereby reducing it to products of invariants of the previous kind. Notice that with this definition the Boltzmann Gibbs weights (and the P_α) are not $1/N$ dominant.

We will prove that for all $1/N$ dominant observables the average value A factorizes :

$$A = \overline{\sum_{\alpha_1, ... \alpha_k} P_{\alpha_1} ... P_{\alpha_k} \prod_{1 \leq r < s \leq k} \delta(q^{\alpha_r \alpha_s} - q^{rs})} \, X \qquad (15)$$

where X is reproducible and self averaging and measures the observable $\mathcal{O}$ for any k-uple of states that have fixed mutual overlaps q^{rs}.

In this section we will discuss the $1/N$ dominant observables without the J_{ij} (in section 4 we shall use invariants including the J_{ij}'s). Without loss of generality we can consider the observable

$$\mathcal{O}_\eta = \exp \frac{\eta}{N} \sum_i \sigma_i^{\alpha_1} ... \sigma_i^{\alpha_k} . \qquad (16)$$

The first step in the computation of A is to express it as a usual statistical mechanics average using the method introduced in [1] :

$$A = \overline{\left\langle \mathcal{O}_\eta \prod_{1 \leq r < s \leq k} \delta\left(\frac{1}{N} \sum_i \sigma_i^r \sigma_i^s - q^{rs} \right) \right\rangle_{(k)}} \qquad (17)$$

where the average $\langle \quad \rangle_{(k)}$ is the standard Gibbs average taken for a system of k identical non interacting copies of the original system, with Hamiltonian :

$$H^{(k)} = \sum_{r=1}^{k} H_{\text{S.K.}}(J, \sigma^r) . \tag{18}$$

The average in (17) can then be computed by any method of statistical mechanics, for instance by Monte Carlo [12]. Here we shall use the replica method to do the average over the disorder in (17). The systems $1, ..., k$ are any systems among n, and one should average over the different choices [14]. This gives :

$$A = \lim_{n \to 0} \frac{1}{n(n-1) \dots (n-k+1)} \sum_{a_1, \dots, a_k}' \text{Tr}_{\{\sigma^a\}} \times$$

$$\times \exp\left[- \beta \sum_{r=1}^{k} H_{\text{S.K.}}(J, \sigma^r) + \frac{\eta}{N} \sum_i \sigma_i^{a_1} \dots \sigma_i^{a_k} \right]$$

$$\times \prod_{1 \leqslant r < s \leqslant k} \delta\left(\frac{1}{N} \sum_i \sigma_i^{a_r} \sigma_i^{a_s} - q^{rs} \right) \tag{19}$$

where the $\sum'_{a_1, \dots, a_k}$ is a sum over the indices $a_r = 1, ..., n$, with the constraint that all the indices be different from each other.

Formulae (17) to (19) are the basic steps expressing the physical quantity A defined in (13) as a trace over spin variables in a replicated system. We shall not repeat them for each new quantity we compute, but just summarize these steps by the expression « ... going into replica space... ».

The evaluation of (19) is very cumbersome but relatively straightforward.

We shall show how to do it in one very simple example and from it induce the general result. Let us therefore approximate $q(x)$ by

$$q(x) = \begin{cases} q_0 & 0 \leqslant x \leqslant x_1 \\ q_1 & x_1 \leqslant x \leqslant x_2 \\ q_2 & x_2 \leqslant x \leqslant 1 \end{cases} \tag{20}$$

and let us consider the special case

$$\mathcal{O}_\eta = \exp \frac{\eta}{N} \sum_i \sigma_i^1 \sigma_i^2 \sigma_i^3 \tag{21}$$

with fixed overlaps :

$q^{12} = q_2$ and $q^{23} = q_1$ so that $q^{31} = q_1$ is enforced by ultrametricity.

Notice that q_2 is the maximal overlap so that the replicas a_1 and a_2 are in the same state. Therefore we are calculating :

$$\sum_{\alpha, \beta} P_\alpha^2 P_\beta \, \delta(q^{\alpha\beta} - q_1) \exp \frac{\eta}{N} \sum_i (m_i^\alpha)^2 m_i^\beta . \tag{22}$$

Going into replica space we obtain :

$$A = \lim_{n \to 0} \frac{1}{n(n-1)(n-2)} \sum_{a_1 a_2 a_3}' \exp N \frac{\beta^2}{4} \left(n - \sum_{a \neq b} Q_{ab}^2 \right) \times$$

$$\times \delta_{Q_{a_1 a_2}, q_2} \delta_{Q_{a_1 a_3}, q_1} \left(\text{Tr}_{\{\sigma^a\}} \left\{ \exp\left[- \beta H(Q, \sigma) + \frac{\eta}{N} \sigma^{a_1} \sigma^{a_2} \sigma^{a_3} \right] \right\} \right)^N \tag{23}$$

where $H(Q, \sigma)$ is the one site Hamiltonian defined in (12).

The contribution to A from each triplet of replica indices a_1, a_2, a_3 satisfying the constraints $Q_{a_1 a_2} = q_2$, $Q_{a_1 a_3} = q_1$ is the same, we call it A_2.

Using

$$\exp \frac{\eta}{N} \sigma^{a_1} \sigma^{a_2} \sigma^{a_3} = \cosh \frac{\eta}{N} + \sigma^{a_1} \sigma^{a_2} \sigma^{a_3} \sinh \frac{\eta}{N} \tag{24}$$

we obtain :

$$A = A_1 \times A_2$$

$$A_1 = \lim_{n \to 0} \frac{1}{n(n-1)(n-2)} \sum_{a_1 a_2 a_3}' \delta_{Q_{a_1 a_2}, q_2} \delta_{Q_{a_1 a_3}, q_3}$$

$$A_2 = \left(\cosh \frac{\eta}{N} \text{Tr} \, e^{-\beta H(Q, \sigma)} + \sinh \frac{\eta}{N} \text{Tr} \, \sigma^r \sigma^s \sigma^t e^{-\beta H(Q, \sigma)} \right)^N \tag{25}$$

with $Q_{rs} = q_2$, $Q_{rt} = q_1$.

The first term is simply the probability of finding 3 states with specified overlaps :

$$A_1 = \overline{\sum_{\alpha, \beta} P_\alpha^2 P_\beta \, \delta_{q^{\alpha\beta}, q_1}} . \tag{26}$$

In the second term we take the limits $N \to \infty$, $n \to 0$. Using :

$$\lim_{n\to 0} \mathrm{Tr}\, e^{-\beta H} = 1 ;$$

$$\lim_{n\to 0} \mathrm{Tr}\, \sigma^r \sigma^s \sigma^t e^{-\beta H} = \langle\!\langle\, \sigma^r \sigma^s \sigma^t \,\rangle\!\rangle \qquad (27)$$

we find :

$$A_2 = \exp \eta \langle\!\langle\, \sigma^r \sigma^s \sigma^t \,\rangle\!\rangle . \qquad (28)$$

The factorization $A = A_1 A_2$ is an example of the general factorization valid for all $1/N$ dominant observables as announced in (15). As the result (28) for A_2 is valid whatever η, we find finally in this simple example that $1/N \sum_i (m_i^\alpha)^2 m_i^\beta$ is independent of the states α, β which are chosen, provided that $q^{\alpha\beta}$ is fixed equal to q_1. Its value is given by the one site average :

$$\frac{1}{N} \sum_i (m_i^\alpha)^2 m_i^\beta = \langle\!\langle\, \sigma^r \sigma^s \sigma^t \,\rangle\!\rangle \quad \text{where} \quad \begin{aligned} Q_{rs} &= q_2 \\ Q_{rt} &= q_1 . \end{aligned} \qquad (29)$$

The same kind of argument applies to any $1/N$ dominant observable which doesn't contain the J_{ij}'s. It enables one to prove that these observables are reproducible once all the mutual overlaps between the states are fixed, and their values are given in general by products of one site averages.

We must now calculate

$$\langle\!\langle\, \sigma^r \sigma^s \sigma^t \,\rangle\!\rangle . \qquad (30)$$

The general technique we shall use is inspired by the work of de Almeida and Lage [7] who first computed the distribution of local magnetization. It takes full advantage of the ultrametric structure of the matrix Q_{ab} in Parisi's R.S.B., which allows an easy description of this matrix as a tree (Fig. 2) : replica indices are the extrema of the branches, and the value of the matrix element Q_{ab} depends only on the closest common ancestor to a and b. The number of iterations in Parisi's R.S.B. scheme is the number of branching levels in the tree (3 in our case, M in general). At level k, the matrix element is q_k, and the number of branchings for each branch is n_k/n_{k+1}. Let us first discuss our « tree method » on the example.

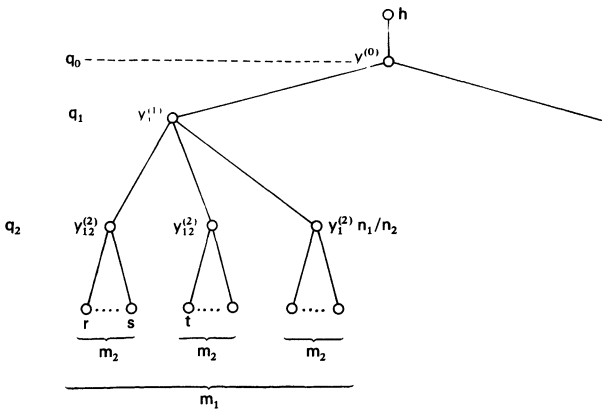

Fig. 2. — The regular tree of replica. The diagram describes the Parisi matrix Q_{ab} with three replica symmetry breakings.

The method [10, 15, 7, 8, 9] can be more easily described if we change the labelling of the replicas using $M = 3$ (number of branching levels) indices instead of just one (see Fig. 2) :

$$\begin{aligned} a \to u, v, w \qquad & 1 \leqslant u \leqslant \frac{n_0}{n_1} \\ 1 \leqslant a \leqslant n_0 = n \qquad & 1 \leqslant v \leqslant \frac{n_1}{n_2} \\ & 1 \leqslant w \leqslant \frac{n_2}{n_3} \end{aligned} \qquad (31)$$

while $r = (1, 1, 1)$; $s = (1, 1, 2)$; $t = (1, 2, 1)$.

Then :

$$Q_{ab}\, \sigma^a \sigma^b = q_0 \Big(\sum_{u,v,w} \sigma^{uvw} \Big)^2 + (q_1 - q_0) \sum_u \Big(\sum_{v,w} \sigma^{uvw} \Big)^2 + (q_2 - q_1) \sum_{u,v} \Big(\sum_w \sigma^{uvw} \Big)^2 - q_2 \sum_{uvw} (\sigma^{uvw})^2 . \qquad (32)$$

To calculate the trace over the σ we use the Hubbard Stratanovich transformation for each of the terms in (32). We introduce :

$$y^{(0)} ; \quad y_u^{(1)} ; \quad y_{u,v}^{(2)} \qquad (33)$$

so that :

$$\begin{aligned} \langle\!\langle\, \sigma^r \sigma^s \sigma^t \,\rangle\!\rangle = & \int \frac{dy^{(0)}}{\sqrt{2\pi q_0}} \prod_u \left(\frac{dy_u^{(1)}}{\sqrt{2\pi(q_1 - q_0)}} \prod_v \frac{dy_{u,v}^{(2)}}{\sqrt{2\pi(q_2 - q_1)}} \right) \times \\ & \times \exp\left(- \frac{(y^{(0)} - h)^2}{2 q_0} - \sum_u \frac{(y_u^{(1)} - y^{(0)})^2}{2(q_1 - q_0)} - \sum_{u,v} \frac{(y_{u,v}^{(2)} - y_u^{(1)})^2}{2(q_2 - q_1)} \right) \times \\ & \times \mathrm{Tr}_{\{\sigma^{uvw}\}} \left[\sigma^{111}\, \sigma^{112}\, \sigma^{121} \exp \beta \sum_{uvw} y_{uv}^{(2)}\, \sigma^{uvw} \right] . \end{aligned} \qquad (34)$$

The $\mathrm{Tr}_{\{\sigma^{uvw}\}}$ is equal to :

$$\prod_{u,v} (\cosh \beta y_{u,v}^{(2)})^{n_2} \tanh^2 \beta y_{1,1}^{(2)} \tanh \beta y_{1,2}^{(2)} . \tag{35}$$

The integrations over the y's are done in successive steps, going up the tree in figure 2. At the first step we integrate over n/n_2 variables, the $y_{u,v}^{(2)}$. We get :

$$\left[\prod_{u=2}^{n/n_1} T(y_u^{(1)})\right] \cdot [T'(y_1^{(1)})] \tag{36}$$

with :

$$T(y_u^{(1)}) = \left(\int \frac{dy}{\sqrt{2\pi(q_2 - q_1)}} \exp\left[-\frac{(y - y_u^{(1)})^2}{2(q_2 - q_1)}\right] \cosh^{n_2} \beta y\right)^{\frac{n_1}{n_2}}$$

$$T'(y_1^{(1)}) = \int \frac{dy}{\sqrt{2\pi(q_2 - q_1)}} \exp -\frac{(y - y_1^{(1)})^2}{2(q_2 - q_1)} \cosh^{n_2} \beta y \tanh \beta y \times$$

$$\times \int \frac{dy'}{\sqrt{2\pi(q_2 - q_1)}} \exp -\frac{(y' - y_1^{(1)})^2}{2(q_2 - q_1)} \cosh^{n_2} \beta y' \tanh^2 \beta y'$$

$$\times \left(\int \frac{dy''}{\sqrt{2\pi(q_2 - q_1)}} \exp -\frac{(y'' - y_1^{(1)})^2}{2(q_2 - q_1)} \cosh^{n_2} \beta y''\right)^{\frac{n_1}{n_2} - 2} . \tag{37}$$

At the second step we integrate over the $y_u^{(1)}$ and get :

$$\left(\int \frac{dy^{(1)}}{\sqrt{2\pi(q_1 - q_0)}} \exp -\frac{(y^{(1)} - y^{(0)})^2}{2(q_1 - q_0)} T(y^{(1)})\right)^{\frac{n}{n_1} - 1} \times \left(\int \frac{dy^{(1)}}{\sqrt{2\pi(q_1 - q_0)}} \exp -\frac{(y^{(1)} - y^{(0)})^2}{2(q_1 - q_0)} T'(y^{(1)})\right). \tag{38}$$

The final step is to integrate over $y^{(0)}$. The appearance of the diffusion kernel :

$$\frac{1}{\sqrt{2\pi \Delta q}} \exp -\frac{(\Delta y)^2}{2 \Delta q} \tag{39}$$

shows that the calculation can be seen as the result of a branched diffusion process along the tree. This mathematical interpretation is useful to extract the general rules that allow one to arrive to a formula like (38) reading it directly from the tree :

a) The whole tree is an operator which acts on n functions of y to produce a single function of y ;

b) Each oriented line on the tree joining two branching points situated at overlaps q_k, q_{k-1} acts on one function of y at q_k to produce one function at q_{k-1} :

$$(C_{q_k - q_{k-1}}[f])(y) = \frac{1}{\sqrt{2\pi(q_k - q_{k-1})}} \int \exp\left[-\frac{(y - y')^2}{2(q_k - q_{k-1})}\right] f(y')\, dy' ; \tag{40}$$

c) At each branching point the functions arriving to it are simply multiplied at the same value of y ;

d) At the lower end we begin either from $\cosh \beta y . \tanh \beta y$ for those replicas whose spin average value are calculated, $\cosh \beta y$ for the rest ;

e) At the upper end we identify $y = h$ (external magnetic field).

These rules are sufficient to do all computations of one site averages. However in practice they can be simplified, and the limit $n \to 0$ (tree with 0 branch !) can be taken more explicitly, through the following reformulation.

In a given computation of $\langle\!\langle\ \sigma^{a_1} \ldots \sigma^{a_k}\ \rangle\!\rangle$, there is always a *finite* number k of privileged replica indices, and k remains finite even when $n \to 0$.

We call privileged branches those to which the privileged replica indices are attached (see Fig. 3). Climbing up these privileged branches means going through a new diffusion process (which is the diffusion process we defined before, interrupted at each step by the grafting of normal branches), until two or more privileged branches join.

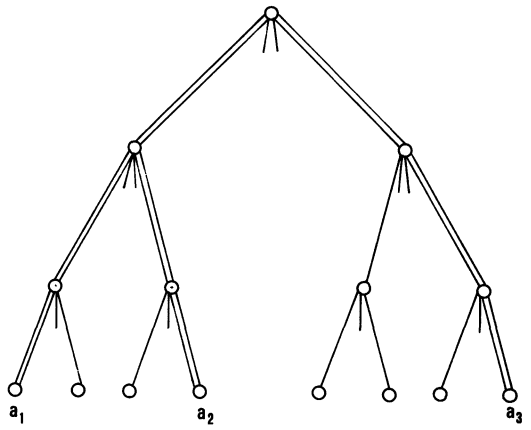

Fig. 3. — The tree of privileged branches. Double lines represent privileged branches. In the figure we chose k = number of privileged states = 3. The single lines correspond to the normal lines that end in the privileged ones. The $n \to 0$ limit can be done once one has summed over all graftings of normal lines.

Once one has identified this new diffusion process, one is able to take the limit $n \to 0$ explicitly, since one can work only on the « backbone » of the tree formed of the (finite number of) privileged branches.

As we saw before, a normal branch is an operator which acts on the initial function

$$z_M(y) = (\cosh \beta y)^{m_M} \tag{41}$$

and gives at each new level when climbing up the tree a new function, with the recursion relation

$$z_{k-1}(y) = (C_{q_k - q_{k-1}}[z_k(y)])^{\frac{n_{k-1}}{n_k}}. \tag{42}$$

One can take explicitly the limit $n \to 0$, in which case the inequalities $n = n_0 \geqslant n_1 \ldots \geqslant n_M \geqslant 1$ must be reversed : $0 = x_0 \leqslant x_1 \ldots \leqslant x_M \leqslant 1$. (We keep the notation x_i for the value of n_i in the $n \to 0$ limit, so that $0 \leqslant x_i \leqslant 1$.)

The evolution along a privileged branch is inferred directly from the general rules a) to e) given above. Going from level k to $k-1$, there is a grafting of $n_{k-1}/n_k - 1$ normal branches on the privileged one, which leads to a recursion relation :

$$f_{k-1}(y) = [C_{q_k - q_{k-1}} z_k(y)]^{\frac{n_{k-1}}{n_k} - 1} \cdot [C_{q_k - q_{k-1}} f_k(y)]. \tag{43}$$

Introducing $\hat{f}_k(y) = \dfrac{f_k(y)}{z_k(y)}$, we obtain :

$$\hat{f}_{k-1} = \frac{C_{q_k - q_{k-1}}(z_k \hat{f}_k)}{C_{q_k - q_{k-1}}(z_k)} \equiv \mathcal{T}_{q_{k-1}, q_k}[\hat{f}_k]. \tag{44}$$

This defined the evolution operator along a privileged branch. If such a branch goes without encountering another privileged branch from a level $q_{l'}$ to $q_l < q_{l'}$, it corresponds to a diffusion kernel :

$$\mathcal{T}_{q_l, q_{l'}} = \mathcal{T}_{q_l, q_{l+1}} \mathcal{T}_{q_{l+1}, q_{l+2}} \ldots \mathcal{T}_{q_{l'-1}, q_{l'}}. \tag{45}$$

We obtain finally the general rules for the computation of a one site average :

I) Draw the backbone tree of privileged branches.
II) To each privileged replica associate $\hat{f}_M = \tanh \beta y$.
III) Go up in the tree, applying the operator $\mathcal{T}$ along each branch.
IV) At a vertex, multiply the incoming functions $\hat{f}$.
V) At the upper end (level q_0) apply the operator $C_{q_0} \equiv \mathcal{T}_{0, q_0}$.

The example we worked out before reads :

$$\langle\!\langle\, \sigma^r \sigma^s \sigma^t \,\rangle\!\rangle \equiv \mathcal{T}_{0, q_1}[(\mathcal{T}_{q_1, q_2}(\tanh \beta y)).(\mathcal{T}_{q_1, q_2}(\tanh^2 \beta y))]. \tag{46}$$

More generally, computing this same $\langle\!\langle\, \sigma^r \sigma^s \sigma^t \,\rangle\!\rangle$ for $q^{rs} = q$, $q^{rt} = q' > q$, in the full R.S.B. scheme gives $\langle\!\langle\, \sigma^r \sigma^s \sigma^t \,\rangle\!\rangle = \mathcal{T}_{0,q}[(\mathcal{T}_{q, q_M}(\tanh \beta y)).(\mathcal{T}_{q, q'}[\mathcal{T}_{q', q_M} \tanh \beta y]^2)]$.

It is also useful to take the continuum limit, i.e. an infinite number of R.S.B. In this limit we write $x_k = x$; $x_{k-1} = x - dx$, and the above recursive relations read :

— for $z : z_k(y) \to \tilde{z}(x, y)$

$$-\frac{\partial \tilde{z}}{\partial x} = \frac{1}{2}\dot{q}(x)\frac{\partial^2 \tilde{z}}{\partial y^2} - \frac{\tilde{z}}{x}\operatorname{Log}\tilde{z}; \tag{47}$$

— for a privileged branch : $\hat{f}_k(y) \to \tilde{f}(x, y)$

$$-\frac{\partial \tilde{f}}{\partial x} = \frac{1}{2}\dot{q}(x)\left[\frac{\partial^2 \tilde{f}}{\partial y^2} + \frac{\partial \ln \tilde{z}}{\partial y}\frac{\partial \tilde{f}}{\partial y}\right]. \tag{48}$$

3. Local magnetization distributions-organization in clusters.

The distribution of local magnetization in state α :

$$\mathcal{P}_\alpha(m) = \frac{1}{N}\sum_i \delta(m_i^\alpha - m) \tag{49}$$

can be deduced from its moments :

$$M_\alpha^{(k)} = \int \mathcal{P}_\alpha(m)\, m^k\, dm = \frac{1}{m}\sum_i (m_i^\alpha)^k. \tag{50}$$

Using the methods of the previous section, one finds that these moments are reproducible, with :

$$M_\alpha^{(k)} = \langle\!\langle\, \sigma^{a_1} \dots \sigma^{a_k} \,\rangle\!\rangle\,; \quad q^{a_i a_j} = q_M\,, \quad 1 \leqslant i < j \leqslant k\,. \tag{51}$$

This one site average thus corresponds to a simple tree with only one privileged branch, so that :

$$M_\alpha^{(k)} = \mathcal{P}_{0,q_M}((\tanh \beta y)^k) \tag{52}$$

and :

$$\mathcal{P}_\alpha(m) = \mathcal{P}(m) = \mathcal{P}_{0,q_M}[\delta(m - \tanh \beta y)]. \tag{53}$$

Let us now compute the correlations between the magnetizations in states α and β :

$$\mathcal{P}_{\alpha,\beta}(m, m') = \frac{1}{N}\sum_i \delta(m_i^\alpha - m)\,\delta(m_i^\beta - m'). \tag{54}$$

From the double moments :

$$M_{\alpha,\beta}^{(k,l)} = \int \mathcal{P}_{\alpha,\beta}(m, m')\, m^k\, m'^l\, dm\, dm' \tag{55}$$

we find that :

— The moments $M^{(k,l)}$ and hence $\mathcal{P}_{\alpha,\beta}$ are reproducible between pairs of states α, β having a fixed overlap $q^{\alpha\beta} = q$:

$$M_{\alpha,\beta}^{(k,l)} = M_q^{(k,l)}; \qquad \mathcal{P}_{\alpha,\beta}(m, m') = \mathcal{P}_q(m, m'). \tag{56}$$

— The values of the moments are :

$$M_q^{(k,l)} = \langle\!\langle\, \sigma^{a_1} \dots \sigma^{a_k}\, \sigma^{b_1} \dots \sigma^{b_l} \,\rangle\!\rangle \tag{57}$$
$$q^{a_i a_j} = q^{b_i b_j} = q_M \qquad 1 \leqslant i < j \leqslant k$$
$$q^{a_1 b_1} = q < q_M\,.$$

This is again easily computed with the tree method : there are two privileged branches which join at level q, so that

$$M_q^{(k,l)} = \mathcal{P}_{0,q}\left\{[\mathcal{P}_{q,q_M}(\tanh \beta y)^k][\mathcal{P}_{q,q_M}(\tanh \beta y)^l]\right\} \tag{58}$$

and

$$\mathcal{P}_q(m, m') = \mathcal{P}_{0,q}\left\{[\mathcal{P}_{q,q_M}\,\delta(m - \tanh \beta y)][\mathcal{P}_{q,q_M}\,\delta(m' - \tanh \beta y)]\right\}. \tag{59}$$

It is now clear how these results can be generalized to more states. The correlations between the local magnetizations in different states are always reproducible when all the mutual overlaps of the states are fixed. From the structure of the corresponding tree one can derive the analog of (58).

Let us now quote two more examples which will be useful in the following physical discussion. If $\alpha_1 \dots \alpha_k$ are k states which have all mutual overlaps $q^{\alpha_i \alpha_j} = q$, one finds :

$$\mathcal{P}_{\alpha_1 \dots \alpha_k}(m_1, \dots, m_k) \equiv \frac{1}{N}\sum_i \delta(m_i^{\alpha_1} - m_1) \dots \delta(m_i^{\alpha_k} - m_k) =$$
$$= \mathcal{P}_{0,q}\left\{[\mathcal{P}_{q,q_M}\,\delta(m_1 - \tanh \beta y)] \dots [\mathcal{P}_{q,q_M}\,\delta(m_k - \tanh \beta y)]\right\}. \tag{60}$$

If α, β, γ are three states with $q^{\alpha\beta} = q'$, $q^{\alpha\gamma} = q^{\beta\gamma} = q < q'$, one finds :

$$\mathcal{P}_{\alpha\beta\gamma}(m, m', m'') \equiv \frac{1}{N}\sum_i \delta(m_i^\alpha - m)\,\delta(m_i^\beta - m')\,\delta(m_i^\gamma - m'') = \mathcal{P}_{0,q'}\left\{[\mathcal{P}_{q',q_M}\,\delta(m'' - \tanh\beta y)] \times \right.$$
$$\left. \times\,[\mathcal{P}_{q',q}\left\{[\mathcal{P}_{q,q_M}\,\delta(m - \tanh\beta y)].[\mathcal{P}_{q,q_M}\,\delta(m' - \tanh\beta y)]\right\}]\right\}. \qquad (61)$$

Our notations here are in terms of the evolution operators $\mathcal{P}_{q,q'}$, which act on functions of y to give other functions of y, as explained in section 2. It will be useful to use notations a bit more explicit, taking advantage of the fact that these operators are linear operators. We write :

$$\begin{aligned} \tilde{\mathcal{P}}_q(m, y) &= \mathcal{P}_{q,q_M}\,\delta(m - \tanh\beta y) \\ \mathcal{P}_{0,q}\,f(y) &= \int dy\, N_q(y)\, f(y)\,. \end{aligned} \qquad (62)$$

The correlation function between k equidistant states at distance Q can then be written as :

$$\mathcal{P}_Q(m_1, ..., m_k) = \int dy\, N_Q(y)\, \tilde{\mathcal{P}}_Q(m_1, y) ... \tilde{\mathcal{P}}_Q(m_k, y) \qquad (63)$$

where $\tilde{\mathcal{P}}_q(m, y)$ appears in the distribution of local magnetizations in one state :

$$\mathcal{P}(m) = \int dy\, N_q(y)\, \tilde{\mathcal{P}}_q(m, y) \qquad (64)$$

for all q, $q_0 \leqslant q \leqslant q_M$, for instance $q = Q$.

Furthermore one can deduce from the definitions of N and $\tilde{\mathcal{P}}$ the following properties [9] :

$$\begin{aligned} &\forall q : N_q(y) \geqslant 0\,; \qquad \int dy\, N_q(y) = 1 \\ &\forall q : \tilde{\mathcal{P}}_q(m, y) \geqslant 0\,; \qquad \int dm\, \tilde{\mathcal{P}}_q(m, y) = 1 \end{aligned} \qquad (65)$$

which indicate that N_q and $\tilde{\mathcal{P}}_q$ are probability distributions of the variables y and m respectively.

The formulae (63) to (65) are the starting point of our physical discussion. They show that at any scale Q, there exists a random variable y with a distribution $N_Q(y)$ such that, when y is fixed, the distribution of local magnetizations in any k equidistant states at distance Q are independent of each other, with distribution $\tilde{\mathcal{P}}_Q(m, y)$. We claim that to each of the N sites of the lattice can be given a value of the parameter y. This is not obvious at this point and will be proven below through the analysis of the « ancestor state ». It will appear even more evident in section 4. Let us now examine the consequences of this assertion :

— at any scale Q, the sites where the variable y takes values between y and $y + \Delta y$ form a cell, $\mathcal{C}_y$;

— the volume of this cell is $|\mathcal{C}_y| = (N_Q(y)\,\Delta y).N$;

— inside the cell, the local magnetizations in k equidistant states at distance Q are independent random variables with the same distribution $\tilde{\mathcal{P}}_Q(m, y)$:

$$\frac{1}{|\mathcal{C}_y|}\sum_{i\in\mathcal{C}_y} \delta(m_i^{\alpha_1} - m_1) ... \delta(m_i^{\alpha_k} - m_k) = \tilde{\mathcal{P}}_Q(m_1, y) ... \tilde{\mathcal{P}}_Q(m_k, y)$$

$$\tilde{\mathcal{P}}_Q(m, y) = \frac{1}{|\mathcal{C}_y|}\sum_{i\in\mathcal{C}_y} \delta(m_i^\alpha - m)\,. \qquad (66)$$

The different cells $\mathcal{C}_k$ are shared by all the states which are equidistant at distance Q. In the ultrametric tree of the states, a partition of cells is associated with each branching at the scale Q (points A, A' ... of Fig. 1). The cells associated with the point A may not be the same as those associated with A' (they are in fact in general different), but they have the same volumes, and the same distribution of local magnetizations inside them. A very useful concept is that of the ancestor state : to each branching point in the tree of the states, such as A in figure 1, we associate a configuration of local magnetizations $m_i^{(A)}$ which is uniform in the corresponding cells $\mathcal{C}_y^{(A)}$. This is not a pure state of the system, we call it the ancestor state. It satisfies :

$$m_i^{(A)} = m_Q(y), \qquad i \in \mathcal{C}_y^{(A)}, \qquad (67)$$

where $m_Q(y) = \int \tilde{\mathcal{P}}_Q(m, y)\, m\, dm$ is the average magnetization in the cell $\mathcal{C}_y^{(A)}$ for the pure states which are descendants of A.

The ancestor state has a global magnetization equal to the global magnetization in any state :

$$\sum_y \frac{|\mathcal{C}_y^{(A)}|}{N} m_Q(y) = \int N_Q(y)\, dy \int \tilde{\mathcal{P}}_Q(m, y)\, m\, dm = \int \mathcal{P}(m)\, m\, dm \equiv M \tag{68}$$

and a self overlap equal to Q, the level of the tree where the point A lies :

$$\sum_y \frac{|\mathcal{C}_y^{(A)}|}{N} [m_Q(y)]^2 = \int N_Q(y)\, dy [m_Q(y)]^2 = Q\,. \tag{69}$$

There is in fact a way to obtain the ancestor state from its descendants : it can be shown using the methods of [2] that the number of descendants of a particular ancestor A at a finite Δq is always infinite in the thermodynamic limit. Hence it is always possible to choose a family F of $\mathcal{N}$ descendents of A, $\alpha_1, \ldots, \alpha_{\mathcal{N}}$, which are all equidistant at distance Q, with $\mathcal{N} \gg 1$. Defining :

$$m_i^F = \frac{1}{\mathcal{N}} \sum_{k=1}^{\mathcal{N}} m_i^{\alpha_k} \tag{70}$$

one obtains from (63) :

$$\forall r : \frac{1}{N} \sum_{i=1}^{\mathcal{N}} (m_i^F)^r = \int dy\, N_Q(y)\, [m_Q(y)]^r \tag{71}$$

which proves that the variable y labels the sites where $m_i = m_Q(y)$; it is easily proven from (63) again that these sites are the same whatever the choice of the family F of $\mathcal{N}$ equidistant descendants of A. They define a unique cell which is exactly the domain where the magnetization of the ancestor is uniform with value $m_Q(y)$.

Finally let us explain what happens when we change from a scale Q to $Q' > Q$. Let us consider three states $\alpha_1, \alpha_2, \alpha_3$ such that $q^{\alpha_1\alpha_2} = q^{\alpha_1\alpha_3} = Q$, $q^{\alpha_2\alpha_3} = Q'$. The correlation of local magnetizations in these three states depends only on Q and Q', it can be written as :

$$\mathcal{P}_{Q,Q'}(m_1, m_2, m_3) = \int dy\, N_Q(y)\, \tilde{\mathcal{P}}_Q(m_1, y) \cdot \int dy'\, G(Q, Q', y, y')\, \tilde{\mathcal{P}}_Q(m_2, y')\, \tilde{\mathcal{P}}_Q(m_3, y') \tag{72}$$

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where $G(Q, Q', y, y')$ is the integral kernel associated with the evolution operator $\mathcal{P}_{Q,Q'}$ defined in (45).

This formula, which can be obviously generalized to the correlations of any k states at distances Q, l states at distance Q' shows that :

— At the scale $Q' > Q$, any cell $\mathcal{C}_y$ of the scale Q can be cut into subcells $\mathcal{C}_{y,y'}$.

— The probability that a site in the cell $\mathcal{C}_y$ be in a definite subcell $\mathcal{C}_{y,y'}$ is :

$$\frac{|\mathcal{C}_{y,y'}|}{|\mathcal{C}_y|} = G(Q, Q', y, y')\, \Delta y'\,. \tag{73}$$

— States at distance Q' have uncorrelated local magnetizations inside each subcell $\mathcal{C}_{y,y'}$, with the same distribution $\tilde{\mathcal{P}}_{Q'}(m, y')$, while states at distance Q have uncorrelated local magnetizations inside the whole cell $\mathcal{C}_y$.

We notice that the coherence of this picture is enforced by the following identities :

— The subcells form a partition of the cell :

$$\int dy'\, G(Q, Q', y, y') = 1\,.$$

— The distribution of magnetizations in the cell is induced by the distribution in the subcells :

$$\int dy'\, G(Q, Q', y, y')\, \tilde{\mathcal{P}}_{Q'}(m, y') = \tilde{\mathcal{P}}_Q(m, y)\,.$$

Of course, the direct analysis of the structure at the scale Q' shows cells $\mathcal{C}'_{y'}$ of volume $N_{Q'}(y').(\Delta y').N$ where the distribution of magnetizations is $\tilde{\mathcal{P}}_{Q'}(m, y')$. These cells $\mathcal{C}'_{y'}$ are reunions of the $\mathcal{C}_{y,y'}$ corresponding to different y :

$$\mathcal{C}_{y'} = \bigcup_y \mathcal{C}_{y,y'}\,;$$

$$|\mathcal{C}'_{y'}| = N \cdot \int dy\, N_Q(y)\, G(Q, Q', y, y') \cdot \Delta y'\,. \tag{74}$$

Thus a direct study at the scale Q' makes one loose the information about the cells at the scale Q. The maximal information is obtained from the sequence of domains embedded into other domains at all the scales starting from the lowest one ($Q = Q_{\min}$).

4. Couplings between the cells and T.A.P. equations.

In this section we study in details the structure of the cells and the way in which they are coupled one to another. As we saw before, it is crucial in order not to loose any information to keep trace of the partitioning into subcells at *every scale*. Therefore we introduce here a resolution on the function $q(x)$, assuming that it can be approximated by a stepwise constant function with M replica symmetry breakings :

$$q(x) = q_l, \qquad x_l \leqslant x \leqslant x_{l+1} \tag{75}$$

where $l = 0, ..., M$; $x_0 = 0, x_{M+1} = 1$.

The resolution in the parameters y necessary for the definition of the cells is also self understood in the following.

The discussion in the previous section tells us that :

— Any pure state α is characterized by a number of cells $\mathcal{C}^{\alpha}_{y_0 y_1 ... y_M}$ inside which the magnetization m_i^{α} is constant and takes the value $\tanh \beta y_M$.

— Any ancestor state A living at the scale (age) q_l is characterized by cells $\mathcal{C}^{(A)}_{y_0 y_1 ... y_l}$ inside which the magnetization $m_i^{(A)}$ is uniform and takes the value $m_{q_l}(y_l)$.

— The cells $\mathcal{C}^{\alpha}$ of all the descendants α of a given ancestor A are disjoint subcells of the $\mathcal{C}^{A}$ and :

$$\bigcup_{y_{l+1}, ..., y_M} \mathcal{C}^{\alpha}_{y_0 ... y_l y_{l+1} ... y_M} = \mathcal{C}^{(A)}_{y_0 ... y_l} . \tag{76}$$

The set of all spin configurations with such an arrangement can be constructed in the following way :

a) At every site of the lattice imagine simultaneous independent stochastic processes according to the Langevin's equation [18] :

$$\frac{dy_i}{dx}(x) = \beta x \frac{dq}{dx} m_{q(x)}(y_i) + z_i(x) \sqrt{\dot{q}(x)} \tag{77}$$

where $z(x)$ is a Gaussian noise with

$$\overline{z_i(x)} = 0$$
$$\overline{z_i(x)\, z_j(x')} = \delta(x - x')\, \delta_{ij} \tag{78}$$

and the initial condition is $y_i(0) = h_{ext}$, $q(0) = 0$, $q(0^+) = q_0$.

b) Assign the magnetization $m_i = \tanh [\beta y_i(1)]$ to each site.

The discretization we were referring to above is simply the discretization of the Langevin's equation. The ramification of the tree is a trivial consequence of the possible random choices that one can make at every step of x. The ancestor results from integrating the Langevin equation up to x_0 and assigning the magnetization $m_{q(x_0)}(y_i(x_0))$ at each site i.

However it is important to notice that only a small part of the configurations so generated are pure states of the system.

A case which is particularly interesting is the ancestor at the smallest possible scale $q_{min} = q_0$. It is a common ancestor to all the pure states of the system (we call it grand ancestor G-A) and thus its cells $\mathcal{C}^{G-A}_{y_0}$ are common to all the states, they form a universal partition of the system. (This partition is also the only one which exists above the critical temperature in a field, where $q(x) = q_0$, $0 \leqslant x \leqslant 1$ and there is only one state.)

In order to understand the tree of states better, we have computed the average couplings between the cells associated with a given ancestor. First let us consider the case of the largest cells, the ones of the grand ancestor. We define :

$$J_{y_0, y_0'} = \frac{N}{|\mathcal{C}_{y_0}|\,|\mathcal{C}_{y_0'}|} \sum_{\substack{i \in \mathcal{C}^{G-A}_{y_0} \\ j \in \mathcal{C}^{G-A}_{y_0'}}} J_{ij} . \tag{79}$$

These couplings between the $\mathcal{C}^{G-A}_{y_0}$ can be computed from the reproducible and self averaging functions :

$$M_{s,t} = \frac{1}{N} \sum_{i,j} m_i^{\alpha_1} ... m_i^{\alpha_s} J_{ij} m_j^{\beta_1} ... m_j^{\beta_t} \tag{80}$$

where the α's and the β's are $s + t$ pure states which are all equidistant at distance q_0. Proceeding as in section 3 by taking a large family $\mathcal{N}$ of such states and averaging over the possible choices of α_i, we get :

$$\frac{1}{\mathcal{N}} \sum_{\alpha=1}^{\mathcal{N}} m_i^{\alpha} = m_i^{G-A}$$

$$M_{s,t} \underset{\mathcal{N} \to \infty}{\sim} \frac{1}{N} \sum_{i,j} [m_i^{G-A}]^s J_{ij} [m_j^{G-A}]^t \tag{81}$$

which reads

$$M_{s,t} = \sum_{y_0, y_0'} \frac{|\mathcal{C}^{G-A}_{y_0}|\,|\mathcal{C}^{G-A}_{y_0'}|}{N^2} [m_{q_0}(y_0)]^s [m_{q_0}(y_0')]^t J_{y_0 y_0'} . \tag{82}$$

We see that the $J_{y_0, y_0'}$ are self-averaging quantities which can be inferred from the $M_{s,t}$.

The quantities $M_{s,t}$ are computed in the appendix. One obtains :

$$J_{y_0, y_0'} = m_{q_0}(y_0') \frac{y_0 - h}{q_0} + m_{q_0}(y_0) \frac{y_0' - h}{q_0} . \tag{83}$$

Before analysing this formula let us derive the corresponding formula for the domains of an ancestor at a scale $q > q_0$. For instance at the scale q_1, one must compute the functions :

$$M^{u,v}_{s,t} = \frac{1}{N} \sum_{i,j} m_i^{\alpha_1} ... m_i^{\alpha_s} m_i^{\gamma_1} ... m_i^{\gamma_u} J_{ij} m_j^{\beta_1} ... m_j^{\beta_t} m_j^{\delta_1} ... m_j^{\delta_v} \tag{84}$$

where $\alpha_1, ..., \alpha_s, \beta_1, ..., \beta_t, \gamma_1, \delta_1$ are $s + t + 2$ states all equidistant at distance q_0, $\gamma_1, ..., \gamma_u$ are all equidistant at distance q_1, and $\delta_1 ... \delta_v$ are all equidistant at distance q_1. One has then :

$$M^{s,t}_{u,v} = \frac{1}{N^2} \sum_{y_0,y'_0} | \mathcal{C}^{G-A}_{y_0} | \, | \mathcal{C}^{G-A}_{y'_0} | \, m_{q_0}(y_0)^s \, m_{q_0}(y'_0)^t \times \sum_{y_1,y'_1} \frac{| \mathcal{C}^{(A)}_{y_0,y_1} | \, | \mathcal{C}^{(A)}_{y'_0,y'_1} |}{| \mathcal{C}^{G-A}_{y_0} | \, | \mathcal{C}^{G-A}_{y'_0} |} J^{A}_{y_0y_1,y'_0y'_1} . m_{q_1}(y_1)^u \, m_{q_1}(y'_1)^v \tag{85}$$

where :

$$J^{A}_{y_0y_1,y'_0y'_1} \equiv \frac{N}{| \mathcal{C}^{(A)}_{y_0,y_1} | \, | \mathcal{C}^{(A)}_{y'_0,y'_1} |} \sum_{\substack{i \in \mathcal{C}^{(A)}_{y_0 y_1} \\ j \in \mathcal{C}^{(A)}_{y'_0 y'_1}}} J_{ij} . \tag{86}$$

A first striking result is obtained from the reproducibility of the moments $M^{s,t}_{u,v}$. Although the subcells $\mathcal{C}^{(A)}_{y_0,y_1}$ associated with different ancestors at the scale q_1 depend on the ancestor, the average coupling between them, $J^{A}_{y_0y_1,y'_0y'_1}$ is independent of A. Using the result of the computation of $M^{s,t}_{u,v}$ done in the appendix, we find :

$$J_{y_0y_1,y'_0y'_1} = J_{y_0,y'_0} + \frac{y'_1 - y'_0}{q_1 - q_0}(m_{q_1}(y_1) - m_{q_0}(y_0)) + \frac{y_1 - y_0}{q_1 - q_0}(m_{q_1}(y'_1) - m_{q_0}(y'_0)) - \\ - \beta x_1 (m_{q_1}(y_1) \, m_{q_1}(y'_1) - m_{q_0}(y_0) \, m_{q_0}(y'_0)) . \tag{87}$$

This formula can be generalized to the couplings between domains at an arbitrary scale. Once again, these couplings are independent of the ancestor state which is considered ; the result is best expressed by the recursive relation :

$$J_{y_0 \ldots y_l, y'_0 \ldots y'_l} = J_{y_0 \ldots y_{l-1}, y'_0 \ldots y'_{l-1}} + \frac{y_l - y_{l-1}}{q_l - q_{l-1}}(m_{q_l}(y'_l) - m_{q_{l-1}}(y'_{l-1})) + \frac{y'_l - y'_{l-1}}{q_l - q_{l-1}}(m_{q_l}(y_l) - m_{q_{l-1}}(y_{l-1})) - \\ - \beta x_l [m_{q_l}(y_l) \, m_{q_l}(y'_l) - m_{q_{l-1}}(y_{l-1}) \, m_{q_{l-1}}(y'_{l-1})] . \tag{88}$$

Several properties can be deduced from equations (87)-(88).

a) There exist in the ancestor states T.A.P.-like equations [16] :

$$\sum_{y'_0,\ldots,y'_l} \frac{| \mathcal{C}_{y'_0 \ldots y'_l} |}{N} J_{y_0 \ldots y_l, y'_0 \ldots y'_l} \, m_{q_l}(y'_l) = y_l - h + m_{q_l}(y_l) \sum_{y'_0,\ldots,y'_l} \frac{| \mathcal{C}_{y'_0 \ldots y'_l} |}{N} \frac{\partial}{\partial y'_l} m_{q_l}(y'_l) \tag{89}$$

in which y_l is the local field, and creates a local magnetization $m_{q_l}(y_l)$. This suggests new ways of approaching the solution of the S.K. model by solving these equations first for the ancestor and then for the descendants.

b) The matrix of the couplings between the cells possesses the remarkable property :

$$\sum_{y'_{l+1} \ldots y'_M} J_{y_0 \ldots y_M, y'_0 \ldots y'_M} \, | \mathcal{C}_{y'_0 \ldots y'_M} | = \\ = J_{y_0 \ldots y_l, y'_0 \ldots y'_l} \, | \mathcal{C}_{y'_0 \ldots y'_l} | \tag{90}$$

whatever the value of $y_{l+1}, ..., y_M$. Such a structure should be found in a large random matrix, remembering that from (86) the $J_{y_0 \ldots y_M, y'_0 \ldots y'_M}$ are themselves averages of J_{ij}'s over *macroscopic* cells.

c) In the limit where the magnetic field is large, one can identify easily the universal cells of the grand-ancestor state. They are regions where the quantity :

$$G_i = \sum_j J_{ij} \tag{91}$$

is uniform.

In general G_i is a random variable whose site distribution follows from the central limit theorem and is a Gaussian with width unity :

$$\frac{1}{N} \sum_i \delta(G_i - G) = \frac{1}{\sqrt{2\pi}} \exp(-G^2/2) . \tag{92}$$

We have computed the distribution of G_i inside a given cell $\mathcal{C}^{G-A}_{y_0}$. Its average value follows from (83) :

$$G_{y_0} = \frac{1}{| \mathcal{C}^{G-A}_{y_0} |} \sum_{i \in \mathcal{C}^{G-A}_{y_0}} G_i = M \cdot \frac{y_0 - h}{q_0} . \tag{93}$$

A more elaborate computation based on techniques introduced in [17] enables one to prove that the distribution of G_i inside the cell $\mathcal{C}^{G-A}_{y_0}$ is in fact a Gaussian, of average G_{y_0}, and of width

$$\frac{1}{| \mathcal{C}^{G-A}_{y_0} |} \sum_{i \in \mathcal{C}^{G-A}_{y_0}} (G_i - G_{y_0})^2 = 1 - \frac{M^2}{q_0} . \tag{94}$$

For general values of the temperature and the magnetic

field $M^2 \leqslant q_0$, which implies that the G_i fluctuate inside each cell of the grand ancestor. On the other hand, for $h \to \infty$ the fluctuations vanish, and sites having the same value of G_i are in the same cell $\mathcal{C}^{G-A}$.

d) There is no frustration in the interactions between the cells of the grand-ancestor :

$$J_{y_0,y_0'} \overset{h\to 0}{\sim} 2\left[\frac{\partial}{\partial y} m_{q_0}(y)\right]\Big|_{y=0} \frac{y_0}{\sqrt{q_0}} \frac{y_0'}{\sqrt{q_0}} \qquad (95)$$

which is of the Mattis type.

5. Conclusions.

We have shown that the surprisingly high level of organization of the space of equilibrium states of a spin glass, reflected by ultrametricity, can be translated directly into a hierarchical organization among the sites. In fact the sequence of cells imbedded into one another associated with each equilibrium state can be used to define a distance between sites, and for such a distance the real space of sites is ultrametric.

It is very interesting to observe that all the states within a given distance share a common sequence of cells : this allows for a coarsed grained description of a spin glass which can be then refined as much as one wants. Certainly this should have interesting consequences in the use of spin glass-like models to build up associative memories [19]. For instance at the crudest level of description, one is given only the cells (of the « grand ancestor state ») which are common to all the states. These cells are the only ones which exist in a field above T_c. An attractive conjecture would be that there exist some way of crossing the de Almeida-Thouless line such that these cells are left invariant, the system developing new cells inside them as the temperature is lowered.

The Fokker-Planck like differential equation which appears naturally in Parisi's solution receives a purely static interpretation : it describes the sizes of the subcells within each cell at every scale.

The average couplings within the cells are precisely the ones which are required for the existence of generalized TAP equations for the ancestor states which have constant magnetization inside each cell. Furthermore, they possess a hierarchical structure which is reminiscent of ultrametricity : the couplings $J_{y_0y_1\ldots y_l,y_0'y_1'\ldots y_l'}$ between the cells $\mathcal{C}_{y_0y_1\ldots y_l}$ and $\mathcal{C}_{y_0'y_1'\ldots y_l'}$ are such that $\sum_{y_k\ldots y_l'} J_{y_0\ldots y_l,y_0'\ldots y_l'}$ is independent of $y_k, \ldots, y_l, (0 < k \leqslant l)$.

This means that for instance the average coupling between a subcell $\mathcal{C}_{y_0y_1}$ of the cell $\mathcal{C}_{y_0}$, and another cell $\mathcal{C}_{y_0'}$ is independent of y_1, i.e. independent of the subcell ! These are the results we obtain by working out the detailed consequences of Parisi's solution. They open the way to a direct test of the validity of this solution : one must find out whether such a structure can indeed be found in a large random matrix.

Finally we would like to point out that, in the process of concentrating on self averaging and reproducible quantities, one must leave aside the weights P_α of the equilibrium states. This paper is in that sense complementary to reference [6] where this aspect is addressed.

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Appendix.

In this appendix we compute the reproducible correlation functions defined in section 4, from which one can deduce the coupling between the domains.

Let us start with the function $M_{s,t}$. The average :

$$\sum_{\substack{\alpha_1,\ldots,\alpha_s \\ \beta_1,\ldots,\beta_t}} \overline{P_{\alpha_1} \ldots P_{\alpha_s} P_{\beta_1} \ldots P_{\beta_t} M_{s,t}^{\alpha_1\ldots\alpha_s,\beta_1\ldots\beta_t} \prod_{1\leqslant i<j\leqslant s} \delta(q^{\alpha_i\alpha_j} - q_0)} \times$$

$$\times \overline{\prod_{1\leqslant i<j\leqslant t} \delta(q^{\beta_i\beta_j} - q_0) \prod_{1\leqslant i\leqslant s;\, 1\leqslant j\leqslant t} \delta(q^{\alpha_i\beta_j} - q_0)} \qquad (A.1)$$

is expressed in replica space as :

$$\lim_{n\to 0} \frac{1}{n(n-1)\ldots(n-(s+t)+1)} \sum_{a_1\ldots a_s,b_1\ldots b_t}{}' \frac{1}{N} \sum_{i,j} \overline{\langle \sigma_i^{a_1} \ldots \sigma_i^{a_s} J_{ij} \sigma_j^{a_{s+1}} \ldots \sigma_j^{a_{s+t}} \rangle} \qquad (A.2)$$

where the replica indices a_i, b_j are all distinct, with $Q_{a_ia_j} = Q_{b_ib_j} = Q_{a_ib_j} = q_0$. As was explained in [17], the J_{ij} can be integrated by parts which amounts to replacing it by $\frac{\beta}{N} \sum_{c=1}^{n} \sigma_i^c \sigma_j^c$. Finally the reproducible part can be

factored out of (A.2) as in section 2, it gives :

$$M_{s,t} = \beta \sum_{c=1}^{n} \langle\langle \sigma^{a_1} \ldots \sigma^{a_s} \sigma^{t} \rangle\rangle \langle\langle \sigma^{b_1} \ldots \sigma^{b_t} \sigma^{c} \rangle\rangle \tag{A.3}$$

where $a_1 \ldots a_s$, $b_1 \ldots b_t$ are any $s + t$ replica indices all different from each other, such that : $Q_{a_i a_j} = Q_{b_i b_j} = Q_{a_i b_j} = q_0$.

We have explained in section 2 how one can compute such quantities as $\langle\langle \sigma^{a_1} \ldots \sigma^{a_s} \sigma^{c} \rangle\rangle$ using the « tree method » when all the overlaps $Q_{a_i c}$ are fixed. For $Q_{a_1 c} = q_l$, one finds :

$$\langle\langle \sigma^{a_1} \ldots \sigma^{a_s} \sigma^{c} \rangle\rangle = \int dy_0 \, N_{q_0}(y_0) \, m_{q_0}(y_0)^{s-1} \int dy_l \, G(q_0, q_l, y_0, y_l) \, m_{q_l}(y_l)^2 \, . \tag{A.4}$$

The only problem which is left is the one of counting the number of ways in which c can be chosen with given overlaps $Q_{a_i c}$ and $Q_{b_j c}$ with the other fixed indices, which is simple since for instance $Q_{b_j c} = q_0$ if $Q_{a_1 c} = q_l > q_0$. The result is :

$$\frac{1}{\beta} M_{s,t} = s \left\{ \int dy_0 \, N_{q_0}(y_0) \, m_{q_0}(y_0)^{t+1} \right\} \times \left\{ \int dy'_0 \, N_{q_0}(y'_0) \, m_{q_0}(y'_0)^{s-1} \left[1 - \sum_{l=0}^{M} (x_{l+1} - x_l) \, q(0, l, y) \right] \right\} + (s \rightleftarrows t) \tag{A.5}$$

where $(s \rightleftarrows t)$ is the term obtained from the first one by the symmetry operation $\begin{bmatrix} s \to t \\ t \to s \end{bmatrix}$, and

$$q(l', l, y) = \int dy_l \, G(q_{l'}, q_l, y, y_l) \, [m_{q_l}(y_l)]^2 \, . \tag{A.6}$$

This can be simplified by use of the formula :

$$\frac{1}{\beta} \frac{\partial}{\partial y} m_{q_l}(y) = 1 - x_l [m_{q_l}(y)]^2 + \sum_{l'=l}^{K} (x_{l'} - x_{l'+1}) \, q(l, l', y) \tag{A.7}$$

which can be proven by induction starting from the case $y = M$.

We have finally

$$M_{s,t} = \int dy_0 \, dy'_0 \, N_{q_0}(y_0) \, N_{q_0}(y'_0) \left\{ m_{q_0}(y'_0) \frac{\partial}{\partial y_0} + m_{q_0}(y_0) \frac{\partial}{\partial y'_0} \right\} \{ m_{q_0}(y_0)^s \, m_{q_0}(y'_0)^t \} \, . \tag{A.8}$$

The computation of the moments $M^{s,t}_{u,v}$ defined in (84) follows the same lines. One finds

$$M^{s,t}_{u,v} = \beta \sum_{e} \langle\langle \sigma^{a_1} \ldots \sigma^{a_s} \sigma^{c_1} \ldots \sigma^{c_u} \sigma^{e} \rangle\rangle \langle\langle \sigma^{b_1} \ldots \sigma^{b_t} \sigma^{d_1} \ldots \sigma^{d_v} \sigma^{e} \rangle\rangle \tag{A.9}$$

where $a_1 \ldots a_s$, $b_1 \ldots b_t$, c_1, d_1 all have the same mutual overlap q_0, while $c_1, \ldots, c_u$ all have the mutual overlap q_1, and $d_1 \ldots d_v$ all have the mutual overlap q_1. The problem of counting the number of ways in which replica e can be chosen with fixed overlaps from the a, b, c, d is a bit more tedious than in the case of $M_{s,t}$, but not difficult. One finds :

$$\begin{aligned} M^{s,t}_{u,v} = \int dy_0 \, dy'_0 \, N_{q_0}(y_0) \, N_{q_0}(y'_0) \int dy_1 \, dy'_1 \, G(q_0, q_1, y_0, y_1) \, G(q_0, q_1, y'_0, y'_1) \times \\ \times m_{q_0}(y_0)^s \, m_{q_0}(y'_0)^t \, m_{q_1}(y_1)^u \, m_{q_1}(y'_1)^v \times \left[\frac{y_0 - h}{q_0} m_{q_0}(y'_0) + \frac{y'_0 - h}{q_0} m_{q_0}(y_0) \right. \\ + \frac{y_0 - y_1}{q_0 - q_1} (m_{q_1}(y'_1) - m_{q_0}(y'_0)) + \frac{y'_0 - y'_1}{q_0 - q_1} (m_{q_1}(y_1) - m_{q_0}(y_0)) \\ \left. - \beta x_1 (m_{q_1}(y_1) \, m_{q_1}(y'_1) - m_{q_0}(y_0) \, m_{q_0}(y'_0)) \right] . \end{aligned} \tag{A.10}$$

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