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Ageing classification in glassy dynamics

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Abstract. We study the out-of-equilibrium dynamics of several models exhibiting ageing. We attempt to identify various types of ageing systems using a phase space point of view. We introduce a trial classification, based on the overlap between two replicas of a system, which evolve together until a certain waiting time, and are then totally decoupled. In this way we investigate two types of systems, domain growth problems and spin glasses, and we show that they behave differently.

1. Introduction

The dynamics of spin glasses and other disordered systems exhibits a very much studied phenomenon known as 'ageing': the behaviour of the system depends on its history, and experiments show a typical out-of-equilibrium regime on all (accessible) time scales [1]. In the simplest case one quenches the system into its low-temperature phase at time $t = 0$, and the dynamics of the system depends on its age, i.e. the time elapsed since the quench. This type of behaviour can be studied, for example, by looking at the correlation function of some local observable $O(t)$, $C(t, t') = \langle O(t)O(t')\rangle$, or at the response of such an observable to a change in a conjugated external field $h(t')$: $r(t, t') = \langle \partial O(t)/\partial h(t')\rangle$. While in the usual equilibrium behaviour these two-times quantities obey time-translational invariance (TTI) ($C(t, t') = C(t - t')$, $r(t, t') = r(t - t')$) and the fluctuation–dissipation theorem (FDT) relating correlation and response, one frequently observes in off-equilibrium dynamics a dependence on $C(t, t') \simeq t^{-\alpha}C(t'/t)$, which is referred to as ageing behaviour, and a violation of FDT.

This kind of ageing behaviour is not restricted to spin glasses: the persistence of out-of-equilibrium effects even after very long times has been observed in many other systems, either experimental systems [2], or in computer simulations [3]. In some cases ageing could be studied analytically [4–9].

The kind of loose definition of ageing that we have used so far seems to be ubiquitous and to hide a variety of very distinct physical situations. While the mean-field spin glass is known to be a complicated system with a rough free-energy landscape with many metastable states, ageing also occurs in much simpler problems like the random walk [7], the coarsening of domain walls in a ferromagnet quenched below its critical temperature [10], or some problems with purely entropic barriers [8, 9], all problems in which the free-energy landscape seems to be very simple.

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It is interesting to find a way to distinguish between these different types of ageing, and this paper takes some steps towards such a classification. A first classification of ageing has already been proposed in the literature [5]. In the mean-field dynamics of spin glasses, it has been shown that the response function exhibits an anomaly in the low-temperature phase [13, 14]. While it looks mysterious in the framework of equilibrium dynamics, this anomaly is well understood if one studies off-equilibrium dynamics [5]. The anomaly is defined there as

$$\bar{\chi} = \lim_{t\to\infty} \int_0^t \mathrm{d}t'\, r(t,t') - \int_0^\infty \lim_{t_w\to\infty} r(t_w+\tau, t_w)\, \mathrm{d}\tau\ . \tag{1}$$

It measures the difference between the susceptibility of the system at large times and the susceptibility of a hypothetical system which would be at equilibrium. A non-zero anomaly shows the existence of a long-term memory of the system to some perturbations occurring at any time. Systems with such an anomaly certainly exhibit strong ageing effects.

In spite of its nice mathematical structure, the anomaly is, in general, not easy to control and compute (analytically or numerically). In this paper we want to propose another tool for the classification of ageing. We shall use an overlap $Q_{t_w}(t_w+t, t_w+t)$ between two identical copies of the system, which are constrained to evolve from the same initial configuration and with the same thermal noise between the initial quench and a time t_w, and then evolve with different realizations of the thermal noise between t_w and t_w+t. This quantity was introduced in [12], and in a study of the ageing dynamics of the spherical spin glass by Cugliandolo and Dean [15] (slightly different objects involving two copies of the system evolving with the same noise have also been studied before [11]). We argue that the asymptotic value of this overlap in the double limit $\lim_{t_w\to\infty}\lim_{t\to\infty} Q_{t_w}(t_w+t, t_w+t)$ distinguishes between different types of ageing. In a first class of systems the limit is finite (equal to the Edwards–Anderson parameter q_{EA} in the cases we have studied so far). This class, which we call type I, includes the models with coarsening of domain walls: we show it explicitly hereafter in the case of the $O(n)$ model with $n\to\infty$, and within some widely used assumptions for the domain growth in the non-conserved scalar order parameter case. The second class, ageing systems of type II, contains the spin-glass-like problems with complicated free energy landscapes, and we study explicitly the p-spin spherical models or the zero-dimensional version of the manifolds in a random potential. For this class, the limit $\lim_{t_w\to\infty}\lim_{t\to\infty}$ of the overlap is equal to the minimum possible overlap (i.e. zero for the p-spin spherical model (with $p>2$) and q_0 for the zero-dimensional manifolds).

Besides suggesting a first (rough) classification, this overlap function may turn out to give some intuitive ideas about the energy landscape in which the system evolves, and its complexity. For example, if we think of a system falling down a ‘gutter’, it is clear that it will continuously go away from its position at t_w (the correlation function decreases to zero), but two copies separated at t_w will not be able to separate indefinitely, and the overlap will have a limit at long times which can depend on t_w: as t_w grows, the system gets closer and closer to equilibrium. Type I systems seem to have such a behaviour.

In contrast, a rugged landscape with many bifurcations, and many different paths, will allow two copies to really move away from one another, so the overlap will decay to its minimum possible value, for any finite t_w: the distance between the two replicas becomes the largest possible one.

For one-time quantities, like the energy for example, type I and type II systems seem to have a similar kind of freezing. The ageing behaviour (the study of two-time quantities) shows that this freezing is not full. Besides, the study of the overlap between two copies shows that the freezing in type I systems is in some sense more ‘robust’ than in type II.

The classification induced by the asymptotic value of the overlap function might coincide with the one quoted above, using the anomaly of the response. Indeed, all type II systems that we study are known to possess a non-vanishing anomaly. In type I systems, the anomaly has been computed so far only in the $O(n)$ model with $n \to \infty$, where it does vanish (or equivalently for the $p = 2$ spherical model [15]). The existence of a general relation between these two criteria remains to be studied. At an intuitive level it may look plausible: if one thinks of a type I system as evolving in a phase space gutter, it should not have a long term memory of a perturbation. On the other hand, a type II system evolving in a rugged landscape will be continuously bifurcating and a change of direction will be remembered at long times. As we are aware that such intuitive arguments can be very misleading, we just mention them here as a motivation to further studies of the response anomaly in various ageing systems.

The paper is organized as follows. In section 2, we define the dynamics and various quantities we study, and present the general features of equilibrium dynamics. Section 3 is devoted to the study of various problems of domain growth, with analytical and numerical results. Type II systems are studied in section 4, where we analyse, in particular, the behaviour of the zero-dimensional version of the random manifold problem and of Bouchaud's model of phase space traps [16, 17]. The last section contains our conclusions.

2. Definitions, equilibrium dynamics

We consider systems described by a field $\phi(\boldsymbol{x})$ in a d-dimensional space (we shall also consider spin systems, with obvious generalizations of the definitions). Given a Hamiltonian $H[\phi] = \int \mathrm{d}^d x\, \mathcal{H}(\phi(\boldsymbol{x}))$, we assume a Langevin dynamics at temperature T:

$$\frac{\partial \phi(\boldsymbol{x}, t)}{\partial t} = -\frac{\partial H}{\partial \phi(\boldsymbol{x}, t)} + \eta(\boldsymbol{x}, t) \tag{2}$$

where η is a white noise, with $\langle \eta(\boldsymbol{x}, t)\eta(\boldsymbol{x}', t')\rangle = 2T\delta^d(\boldsymbol{x} - \boldsymbol{x}')\delta(t - t')$ ($\langle\,\rangle$ means an average over this thermal noise).

The quantities we are mostly interested in are the following:

- the two-time autocorrelation function $C(t, t')$: this is the mean overlap between the configurations of the field at times t and t',

$$C(t, t') = \frac{1}{V} \int \mathrm{d}^d x\, \langle \phi(\boldsymbol{x}, t)\phi(\boldsymbol{x}, t')\rangle \tag{3}$$

- the response function

$$r(t, t') = \frac{1}{V} \int \mathrm{d}^d x \left\langle \frac{\delta \phi(x, t)}{\delta \eta(x, t')} \right\rangle \tag{4}$$

- the overlap function $Q_{t_w}(t, t')$: the system evolves during a certain time t_w; at t_w a copy is made, and the two systems obtained, labelled by $^{(1)}$ and $^{(2)}$, evolve independently; $Q_{t_w}(t, t')$ is then the overlap between the configuration of one copy at time t and the other at time t':

$$Q_{t_w}(t, t') = \frac{1}{V} \int \mathrm{d}^d x\, \langle \phi^{(1)}(\boldsymbol{x}, t)\phi^{(2)}(\boldsymbol{x}, t')\rangle\,. \tag{5}$$

Of course, for $t \leqslant t_w$ or $t' \leqslant t_w$, $Q_{t_w}(t, t') = C(t, t')$.

Before turning to out-of-equilibrium dynamics, let us first show that the overlap $Q_{t_w}(t, t')$ is simply related to the correlation in the case of equilibrium dynamics.

If a system is evolving among a set of states, according to a master equation, with transition rates obeying detailed balance, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t} p_i(t) = \sum_j T_{ij} p_j(t) \qquad T_{ij} p_j^{\mathrm{eq}} = T_{ji} p_i^{\mathrm{eq}} \tag{6}$$

where $p_i(t)$ is the probability of being in state i at time t, the formal solution is

$$p_i(t) = \sum_j \langle i | \mathrm{e}^{\mathcal{T} t} | j \rangle p_j(0) \tag{7}$$

where $\langle i | \mathrm{e}^{\mathcal{T} t} | j \rangle$ are the matrix elements of the evolution operator $\mathrm{e}^{\mathcal{T} t}$ ($\langle i | \mathcal{T} | j \rangle = T_{ij}$). The detailed balance implies that

$$\langle j | \mathrm{e}^{\mathcal{T} t} | i \rangle p_i^{\mathrm{eq}} = \langle i | \mathrm{e}^{\mathcal{T} t} | j \rangle p_j^{\mathrm{eq}} . \tag{8}$$

If we express this property in terms of the overlap between two replicas evolving in equilibrium dynamics, we obtain (see figure 1):

$$Q_{\mathrm{as}}(s, t) = C_{\mathrm{as}}(s + t) \tag{9}$$

where we have defined $Q_{\mathrm{as}}(s, t) = \lim_{t_w \to \infty} Q_{t_w}(t_w + s, t_w + t)$ and $C_{\mathrm{as}}(t) = \lim_{t_w \to \infty} C(t_w + t, t_w)$.

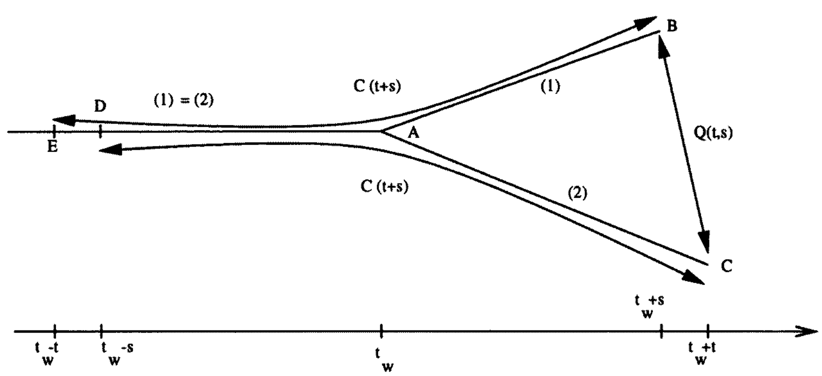

Figure 1. The equilibrium value of the overlap, when $t_w \to \infty$ between the first replica at time $t_w + s$ (point B) and the second at time $t_w + t$ (point C) is the same as the overlap between D (time $t_w - s$, before the separation) and C, or between E (time $t_w - t$) and B.

There exist other interesting large-time limits in the problem which exhibit ageing. In particular, the interesting property of weak-ergodicity breaking [16], defined by $\lim_{t \to \infty} C(t_w + t, t_w) = 0$, expresses the fact that such systems never reach equilibrium. In the following we will therefore be interested in the function

$$S(t_w) \equiv \lim_{t \to \infty} Q_{t_w}(t_w + t, t_w + t) \tag{10}$$

and, in particular, in its large-t_w limit $S_\infty = \lim_{t_w \to \infty} \lim_{t \to \infty} Q_{t_w}(t_w + t, t_w + t)$. We shall show that this limit depends on the type of system one considers and allows for a distinction of various ageing types.

3. Domain-growth processes

A phenomenon which is often considered as a typical example of an out-of-equilibrium dynamical evolution is the phase-ordering kinetics [10, 19]. It is the domain growth process for an infinite system with different low-temperature ordered phases, suddenly quenched from a disordered high-temperature region into an unstable state at low temperature. Here we shall keep to the dynamical evolution of systems with a non-conserved order parameter [10, 19].

The system we study is described by an n-component vector field $\phi(\boldsymbol{x}, t)$ representing the density of magnetization at the point $\boldsymbol{x}$ of a d-dimensional space, as a function of time. The system is prepared at high temperature, where $\langle\phi\rangle = 0$ and then rapidly quenched at $t = 0$ in a low-temperature region, where there is more than one energetically favourable state with $\langle\phi\rangle \neq 0$. This situation is well described by a typical coarse-grained free energy:

$$F = \int \mathrm{d}^d\boldsymbol{x}\left[\tfrac{1}{2}(\nabla\phi)^2 + \boldsymbol{V}(\phi)\right] \tag{11}$$

where the first term represents the energy cost of an interface between two different phases and $V[\phi]$ is a potential with minima at different values of ϕ. The state with $\langle\phi\rangle = 0$ is unstable at low temperature, so the system evolves by forming larger and larger domains of a single phase; at a late stage of growth, the typical pattern of domains is self-similar and the characteristic size of a domain is $L(t)$. This evolution can be studied, for instance, through a Langevin dynamics with thermal noise η:

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \nabla^2\phi - \boldsymbol{V}'(\phi) + \eta\,. \tag{12}$$

Equilibrium is not achieved until $L(t)$ reaches the size of the sample. In an infinite sample one thus observes an ageing behaviour in the correlation function. Roughly speaking the system remembers its age t_w through the value of its typical domain size $L(t)$.

3.1. The $O(n)$ model

Interestingly enough, one of the few exactly solved models of coarsening [10], namely the case of the $O(n)$ model with n large and a constraint $\phi^2 = n$, is also related to a problem which looks like a spin-glass system. Indeed consider the following spin-glass Hamiltonian:

$$H = -\sum_{ij} J_{ij} s_i s_j \tag{13}$$

where s_i are real spins with a spherical constraint $\sum_i s_i^2 = n$, and J_{ij} are random couplings. This model is usually called the $(p = 2)$ spherical spin glass [20]. Its Langevin dynamics,

$$\frac{\mathrm{d}s_i}{\mathrm{d}t} = \sum_j J_{ij} s_j(t) - z(t)s_i(t) + \eta_i(t) \tag{14}$$

where $z(t)$ is a Lagrange parameter enforcing the spherical constraint, can also be written in the basis where the J_{ij} matrix is diagonal:

$$\frac{\mathrm{d}s_\lambda}{\mathrm{d}t} = (\lambda - z(t))s_\lambda(t) + \eta_\lambda(t)\,. \tag{15}$$

Then the dynamical equation reduces to the one of the $O(n)$ model in Fourier space with $\lambda = -k^2$. The only important piece of information on the J matrix is the behaviour of its spectrum near its largest eigenvalue λ^*. The case of a square root singularity, such as for instance the Wigner law, is equivalent to a $d = 3$ coarsening problem. Clearly the spherical spin glass does not really have a spin-glass-like behaviour (this has been known for a long time in statics [20]); however, it exhibits an ageing dynamics, which is related to the growth of the correlation length in the $O(n)$ model. Recently, Cugliandolo and Dean [15] have performed a detailed computation of the dynamics of this problem, in which they computed all the relevant quantities of interest for our discussion. They found ageing in the correlation function, but the anomaly of the response as defined in (1) vanishes, which shows that the memory effects are weak. besides, the overlap limit $S(t_w) = \lim_{t\to\infty} Q(t_w+t, t_w+t)$ is a continuous function of t_w, taking values between $q_{\rm EA}^2$ and $q_{\rm EA}$, with $S_\infty = q_{\rm EA}$, and, of course, $\lim_{t_w\to\infty} Q(t_w+t, t_w+t') = C_{\rm as}(t+t')$.

Physically, the system evolves in time through two processes: a thermal noise which affects evenly all the components s_λ, superimposed onto a deterministic evolution which amounts to reinforcing the eigenmodes closer to the λ^*, and this deterministic part is dominant over the thermal noise.

The fact that this system has the property $S_\infty = q_{\rm EA}$ together with the weak ergodicity breaking property $\lim_{t_w\to\infty}\lim_{t\to\infty} C(t_w+t, t_w) = 0$ suggests the existence of a kind of gutter in phase space: two replicas, even decoupled, remember forever that they are evolving in the same canal.

3.2. Scalar order parameter: analytic study

We now turn to the domain growth problem in the case of a scalar order parameter. This problem cannot be solved exactly but we shall use a well known approximation [18], recently developed by Bray and Humayun [21]. We refer the reader to Bray's review [10] for a detailed presentation of the method. The idea is to take advantage of the universality of domain growth in the scaling regime: after an initial regime of fast growth, the order parameter saturates at the equilibrium value inside a domain and the only way for the system to further decrease the free energy is the reduction of the surface of walls between different domains. Therefore, the dynamical properties of the system at a late stage of growth are given by the motions of the walls and, in particular, by their curvature; the particular shape of the potential $V[\phi]$, provided it has well separated minima, is not crucial. If the growth is influenced by an external field, the difference between the minima introduced by the field will be the relevant variable. The universality gives the freedom to choose an appropriate form for the potential in the free energy, and also a special form for the thermal noise, which makes the analysis more tractable. Specifically, the Langevin equation is replaced by

$$\frac{\partial\phi(\boldsymbol{x},t)}{\partial t} = \nabla^2\phi - V_0'[\phi] + \eta(\boldsymbol{x},t)V_1'[\phi] \tag{16}$$

where $\eta(\boldsymbol{x},t)$ is the Gaussian white noise with zero mean and correlator:

$$\langle\eta(\boldsymbol{x},t)\eta(\boldsymbol{x}',t')\rangle = 2T\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t')\,. \tag{17}$$

The field $\phi(\boldsymbol{x},t)$ is parametrized by an auxiliary field $m(\boldsymbol{x},t)$, through

$$\phi[m] = \phi_0\left(\frac{2}{\pi}\right)^{1/2}\int_0^m \mathrm{d}x\,\exp-(x^2/2) = \phi_0\,\mathrm{erf}[m/\sqrt{2}]\,. \tag{18}$$

With the following choice of the two potentials:

$$V_0[\phi] = \frac{\phi_0^2}{\pi}\exp\left(-2\left[\mathrm{erf}^{-1}\left[\frac{\phi}{\phi_0}\right]\right]^2\right) \qquad V_1[\phi] = \frac{\phi_0^2}{\sqrt{\pi}}\exp\left(\sqrt{2}\,\mathrm{erf}^{-1}\left[\frac{\phi}{\phi_0}\right]\right) \tag{19}$$

the field m satisfies a very simple equation:

$$\frac{\partial m}{\partial t} = \nabla^2 m + (1 - (\nabla m)^2)m + \eta\,. \tag{20}$$

With the wall profile function (18), the field $m(\boldsymbol{x}, t)$ measures the distance of the point $\boldsymbol{x}$ from the interface: at infinite distance from the wall, the field ϕ saturates to its equilibrium value. Moreover, the potential V_0 has the required two-wells shape at the two equilibrium values. The choice of the potential V_1 does not alter this shape and, as can be seen from (16) and from the fact that $V_1'[\phi] = \phi'$, it corresponds to a thermal noise acting only on the interface. This is an approximation which is not able, for instance, to reproduce the process of nucleation of a bubble. In other words, the value of q_{EA} in this case remains fixed at the $T = 0$ value, $q_{\mathrm{EA}} = \phi_0^2$. However, we expect that this approximation will not affect our main conclusions concerning the various large-time limits of the overlap.

The physical situation of a rapid quench will be represented by taking the boundary condition for $m(\boldsymbol{x}, t)$ to be Gaussian with zero mean and correlator:

$$\langle m(\boldsymbol{x}, 0) m(\boldsymbol{x}', 0)\rangle = \delta(\boldsymbol{x} - \boldsymbol{x}')\,. \tag{21}$$

Equation (20) can be solved by neglecting the non-linear term or, more correctly, by taking into account its mean value. Let us neglect it in a first approach. equation (20) can then be solved, giving

$$m(\boldsymbol{x}, t) = \int_{|\boldsymbol{k}|<\mathrm{e}^{-1}} \frac{\mathrm{d}^d\boldsymbol{k}}{2\pi^d} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{x}} \left[\mathrm{e}^{(1-k^2)t} m(\boldsymbol{k}, 0) + \int_0^t \mathrm{d}t'\, \mathrm{e}^{(1-k^2)(t-t')} \eta(\boldsymbol{k}, t')\right] \tag{22}$$

where e is a cut-off given by the width of the interface. The linearity of the equation and the independence of the boundary condition and the noise preserve the Gaussian character of the probability distribution for the field m. Mean values of functions of the field ϕ can be computed in terms of the evolution of the first and second moments for the Gaussian distribution of m. To compute the correlation function $C(\tau + t_w, t_w)$ we introduce two fields $m_1 = m(\boldsymbol{x}, \tau + t_w)$ and $m_2 = m(\boldsymbol{x}, t_w)$. When computing the overlap $Q(\tau + t_w, \tau + t_w)$ the fields m_1 and m_2 denote, respectively, $m^{(1)}(\boldsymbol{x}, \tau + t_w)$ and $m^{(2)}(\boldsymbol{x}, \tau + t_w)$. In both cases the joint distribution of m_1 and m_2 is a Gaussian $P(x_1, x_2)$, which we parametrize as

$$P(x_1, x_2) = \frac{\gamma}{2\pi\sqrt{\sigma_1\sigma_2}} \exp\left[-\frac{\gamma^2}{2}\left(\frac{x_1^2}{\sigma_1} + \frac{x_2^2}{\sigma_2} - \frac{2 f x_1 x_2}{\sqrt{\sigma_1\sigma_2}}\right)\right] \tag{23}$$

with $\sigma_i = \langle m_i^2\rangle$, $c_{12} = \langle m_1 m_2\rangle$ and $f = c_{12}/\sqrt{\sigma_1\sigma_2}$, $\gamma = 1/\sqrt{1 - f^2}$. In terms of this distribution the correlation (or overlap) is given by

$$\langle \phi[m_1]\phi[m_2]\rangle = \int_{-\infty}^{+\infty} \mathrm{d}x_1\, \mathrm{d}x_2\, \phi(x_1)\phi(x_2) P(x_1, x_2) = \phi_0^2 \frac{2}{\pi} \arcsin f \tag{24}$$

where the function (18) has been replaced by $\phi[m] = \phi_0^2\,\mathrm{sign}[m]$, a good approximation in the large-time regime. The calculation therefore reduces to the computation of the parameter f in the covariance matrix of the probability distribution $P(x_1, x_2)$, which is easily obtained from (22). Defining

$$\mathcal{F}(a, b) = \int_0^a \mathrm{d}\sigma\, \mathrm{e}^{-\sigma} \left(1 - \frac{\sigma}{b}\right)^{-d/2} \left(\mathrm{erf}\left(\frac{\sqrt{b - \sigma}}{e}\right)\right)^d \tag{25}$$

we obtain, for $t_w \gg 1$:

$$\begin{aligned} C(\tau + t_w, t_w) = \phi_0^2 \frac{2}{\pi} \arcsin \Bigg\{ &\left(\frac{4(\tau + t_w)t_w}{(\tau + 2t_w)^2}\right)^{d/4} \\ &\times \frac{1 + T\mathcal{F}(2t_w, \tau + 2t_w)}{[1 + T\mathcal{F}(2(\tau + t_w), 2(\tau + t_w))]^{1/2} [1 + T\mathcal{F}(2t_w, 2t_w)]^{1/2}} \Bigg\} \end{aligned} \tag{26}$$

and

$$Q(\tau+t_w,\tau+t_w)=\phi_0^2\frac{2}{\pi}\arcsin\left\{\frac{1+T\mathcal{F}(2t_w,2(\tau+t_w))}{1+T\mathcal{F}(2(\tau+t_w),2(\tau+t_w))}\right\}. \tag{27}$$

The asymptotic behaviours of Q and C are very similar to those studied above in the domain growth of an $n\to\infty$ component order parameter (or $p=2$ spherical model): the asymptotic relation between Q and C at $t_w\to\infty$ with τ finite is satisfied and, for fixed t_w and τ large, C has the limiting behaviour

$$C(\tau+t_w,t_w)\sim\left(\frac{t_w}{\tau}\right)^{d/4} \tag{28}$$

while Q does not go to zero and the limiting value $S(t_w)$ is a continuous function of t_w, approaching the equilibrium value q_{EA} as t_w grows: $S_\infty=q_{\mathrm{EA}}$. So within this approximation this coarsening problem falls into the type I classification.

Note that when one includes the effect of the gradient squared term in (22), treated as an average term (as in [10]), the result is similar except for a change in the numerical value of the function $\mathcal{F}$ in (25), where a term $c\sigma^{-2}$ is present, instead of $\exp(-\sigma)$.

In order to check this approximate analytic treatment, we have performed numerical simulations of domain growth in two dimensions, for a scalar field evolving with a Langevin equation, and also for Ising spins on a regular two-dimensional lattice, with Glauber dynamics [27].

3.3. Scalar order parameter: numerical studies

We have simulated the evolution of a scalar field ϕ on a two-dimensional square lattice, according to the Langevin equation (12), with a quartic V_0 and a bold discretization scheme:

$$\begin{aligned}\phi(i,j,t+1)=\phi(i,j,t)+(&\phi(i+1,j,t)+\phi(i-1,j,t)+\phi(i,j+1,t)+\phi(i,j-1,t)\\&-4\phi(i,j,t)+\phi(i,j,t)-\phi(i,j,t)^3)h+\eta(i,j,t)\end{aligned} \tag{29}$$

where η is a Gaussian noise with zero mean and variance $2Th$, h being the time step used. We proceed by parallel updating of the field, and vary the time step h. At $t=0$, $\phi(i,j)$ are taken as independent random variables uniformly distributed between -1 and 1. We let the system evolve during t_w according to (29), make a copy of it, and let the two copies evolve independently, i.e. with independent thermal noises. We record the correlation of each of the copies with the system at time t_w and the overlap between the replicas.

We present simulations at fixed temperature: we record the overlap and the correlation function for different values of t_w, The linear size of the system was of 200 sites, and one run was made with a 400×400 lattice. Each simulation was made with three different values of h ($h=0.02$, 0.04 and 0.08), to check that the results did not depend on the time step used. We also checked that the t/t_w scaling is well obeyed for the correlation function for large enough t_w; for the overlap, no such scaling is found. We plot the overlap at time t_w+t versus the correlation between times t_w and t_w+t.

The Q versus C curves show quite clearly that the overlap, after a transient regime where it decays faster than the correlation, has a finite limit as C goes to zero. This limit grows with β and with t_w.

This result agrees with the previous analytic study, as far as the asymptotic behaviour of the overlap and correlation are concerned. We have also performed simulations of a two-dimensional Ising spin system (with nearest-neighbour ferromagnetic interactions), with Metropolis dynamics with random updating†: at each sweep through the lattice, spins are updated in random order, but this order is the same for both replicas. The results (see figure 3) agree with those obtained by Langevin dynamics.

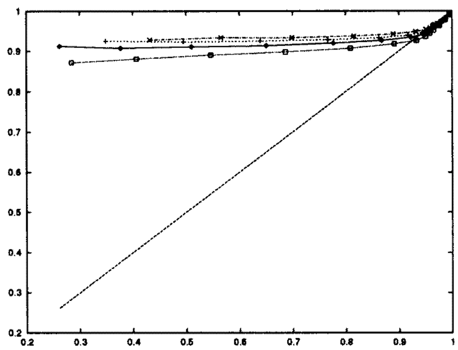

Figure 2. Overlap $Q(t_w + t, t_w + t)$ versus correlation $C(t_w + t, t_w)$ for the scalar field in two dimensions, for $\beta = 6$ and different waiting times (from bottom to top, 10, 20, 50 and 180 MC steps).

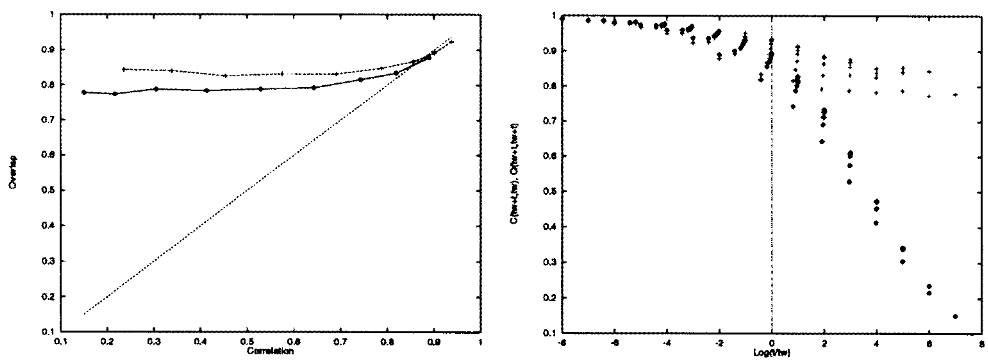

Figure 3. Left: overlap $Q(t_w + t, t_w + t)$ versus correlation $C(t_w + t, t_w)$ for the Ising model in two dimensions, for $\beta = 2$, $t_w = 2^3$ (bottom) and $t_w = 2^4$ (top); Right: correlation and overlap for the Ising model in two dimensions, for different values of t_w and t: $t_w = 2^3, 2^4, \ldots, 2^9$, and $t = 2, \ldots, 2^{10}$.

Note that we also made computer simulations for a model introduced in [25], consisting of an Ising ferromagnet on a cubic lattice, with weak next-nearest-neighbour antiferromagnetic couplings; in this model, the growth is slowed from a power law to a logarithmic behaviour; nevertheless, we find for the correlation and overlap functions a similar behaviour as for the simple ferromagnet.

3.4. The XY-model in one dimension

A simple and soluble model where domain growth can be studied without approximations on the potential is the XY model in $d = 1$ [26]. Namely, the system has no phase transition for $T > 0$, but at very low temperature the correlation length L_{eq} is very large. Then at time scales where the size of the domains is small compared to the correlation length, the system presents the typical non-equilibrium features of a multiple phase system.

† Notice that the choice of dynamics is important: as soon as the chosen algorithm is not deterministic at zero temperature (as is the case, for example, if we take Glauber dynamics), the overlap will decrease to zero even at $T = 0$.

In this simple model, the order parameter is a two-dimensional vector field $\phi(x,t)$ of fixed length $\phi^2 = 1$ and the coarse-grained free energy is

$$F = \int \mathrm{d}x \, \frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 . \tag{30}$$

Using the non-linear mapping $\phi(x,t) = (\cos\theta(x,t), \sin\theta(x,t))$, the Langevin evolution equation for the field $\theta(x,t)$ can be written easily and solved explicitly without approximation. The physical situation of a rapid quench from a disordered phase to a very low temperature can be included in the formalism by taking the boundary condition $\theta(x,0)$ to be Gaussian with zero mean and correlated at distance ξ. As for the scalar order parameter model, the linearity of the Langevin equation preserves the Gaussian character of the probability distribution for the field θ and the problem can be solved by computing the time evolution of the moments. Let us now consider a quench to a very low temperature, i.e. a situation where the equilibrium correlation length L_{eq} is very large, and the time needed for the domains to reach this size, t_{eq}, is also very large: in fact, $L_{\mathrm{eq}} \simeq T^{-1/2}$, and $t_{\mathrm{eq}} \simeq T^{-2}$. In a time regime where $\tau + t_w$, $t_w \gg 1$ but L_{eq} is very large compared to the size of the domains we have a very simple expression for the correlation function $C(\tau + t_w, t_w)$ and for the overlap $Q(\tau + t_w, \tau + t_w)$ of two replicas separated at time t_w:

$$\begin{aligned} C(\tau + t_w, t_w) = \exp - \frac{1}{\sqrt{\pi}\xi} \bigg[& 2(\tau + 2t_w)^{1/2} - [2(\tau + t_w)]^{1/2} - (2t_w)^{1/2} \\ & - \frac{T\xi}{2} \{ 2[(\tau + 2t_w)^{1/2} - (\tau)^{1/2}] - (2(\tau + t_w))^{1/2} - (2t_w)^{1/2} \} \bigg] \end{aligned} \tag{31}$$

and

$$Q(\tau + t_w, \tau + t_w) = \exp - \frac{1}{\sqrt{\pi}} T (2\tau)^{1/2} . \tag{32}$$

Since the size of the domain evolves as $L(t) \simeq t^{1/4}$, it is clear that there exists at very low temperatures a regime with $1 \ll t_w \ll \tau \ll t_{\mathrm{eq}}$, where the correlation has already decayed to zero while the overlap still has a finite value. Indeed, C decays to zero with a term in the exponential that does not depend on the temperature, but the argument of the exponential for $Q(\tau + t_w, \tau + t_w)$ is $L(\tau)^2/L_{\mathrm{eq}}^2$.

3.5. Conclusion

The previous study shows that the domain growth processes considered here are essentially deterministic in nature, and that their phase space is very simple: we indeed exhibit a time regime $1 \ll t_w \ll t \ll t_{\mathrm{eq}}$ (t_{eq} being the equilibration time, which is infinite in the true ageing problems, but remains finite in the one-dimensional XY model) where the system at time $t_w + t$ has already drifted away from its position in phase space at t_w ($C(t_w + t, t_w)$ is very small), while two copies separated at t_w are still evolving together ($Q(t_w + t, t_w + t)$ is finite). These type I systems are characterized by the existence of a finite limit $S_\infty = \lim_{t_w \to \infty} \lim_{t \to \infty} Q(t_w + t, t_w + t)$ (with $S(t_w) = \lim_{t \to \infty} Q(t_w + t, t_w + t)$ growing continuously towards q_{EA} as t_w grows). The system can therefore be thought of as moving along a gutter in phase space. It is reasonable to expect that, in these systems, the influence of the thermal noise will be limited in time, and there will be no anomaly in the response function (this has been shown so far only for the $O(n)$ with $n \to \infty$ model [15]).

4. Type II models

4.1. A particle in a random potential

We now turn to the study of a well known disordered mean-field model, where we expect a different kind of behaviour for the overlap: the toy model described by the Hamiltonian [22, 5]:

$$H = \tfrac{1}{2}\mu \sum_{\alpha} \phi_{\alpha}^2 + V(\phi_1, \ldots, \phi_N) \tag{33}$$

where V is a Gaussian random potential with correlations:

$$\overline{V(\phi)V(\phi')} = -Nf\left(\frac{(\phi-\phi')^2}{N}\right) \tag{34}$$

with

$$f(b) = \frac{(\theta+b)^{1-\gamma}}{2(1-\gamma)}\,. \tag{35}$$

This model describes a particle in a random potential, in N dimensions, but it can also be interpreted as a spin-glass model: the ϕ_α are then soft spins, in a quadratic well $\frac{1}{2}\mu\sum_\alpha \phi_\alpha^2$, and they interact via V; the statics has a low-temperature spin-glass phase, with continuous replica-symmetry breaking for $\gamma < 1$ (long-range correlations of the disorder) or one-step replica-symmetry breaking for $\gamma > 1$ (short-range correlations). Slightly different forms for f also allow us to deal with the dynamical equations of the spherical p-spin [29, 4, 32] model. This system is described by the Hamiltonian [33, 4]

$$\sum_{i_1<\cdots<i_p}^{N} J_{i_1\ldots i_p} s_{i_1}\ldots s_{i_p} \tag{36}$$

with the constraint $\sum_{i=1}^N s_i^2 = 1$, and Gaussian distributed random p-spin interactions. It can be described by a toy model, with

$$f(b) = -\frac{1}{2}\left(1-\frac{b}{2}\right)^p \tag{37}$$

and a small modification of the dynamical equations (A1) and (A2), which amounts to implementing the spherical constraint by a time-dependent Lagrange multiplier $\mu(t)$.

To compute the overlap function, we introduce two replicas $\phi^{(1)}$ and $\phi^{(2)}$, with a Langevin dynamics:

$$\frac{\partial\phi_\alpha^{(i)}(t)}{\partial t} = -\frac{\partial H}{\partial\phi_\alpha^{(i)}(t)} + \eta_\alpha^{(i)}(t) \tag{38}$$

where $\eta^{(1)}$ and $\eta^{(2)}$ are two white noises with $\langle\eta_\alpha^{(i)}(t)\eta_{\alpha'}^{(i)}(t')\rangle = 2T\delta_{\alpha\alpha'}\delta(t-t')$ and $\eta_\alpha^{(1)}(t) = \eta_\alpha^{(2)}(t)$ if $t \leqslant t_w$. For $t > t_w$, $\eta^{(1)}$ and $\eta^{(2)}$ are uncorrelated.

Using standard field-theoretic techniques [13, 30], it is now possible to derive the evolution equations for the correlation and response functions of each replica, $C^{(1)}(t,t') = C^{(2)}(t,t') = C(t,t')$ and $r^{(1)}(t,t') = r^{(2)}(t,t') = r(t,t')$, and for the overlap $Q_{t_w}(t,t')$, in the large-N limit. These quantities are defined by (3)–(5) and the corresponding equations are written in appendix A.

The large-time limiting values of the correlation define $\tilde{q}$, q_0 and q_1 (see [32]):

$$\begin{aligned} &\lim_{t\to\infty} C(t,t) = \tilde{q} \\ &\lim_{t\to\infty} C(t,t') = q_0 \\ &\lim_{\tau\to\infty}\lim_{t\to\infty} C(t+\tau,t) = q_1\,. \end{aligned} \tag{39}$$

We will now study the behaviour of the overlap function in different time regimes.

We first study the regime of asymptotic dynamics which corresponds to taking the limit $t, t' \to \infty$, with $\tau = t - t'$ finite. We thus obtain the functions

$$r_{\mathrm{as}}(\tau) = \lim_{t'\to\infty} r(t'+\tau, t') \qquad C_{\mathrm{as}}(\tau) = \lim_{t'\to\infty} C(t'+\tau, t') \tag{40}$$

$$Q_{\mathrm{as}}(\tau,\tau') = \lim_{t_w\to\infty} Q_{t_w}(t_w+\tau, t_w+\tau')\,. \tag{41}$$

In this regime, time-translational invariance and the fluctuation dissipation theorem (FDT) are obeyed: $\mathrm{Tr}_{\mathrm{as}}(\tau) = -\frac{\partial}{\partial\tau}C_{\mathrm{as}}(\tau)$. It is well known that this asymptotic regime is identical to equilibrium dynamics for systems with long-range correlations of the disorder [30, 5, 6], but it is different for short-range correlations [29, 30, 4]. In both cases we have found, as expected from the general discussion of section 2, that $Q_{\mathrm{as}}(\tau,\tau') = C_{\mathrm{as}}(\tau+\tau')$. In particular, $\lim_{\tau\to\infty} Q_{\mathrm{as}}(\tau,\tau') = \lim_{\tau'\to\infty} Q_{\mathrm{as}}(\tau,\tau') = q_1$.

Let us now consider the ageing regime. This regime corresponds to having time differences, like $t - t_w$, diverge when $t_w \to \infty$. Here we shall consider the overlap function $Q_{t_w}(t,t')$ in the 'double ageing' regime where $t' - t_w$ also diverges.

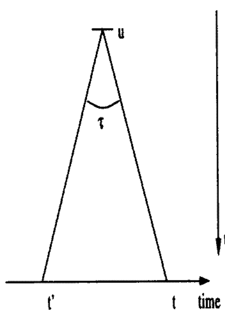

Figure 4. $(t,t') \in D_u$, with $h_u(t')/h_u(t) = \mathrm{e}^{-\tau}$.

There is no full solution of the ageing regime in spin-glass systems. What has been proposed so far, in all cases, is an ansatz about the behaviour of the correlation or response. The first such proposal, by Cugliandolo and Kurchan [4], concerns the case of the p-spin model. They showed that the dynamical equations can be solved in the long-time regime (where one can neglect the time derivatives in (A1)) by the ansatz: $C(t,t') = \mathcal{C}(t'/t)$, $r(t,t') = (x/T)\mathcal{C}'(t'/t)$, or actually by any solution obtained from this through a reparametrization of time $t \to h(t)$, with h an arbitrary increasing function. This solution was subsequently extended to more complicated problems in which the static solution involves a full RSB, like the toy model with long-range correlations of the noise [5] and the SK model [6]. The case of the toy model with short-range correlations of the noise has also been studied recently [32]. The formalisms developed in [5] (non-overlapping time domains) and in [6] (triangular relations) represent the same ansatz but look rather different. Here we shall present the ansatz using mainly the former approach, together with the necessary ingredients for understanding the correspondence between the two formalisms.

Considering first two time quantities like the correlation or response, the ageing regime corresponds to sending t and t' both to infinity, the difference $t - t'$ being itself divergent in this limit. The dynamical equations can be solved in this limit (up to a time reparametrization), neglecting the time derivatives. We consider non-overlapping domains in the (t, t') plane: two times t and t', with $t' < t$, belong to the same domain D_u if we take the limits $t \to \infty$, $t' \to \infty$, with the ratio $h_u(t')/h_u(t)$ finite and fixed to $e^{-\tau}$ $(0 < \tau < \infty)$. The h_u are a family of increasing functions indexed by a parameter u, such that, if $w < u < v$ and the times t, t' belong to D_u, then $h_v(t')/h_v(t) = 0$ and $h_w(t')/h_w(t) = 1$ (a possible choice is $h_u(t) = \exp(t^u)$, in which case D_u is such that $t' = t - (t^{1-u}/u)\tau$). The domain $u = 1$, with $h_1(t) = \exp(t)$, corresponds to the asymptotic regime where FDT and TTI hold. We find it convenient to express the fact that $(t, t') \in D_u$ by the following diagram: it is then easy to show that, if we consider three times $t' < s < t$, with $(s, t) \in D_u$ and $(t', s) \in D_v$, then (t', t) belong to $D_{min(u,v)}$ (see figure 5) which is an ultrametric inequality. If, for example, $v > u$, we have indeed $h_u(t')/h_u(s) = 1$, so $h_u(t')/h_u(t) = h_u(s)/h_u(t)$.

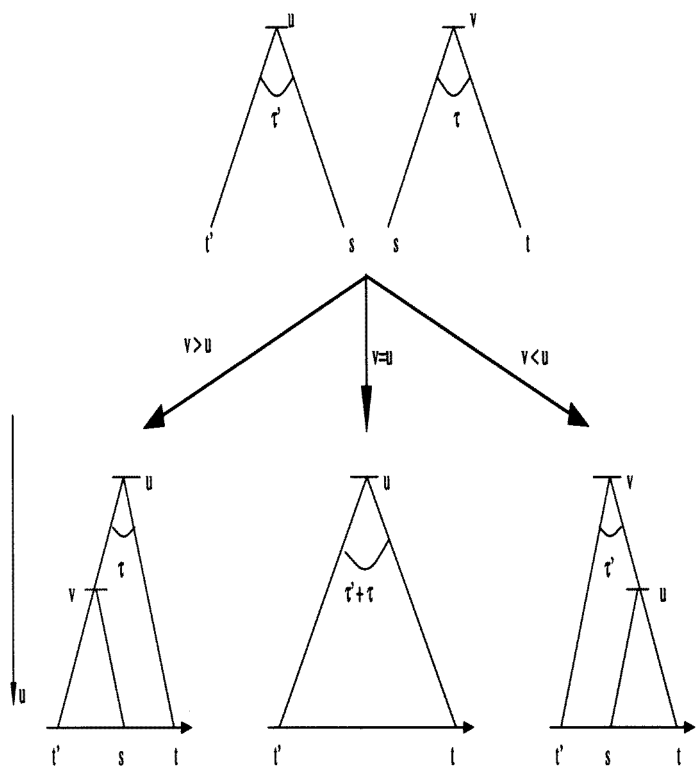

Figure 5. Ultrametric organizations of times.

In each domain D_u, we assume the correlation and response to behave as

$$C(t, t') = C_u(\tau) \qquad r(t, t') = \frac{\mathrm{d}\ln(h_u(t'))}{\mathrm{d}t'} r_u(\tau) \tag{42}$$

with a continuity condition: if D_u and D_v are neighbouring domains, with $u < v$, then $C_u(0) = C_v(\infty)$. Then it is possible to rewrite the equations (A1) (see appendix B for details), and to show [31] that they possess solutions obeying a generalized form of the FDT relation called 'quasi-FDT':

$$x_u \frac{dC_u}{d\tau} = -\mathrm{Tr}_u(\tau)\,. \tag{43}$$

If we now consider three times, $t' < s < t$ (see figure 5) the ansatz implies a simple relation between correlations at times t, t', t, s and t', s. When (t', s) and (s, t) belong to two different domains we have

$$C(t, t') = \min(C(t, s), C(s, t')) \tag{44}$$

and if they are in the same domain D_u, with

$$\frac{h_u(t')}{h_u(s)} = e^{-\tau'} \qquad \frac{h_u(s)}{h_u(t)} = e^{-\tau} \qquad \text{then} \quad \frac{h_u(t')}{h_u(t)} = e^{-\tau'-\tau} \tag{45}$$

so that

$$C(t, t') = C_u(\tau' + \tau) = j_u^{-1}(j_u(C(t, s))\,j_u(C(s, t'))) \tag{46}$$

where $C(t, s) = C_u(\tau)$, $C(s, t') = C_u(\tau')$ and $j_u(z) = \exp(C_u^{-1}(z))$. Equations (44) and (46) form the basis of the formalism of triangular relations introduced in [6], and applied to the toy model in [32]. In appendix C we provide the solution for the overlap function using this formalism.

Since the overlap function $Q_{t_w}(t, t')$ involves three times, we are now looking for a function depending on the domains D_u and $D_{u'}$, $Q_{u,u'}(\tau, \tau')$, where $(t_w, t) \in D_u$ and $(t_w, t') \in D_{u'}$ $(h_u(t_w)/h_u(t) = e^{-\tau}$, $h_{u'}(t_w)/h_{u'}(t') = e^{-\tau'})$.

In appendix B we rewrite the equations (A2) in this frame, and show that they are solved by the following ansatz:

if $u \neq u'$ (for example $u < u'$): $Q_{t_w}(t, t') = C_u(\tau) = \min(C(t, t_w), C(t', t_w))$ (47)

if $u = u'$: $Q_{t_w}(t, t') = C_u(\tau' + \tau) = j_u^{-1}(j_u(C(t, t_w))\,j_u(C(t', t_w)))\,.$ (48)

This ansatz can be easily understood in terms of the previously introduced diagrams: at t_w two 'time-sheets' separate (see figure 6) and two ultrametric systems appear, one for each replica. We stress that this solution exists independently of the actual choice of the disorder correlation, and therefore it is independent of the precise solution of the ageing dynamics: whatever the number of non-overlapping domains appearing in this solution, whatever the actual solutions $C_u(\tau)$, there exists a solution for the overlap function in the ageing regime which is related to the correlation by (48).

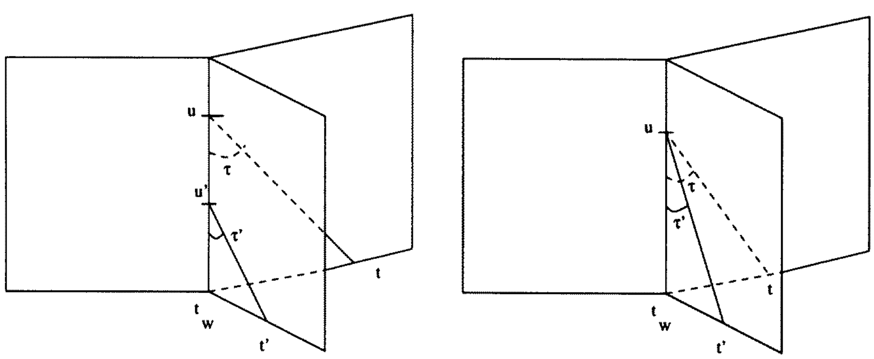

Figure 6. Two-sheets ultrametric structure for times larger than t_w.

Depending on the model, the variable u can be *a priori* continuous, or discrete. It was shown in [5], using the results from [31], that for the long-range model u becomes a continuous variable. In contrast, for short-range models which exhibit statically a one-step replica-symmetry breaking, it has been shown [6, 32] that the ageing dynamics is solved by using a single time domain D_{u^*} (beside the FDT domain D_1).

To summarize, we have shown that, in the ageing regime:

- for the long-range model:

$$Q_{t_w}(t,t') = \min(C(t,t_w), C(t',t_w)) \,. \tag{49}$$

  In particular, it is then clear that the long-time limit of $Q_{t_w}(t,t')$ is q_0;
- for the short-range model, there exists a function j such that

$$Q_{t_w}(t,t') = j^{-1}(j(C(t,t_w))j(C(t',t_w))) \,. \tag{50}$$

  Since $j(q_0) = 0$, and since $\lim_{t\to\infty} C(t,t_w) = \lim_{t'\to\infty} C(t',t_w) = q_0$, we also have $\lim_{t\to\infty} Q_{t_w}(t,t') = \lim_{t'\to\infty} Q_{t_w}(t,t') = q_0$.

In both cases, $\lim_{t_w\to\infty}\lim_{t\to\infty} Q_{t_w}(t,t)$ is q_0, different from $\lim_{t\to\infty}\lim_{t_w\to\infty} Q_{t_w}(t,t)$. Besides, no finite waiting time is sufficient to give a higher limit than q_0 for $\lim_{t\to\infty} Q_{t_w}(t,t)$: an increase in the waiting time only slows down the dynamics, but has no effect on the limiting values. No continuous approach to equilibrium can thus be seen in this way.

Note that since the equations (A1), (A2) are causal, a numerical integration is available, as in [5]. Nevertheless, the integro-differential character of these equations makes it difficult to reach very long times (a huge amount of computer memory is needed). Therefore, the numerical integrations we were able to realize, although fully compatible with the previous study, were not conclusive enough to confirm it.

For the case of the p-spin spherical spin-glass model, an analytic solution of the equations is also available: they are solved by the ansatz corresponding to the short-range model, equation (50). Indeed, we propose for the ageing regime the ansatz $Q_{t_w}(t,t') = Q(t_w/t, t_w/t')$, and we find that the equation giving $Q_{t_w}(t,t')$ can be rewritten so that the three times t_w, t and t' appear only through the ratios t_w/t and t_w/t', and that $Q_{t_w}(t,t') = \frac{1}{q}C(t,t_w)C(t',t_w)$ is the solution of this equation.

We have thus shown that the overlap in the p-spin ($p \geqslant 3$) spherical model exhibits ageing in a similar fashion as the correlation function, and decays to zero for any finite t_w ($q_0 = 0$ for this model). This behaviour is thus very different from the $p = 2$ or the domain-growth case.

4.2. Ageing in traps

A trap model was introduced in [16] and developed in [17] to reproduce off-equilibrium dynamics in glassy systems, and ageing. The model consists of N traps with exponentially distributed energy barriers. This distribution leads to trapping time with infinite mean, and thus to ageing.

In the simplest version, [16] the basic object is $\Pi_1(t,t_w)$, probability that the system has not jumped out of his trap between t_w and $t_w + t$. The overlap between two different states is zero, and the self-overlap is q_{EA}. The correlation function is then $q_{\mathrm{EA}}\Pi_1(t,t_w)$.

We now deal with two systems after t_w: we introduce $\Pi_1^{(2)}(t,t',t_w)$, probability that the first replica has not jumped between t_w and $t_w + t$, and that the second one has not jumped between t_w and $t_w + t'$. The overlap $Q_{t_w}(t_w + t, t_w + t')$ is then simply $q_{\mathrm{EA}}\Pi_1^{(2)}(t,t',t_w)$.

If the system is in trap β (of lifetime τ_β) at time t_w, this probability is $\mathrm{e}^{-t/\tau_\beta}\mathrm{e}^{-t'/\tau_\beta}$, and thus we get $\Pi_1^{(2)}(t,t',t_w) = \Pi_1(t+t',t_w)$. The overlap between the replicas is therefore

$$Q_{t_w}(t_w+t, t_w+t') = C(t_w+t+t', t_w)\,. \tag{51}$$

If we introduce a multilayer tree, the only difference is that we now have a set of $\Pi_j(t,t_w)$ ($j = 1,\ldots,M$), probability that the system has not jumped beyond the jth level of the tree between t_w and t_w+t. It is then clear that the equation (51) is not changed, although the analytic expression for the correlation function depends on the parameters of the tree.

For this particular model, the equilibrium relation is, in fact, satisfied even for out-of-equilibrium dynamics, because of the properties of the chosen exponential decay from the traps. It is then clear that the overlap function goes to zero for large times, for any finite t_w, since $\lim_{t\to\infty} C(t_w+t,t_w) = 0$: $S(t_w) = 0$. We have thus the same scenario as for the toy model: for any finite t_w, the overlap decays to its minimum allowed value, while an infinite t_w gives q_{EA} as a limit.

5. Conclusions

In this paper, we have shown that the overlap between two copies of a system, identical until a waiting time t_w, and then totally independent, is a quantity of interest regarding the geometry of phase space. We have indeed studied this quantity for several models, and shown that its decay is intimately related to the complexity of the landscape and to the type of ageing. For simple systems, the long-time limit of the overlap can be put closer and closer to the equilibrium limit q_{EA} by changing the time the replicas spend together. In contrast, for systems exhibiting a complex phase space, the limit of the overlap is always the minimum value, i.e. the two replicas are able to separate from each other, no matter how long the waiting time is (if it stays finite). This difference of behaviour can be quantitatively seen by a study of the various long time limits of the overlap: for any system, $\lim_{t\to\infty}\lim_{t_w\to\infty} Q_{t_w}(t_w+t, t_w+t)$ is q_{EA}, but the inverse order of limits, $S_\infty = \lim_{t_w\to\infty}\lim_{t\to\infty} Q_{t_w}(t_w+t,t_w+t)$ (and the behaviour of $S(t_w) = \lim_{t\to\infty} Q_{t_w}(t_w+t, t_w+t)$), distinguishes between ageing in a 'simple' phase space, and type II (spin-glass) systems.

Acknowledgments

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Appendix A. Dynamical equations for the toy model

In the limit of infinite N, the dynamical equations for the two-times correlation and response functions (with $t > t'$) read [5]

$$\begin{aligned}
\frac{\partial r(t,t')}{\partial t} &= -\mu r(t,t') + \int_0^t \mathrm{d}s\, m(t,s)(r(t,t') - r(s,t')) \\
\frac{\partial C(t,t')}{\partial t} &= -\mu C(t,t') + 2\int_0^{t'} \mathrm{d}s\, w(t,s)\, r(t',s) + \int_0^t \mathrm{d}s\, m(t,s)(C(t,t') - C(s,t')) \\
\frac{1}{2}\frac{\mathrm{d}C(t,t)}{\mathrm{d}t} &= -\mu C(t,t) + 2\int_0^t \mathrm{d}s\, w(t,s) r(t,s) \\
&\quad + \int_0^t \mathrm{d}s\, m(t,s)(C(t,t) - C(s,t)) + T
\end{aligned} \tag{A1}$$

and the equations we obtain for Q_{t_w} are

$$\begin{aligned}\frac{\partial Q_{t_w}(t,t')}{\partial t} &= -\mu Q_{t_w}(t,t') + 2\int_0^{t'} \mathrm{d}s\, W_{t_w}(t,s) r(t',s) \\ &\quad + \int_0^t \mathrm{d}s\, m(t,s)(Q_{t_w}(t,t') - Q_{t_w}(s,t')) \\ \frac{1}{2}\frac{\mathrm{d}Q_{t_w}(t,t)}{\mathrm{d}t} &= -\mu Q_{t_w}(t,t) + 2\int_0^{t} \mathrm{d}s\, W_{t_w}(t,s) r(t,s) \\ &\quad + \int_0^t \mathrm{d}s\, m(t,s)(Q_{t_w}(t,t) - Q_{t_w}(s,t)) + T\theta(t_w - t)\end{aligned} \tag{A2}$$

with the same notation as [5]:

$$\begin{aligned} &w(t,t') = f'(b(t,t')), m(t,t') = 4f''(b(t,t'))r(t,t') \\ &b(t,t') = C(t,t) + C(t',t') - 2C(t,t')\end{aligned} \tag{A3}$$

and

$$\begin{aligned} &W_{t_w}(t,t') = f'(B_{t_w}(t,t')) \\ &B_{t_w}(t,t') = C(t,t) + C(t',t') - 2Q_{t_w}(t,t')\,.\end{aligned} \tag{A4}$$

Appendix B. Non-overlapping time domains

We show how to compute one of the integrals of equations (A1), (A2) in the frame of the ansatz described in [5], using the diagrams introduced in figure 4. We apply this method to the overlap equation and show that a similar ansatz is a solution.

For $(t',t) \in D_u$, we parametrize

$$\begin{aligned} &b(t,t') = b_u(\tau) \qquad r(t,t') = \frac{\mathrm{d}\ln(h_u(t'))}{\mathrm{d}t'} r_u(\tau) \\ &m(t,t') = \frac{\mathrm{d}\ln(h_u(t'))}{\mathrm{d}t'} m_u(\tau) \qquad w(t,t') = w_u(\tau)\,.\end{aligned} \tag{B1}$$

The integral $\int_{t'}^t \mathrm{d}s\, m(t,s) r(s,t')$, appearing in (A1), has then three contributions (see figure B1):

- $(t',s) \in D_v$ $(v > u)$, where $m(t,s) = m_u(\tau)\mathrm{d}\ln h_u(s)/\mathrm{d}s$,
- t', t, s in the same domain D_u,
- $(s,t) \in D_v$ $(v > u)$, where $r(s,t') = r_u(\tau)$.

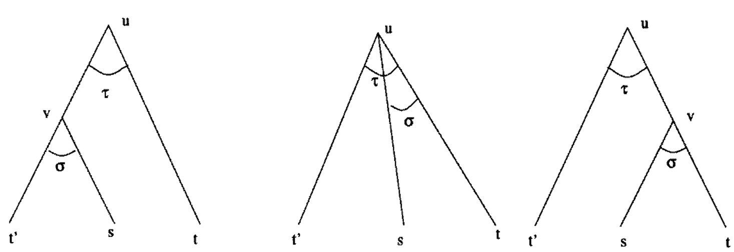

Figure B1. For $t' < s < t$, three regimes can be separated.

$$\int_{t'}^{t} \mathrm{d}s\, m(t,s) r(s,t') = \frac{\mathrm{d}\ln h_u(t')}{\mathrm{d}t'} \left(m_u(\tau) \sum_{v>u} \int_0^\infty \mathrm{d}\sigma\, r_v(\sigma) + \int_0^\tau \mathrm{d}\sigma\, m_u(\sigma) r_u(\tau-\sigma) \right.$$
$$\left. + r_u(\tau) \sum_{v>u} \int_0^\infty \mathrm{d}\sigma\, m_v(\sigma) \right). \tag{B2}$$

Separating in this way all different contributions in the integrals, we obtain the following equation for $b_u(\tau)$:

$$0 = b_u(\tau) \left[-\mu + \sum_{v\leqslant u} \int_0^\infty \mathrm{d}s\, m_v(s) \right] + 2T - \int_0^\tau \mathrm{d}s\, m_u(s)\; b_u(\tau-s)$$
$$-4w_u(\tau) \sum_{v>u} \int_0^\infty \mathrm{d}s\, r_v(s) - \int_0^\infty \mathrm{d}s\; [m_u(\tau+s)\; b_u(s) + 4w_u(\tau+s) r_u(s)]$$
$$+ \sum_{v\geqslant u} \int_0^\infty \mathrm{d}s\; [m_v(s)\; b_v(s) + 4w_v(s)\; r_v(s)]\;. \tag{B3}$$

The same method can be applied to the equation for the overlap function $B_{t_w}(t,t') = B_{u,u'}(\tau,\tau')$. For $u, u' < 1$ (with $u \neq u'$) it reads:

$$0 = B_{u,u'}(\tau,\tau') \left[-\mu + \sum_{v\leqslant u} \int_0^\infty \mathrm{d}s\, m_v(s) \right] + 2T - \int_0^\tau \mathrm{d}s\, m_u(s) B_{u,u'}(\tau-s,\tau')$$
$$-4w_u(\tau) \sum_{u'>v>u} \int_0^\infty \mathrm{d}s\, r_v(s) - 4W_{u,u'}(\tau,\tau') \sum_{v>u'} \int_0^\infty \mathrm{d}s\, r_v(s)$$
$$-4w_u(\tau) \int_{\tau'}^\infty \mathrm{d}s\, r_{u'}(s) - 4 \int_0^{\tau'} \mathrm{d}s\; W_{u,u'}(\tau,\tau'-s) r_{u'}(s)$$
$$- \int_0^\infty \mathrm{d}s\; [m_u(\tau+s) b_u(s) + 4w_u(\tau+s) r_u(s) - m_u(s) b_u(s) - 4w_u(s) r_u(s)]$$
$$+ \sum_{v>u} \int_0^\infty \mathrm{d}s\; [m_v(s) b_v(s) + 4w_v(s) r_v(s)]\;. \tag{B4}$$

If we insert in this equation the ansatz

$$B_{u,u'}(\tau,\tau') = b_u(\tau) \qquad (\text{if } u < u') \tag{B5}$$

we reobtain equation (B3). For $u = u'$, the equation is slightly different, and the ansatz

$$B_{u,u}(\tau,\tau') = b_u(\tau+\tau') \tag{B6}$$

together with the quasi-FDT, again gives back equation (B3), evaluated at $\tau + \tau'$.

Appendix C. Triangular relations

We derive the solution for the overlap function using the formalism of triangular relations [6, 32]. Neglecting time derivatives, and distinguishing ageing and FDT regimes in (A1) and (A2), the final equation for the slow varying part (i.e. non-FDT) of $B_{t_w}(t,t') =$

$C(t,t)+C(t',t')-2Q_{t_w}(t,t')$ reads (with $q=2(\tilde{q}-q_1)$):

$$
\begin{aligned}
0 = B_{t_w}(t,t')&\left[-\mu+\int_0^t \mathrm{d}s\, m(t,s)\right]+2T-\frac{2q}{T}[W_{t_w}(t,t')-f'(q)]\\
&+4\int_0^t \mathrm{d}s\, w(t,s)r(t,s)-4\int_0^{t_w}\mathrm{d}s\, w(t,s)r(t',s)-4\int_{t_w}^{t'}\mathrm{d}s\, W_{t_w}(t,s)\ r(t',s)\\
&+\int_0^t \mathrm{d}s\, m(t,s)b(t,s)-\int_0^{t_w}\mathrm{d}s\, m(t,s)b(t',s)-\int_{t_w}^{t}\mathrm{d}s\, m(t,s)\ B_{t_w}(t',s)
\end{aligned}
\tag{C1}
$$

where now all the times in the equations belong to the ageing regime.

The approach of triangular relations measures the various time domains directly in terms of the distance $b(t,t')$. Indeed, there is a one-to-one correspondence between $b(t,t')$ (at large times) and the functions $b_u(\tau)$. More precisely, if $t,t'\to\infty$ with $b(t,t')$ fixed to B, then t,t' belong to D_u, with $h_u(t')/h_u(t)=\mathrm{e}^{-\tau}$, where u and τ are fixed by $b_u(\tau)=B$. Then, the quasi-FDT can be expressed as

$$
r(t,t')=X[b(t,t')]\frac{\partial b(t,t')}{\partial t'}
\tag{C2}
$$

where $X[b(t,t')]=x_u/(2T)$. Similarly, the ultrametric structure of time domains described in (44), (46) can be written in a compact form as a triangular relation [6, 32]:

$$
b(t,t')=g(b(t,s),b(s,t'))
\tag{C3}
$$

with $g(b,b')=\max(b,b')$ when b and b' belong to different domains (also named 'blobs' [6]), and $g(b,b')=j^{-1}(j(b)j(b'))$ within the same domain.

The ansatz concerning the function B_{t_w} reads

$$
B_{t_w}(t,t')=\gamma[b(t,t_w),b(t',t_w)]
\tag{C4}
$$

where γ is a function to be determined. Setting $b(t,t_w)=b_w$, $b(t',t_w)=b'_w$, $b(t,t')=b$, $b_0=2(\tilde{q}-q_0)$ and

$$
F(b)=-\int_b^q \mathrm{d}s\, X(s)
\tag{C5}
$$

we obtain:

$$
\begin{aligned}
0=\gamma(b_w,b'_w)&\left[-\mu+4\int_{b_0}^q \mathrm{d}s\, f''(s)X(s)\right]+2T-\frac{2q}{T}[f'[\gamma(b_w,b'_w)]-f'(q)]\\
&+4f'(b_0)\ F(b_0)+4\int_{b_0}^q \mathrm{d}s\, f''(s)X(s)\ s+4\int_{b_0}^q \mathrm{d}s\, f'(s)X(s)\\
&+4\int_{b_0}^{b_w}\mathrm{d}s\, f''(s)F[\bar{g}(b,s)]-4\int_{b_0}^{b_w}\mathrm{d}s\, f''(s)X(s)\bar{g}(b,s)\\
&+4\int_{b_w}^{b}\mathrm{d}s\, F[\bar{g}(b,s)]f''[\gamma(b_w,\bar{g}(s,b_w))]\gamma'(b_w,\bar{g}(s,b_w))\bar{g}'(s,b_w)\\
&-4\int_{b_w}^{q}\mathrm{d}s\, f''(s)X(s)\gamma[\bar{g}(s,b_w),b'_w]
\end{aligned}
\tag{C6}
$$

where $\bar{g}$ is the reciprocal function of g: given three times $t'<s<t$, in the limit $t',s,t\to\infty$,

$$
b(t,t')=g(b(t,s),b(s,t')),b(s,t')=\bar{g}(b(t,s),b(t,t'))\,.
\tag{C7}
$$

Equation (C6) is a functional equation that gives γ in terms of f, X and $\bar{g}$.

Using the g-function described above, one can check from (C6) that the solution is $\gamma(x, y) = g(x, y)$ (this means that the relation between the overlap function $B_{t_w}(t, t')$ and the correlation functions $b(t, t_w)$ and $b(t', t_w)$ is the same as the triangular relation between the correlation functions). This way, one gets back the result for the overlap given in (49), (50).

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