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THEORY OF RANDOM SOLID STATES

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Abstract

This text is a non-technical, elementary introduction to the theory of glassy phases and their ubiquity. The aim is to provide a guide, and some kind of coherent view, to the various topics which have been explored in recent years in this very diverse field, ranging from spin or structural glasses to protein folding, combinatorial optimization, neural networks, error correcting codes and game theory.

1.1 A few landmarks

1.1.1 *Structural glasses*

Nature provides for us numerous examples of systems which may condense into an amorphous solid state. Probably the most common case is that of structural glasses, of which the window glass has been known for several millennia; recent reviews can be found in Angell (1995) and Benedetti (1997). Structural glasses consist of a phase of matter in which atoms or molecules are arranged in space in a structure which is frozen in time, apart from some small fluctuations. Yet, contrarily to the case of crystalline solids, the arrangement of these molecules is not a periodic one. It is a 'random' arrangement: although the system exhibits some kind of regularity on small enough scales (in the range of a few inter-atomic distances), this regularity is lost on larger length scales, as attested from the absence of sharp peaks in the diffraction pattern.

A random arrangement of the degrees of freedom, but one which is frozen and does not evolve in time: these are the basic ingredients of what we shall call the random, or amorphous, solid state, and what goes generally under the name of 'glass phase' (I have preferred the former because the term 'glass' is more specialized and might lead to some misunderstanding when we shall move to the random solid states of some systems which are more remote from condensed matter physics). Qualitatively this description is fine, yet the reader should be aware from the beginning of the difficulty of giving more precise definitions. We used the word 'phase of matter' but it may be (and has been) disputed whether this is a really new phase of matter. The glass state might not exist as a true separate phase, but just be describing a liquid with an extremely large viscosity, so that we do not see it flow in the limited time scale of our experiment. The fact that the structure does not evolve in time should not be thought of as implying that the positions of each atom is frozen: because of vacancies for instance the atoms can actually drift, although very slowly if the system is at low temperatures, as they also do in a crystalline phase. The relative positions of the points in space around which an atom is located, these define this frozen structure. The definition of a 'random' arrangement is not a trivial one either, one could have some order which displays no Bragg peaks but can be described with a little amount of information, or else one could be obliged to describe the glass state by giving the average positions of all atoms, which requires an infinite amount of information (in the 'thermodynamic limit' of infinitely large systems).

These are all important subtleties, and we shall partly address them below. Yet it is clear that, judging from its relaxation time, the glass state is at least a quantitatively different state of matter. Actually one very peculiar aspect of glass forming materials, and one which is so important in their manufacturing, is how rapidly this relaxation time, or the viscosity varies with the external conditions. Some increase by more than twelve orders of magnitude of the relaxation time when one diminishes the temperature by 20 per cent around the glass transition temperature are found in the so-called 'fragile' glasses which have the strongest such increase (Angell 1995; Benedetti 1997). At temperatures well below the glass transition temperature their life time is essentially infinite, and some million years old samples have been found. In the regimes where the experimental time is much smaller than the relaxation time the glass state is out of equilibrium and one observes aging phenomena. Inevitably we shall thus need to face the time dependent properties of these systems, which are even more difficult to describe than their equilibrium counterparts.

Beside its special properties, the glass state is important because of its ubiquity. It can be reached in virtually all systems, by many different pathways. Cooling from a liquid phase is a common one. The cooling rate should then be fast enough for the system to be quenched into the glass state, avoiding thus the crystallization (how fast one should quench depends enormously on the system at hand: as we all know, it is much easier to reach a glass state in liquid silica than in a metal). Probably in most systems the crystalline state is the most stable one, although this has not been proven: at zero temperature, the famous conjecture of Kepler stating that the densest packing of hard sphere is the crystalline one (face centered cubic or hexagonal closed packed) has resisted a proof for four centuries (Hales 1998). Showing that the crystalline state is the most stable one at some finite temperature, is thus likely to be a very hard task. The existence of a crystal state is annoying both for experimentalists who must 'beat the crystallization trap', and for theorists, who must find a proper way of studying a metastable state. But this is not more troublesome than studying super-cooled water, or diamond. A more subtle point, to which we shall return, is the fact that it is extremely difficult to prepare a glass in one given 'glass state'. From the mathematical point of view the idea of a glass at thermal equilibrium is a useful concept, and it turns out to be a very useful starting point in order to start a study, but the last word will deal with out of equilibrium dynamics. As we shall see, there are some indications that these two approaches (thermodynamic equilibrium and out of equilibrium dynamics) are intimately related, but the deep reason for this is not so clear, and its search will be a major challenge for the near future.

1.1.2 *From rubber to spin glass and proteins*

Another technologically important glassy material is rubber (Goldbart *et al.* 1996; Zippelius and Goldbart 1998). There, the basic microscopic constituents are long polymeric chains, and the amorphous solid state is obtained by adding cross-links which glue together permanently these chains- a process called vulcanisation which was discovered by Goodyear one and a half centuries ago.

There exists thus a fundamental conceptual difference with the simpler structural glasses described above: vulcanisation has created some permanent links between the polymers, which are located at random positions. Therefore the description of the vulcanised rubber involves some random variables- the positions of the crosslinks. These random variables are given a priori, they depend on the sample which one is studying, and their number is extensive, i.e. it grows linearly with the volume of the sample. This is very different from our previous case. In simple structural glasses one can work with a system of N molecules interacting by pairs (higher order interactions can be added easily without modifying the argument) through a simple potential $V(r_i, r_j)$. The energy function (the Hamiltonian) is very easily described, being just the sum of the pair interactions. What is complicated to describe and study is the amorphous state adopted by the system under fast cooling. On the contrary in rubber, writing down the Hamiltonian for a given sample requires the knowledge of the positions of all the crosslinks, a very long list which you cannot determine, nor store on your hard disk, and which will be different if you move to a new sample. This type of system, where the Hamiltonian depends on an extensive set of random variables, is said to have quenched disorder. The terminology comes from the fact that the monomers which are crosslinked do not evolve in time, they are not thermalized, contrarily to the other atoms of the polymers which have thermal fluctuations.

Quenched disorder is also present in some exotic magnetic alloys called spin glasses (Mézard *et al.* 1987; Fischer and Hertz 1991; Sherrington 2003). These systems are not present in every-day's life, they can be found only in some specialized solid state physics laboratories, and only in small quantity. They have surreptitiously appeared in various odd corners of materials science only a few decades ago, and nobody has been able to foresee any type of reasonable application in the close future, in spite of the strong evolutionary pressure of grant funding which pushes physicist to try and imagine some. Yet, during the last quarter of the XXth century, there have been many thousands of articles dedicated to spin glasses, both experimental and theoretical, and the spin glass problem has been described as a cornucopia (Anderson 1988). The reason is that spin glasses provide a (relatively) simple laboratory for the study of glass phases, which themselves appear in many domains, in physics and beyond.

The archetypical case of a spin glass is an alloy such as $CuMn$, with a concentration of a few per cent of the magnetic manganese atoms diluted in the non magnetic metal, here copper. The magnetic degrees of freedom are the localized magnetic moments of the Mn atoms. They interact with each other through a complicated process, an indirect exchange with the conduction electrons, but the net result is an interaction which either tends to align the magnetic moments- a ferromagnetic interaction, or tends to anti-align them (anti-ferromagnetic). Whether the interaction between two magnetic moments is ferromagnetic or anti-ferromagnetic depends on the distance between the manganese atoms: the coupling oscillates with distance. But the positions of these atoms are frozen in time, on all accessible time scales, and therefore the couplings between the magnetic moments form a set of quenched variables. Neglecting quantum mechanical effects, a good approximation at the temperatures of study, and using anisotropy to reduce the spins to a set of Boolean degrees of freedom, the Ising spins which describe the projection of the spin onto one axis, one soon arrives at a much simpler system indeed, a set of classical Ising spins interacting with random couplings. One can guess that this kind of generic problem of randomly interacting Boolean variables will provide useful insight into several domains of science and indeed it does, as we shall see. But the richness and difficulty of this problem, which we shall briefly survey in the next section, will be a surprise to any newcomer in the field (Mézard *et al.* 1987; Fischer and Hertz 1991; Talagrand 2003b).

Another example of an amorphous solid state, and one of the greatest importance, is offered by proteins (Garel *et al.* 1998). In its native form, a protein is a long polymer which is folded in such a way that the relative positions of the various atoms are frozen, apart from some small vibrations. In general this structure is not a simple periodic one, although one may find some recurrent substructures, 'alpha helices' and 'beta sheets', signaling a degree of local ordering. In a loose sense proteins thus fall into our broad definition of amorphous solid states. Obviously while including this very rich new field one is drifting from the purest mathematical definition of glass phases. One reason is the fact that proteins are finite size objects. Probably the proper level of description to describe protein folding is the one which considers the amino acid groups as basic entities, and the angles along the backbone as the relevant variables (as always when one chooses one level of description, there also exist some effects which require going to a smaller scale description). So we typically face a problem of a few hundreds to a few thousands degrees of freedom. This is enough to justify a statistical mechanics analysis, but it is not Avogadro's number.

Of more fundamental importance is the fact that proteins generally have one conformation which is preferred, the native state. This is the shape that makes them function, this is the shape that they adopt in natural conditions, and into which they will refold if denaturated. Although they also possess many other metastable states, these seem to have rather higher free energies, so that the protein will be able to avoid these other meta-stable states and fold into its native shape, sometimes with the help of some auxiliary, 'chaperon' molecules. Sometimes the free energy gap must be rather precisely tailored in such a way that some change in the external conditions (e.g. concentration of other proteins) will lead to some change in shape and properties of the protein, as has been demonstrated in the case of protein-DNA interactions. This dominance of the native state is at odds with the situation of glasses or spin glasses where the systems can freeze into any of the possible meta-stable states. One reason for this difference is the fact that the proteins are not completely random objects. Although the primary sequence of amino acids constituting a protein often looks random, one should remember that the sequences used in nature constitute a very small subset of the very large number of possible sequences (20^{100} for proteins made of one hundred amino-acids), and a subset which has been carefully selected by evolution, precisely for the ability to fold into a given shape allowing for some function. A totally random sequence of amino acids, with uniform probability of having each of twenty possible ones on each point along the chain, has very little chance of being a useful protein, or even just a molecule able to fold into a well defined native state. One needs some constraints in the sequence to achieve this, and the most obvious one is to have the right proportion of hydrophobic versus hydrophilic amino-acids, in such a way that the molecule, in water, will tend to form a compact globule with the hydrophobic ones buried inside the globule so that they avoid the water. The type of correlations which are needed in the choice of the sequence, in order to have a good chance of building a protein from a random heteropolymer, is a very difficult and open problem. Proteins provide some type of glasses with quenched-in disorder (the primary sequence of amino-acids), but the nature of the probability distribution of this disorder, and how natural evolution selected it, is still unknown.

We shall not attempt an exhaustive enumeration of glassy states of physical matter, numerous examples range from other biological polymers like DNA and RNA, to glasses of electric dipoles, or of vortex lines in high temperature superconductors (Blatter *et al.* 1994). A very rich class to which these vortex systems belong is that of elastic objects, lines, interfaces such as Bloch walls, modulated phases like charge density waves, which have some thermal fluctuations but are also pinned by some external impurities. The ubiquity of such situations in physics is well documented (as should be clear by now), but in addition glass states show up also in far out contexts, further enlarging the domain of study.

1.1.3 *Networks of interacting individuals: global equilibrium*

Imagine a group of N scientists, consider any two of them, and characterize their relationship at a very crude level by stating whether they are friends or not. These colleagues meet at a conference and the organizer, a very wise person, wishes to optimize their repartition in the two available hotels. He will thus make two groups and try to have as much as possible friends grouped in the same hotel and people who hate each other separated. He first collects the data on who is friend with whom. For each pair of people i, j, he assigns a positive interaction constant $J_{ij} = +1$, if they are friends, otherwise their interaction constant is negative, say $J_{ij} = -1$. From this set of interaction constants, which builds up our sample, the organizer tries to optimize the repartition in the following way: he will allocate each person i either in the hotel uphill, in which case he denotes him in his files by the number $S_i = +1$, or in the hotel downhill, labelled then by $S_i = -1$. Obviously, considering two colleagues i and j, there are two optimal repartitions for each situation of friendship, putting them in the same hotel if they are friends or in different hotels if they are not. These are described mathematically by finding the set of values S_i, S_j which minimize the 'pair interaction energy' $-J_{ij}S_iS_j$. Of course in a realistic case it is impossible to satisfy everybody: often the enemies of my enemies are not necessarily my friends, and the situation is then called frustrated, in a sense that it is not possible to satisfy simultaneously all pairs of people (the degree of frustration is measured by the fraction of triplets i, j, k such that the product $J_{ij}J_{jk}J_{ki}$ is negative). Finding the optimal hotel allocation in the set of 2^N possible ones turns out to be a very difficult problem, intractable by the present computers even for such a small number as $N = 200$. This problem is a case of a combinatorial optimization problem which falls into the so called NP-complete class: there are no known algorithms so far which are able to solve this optimization problem in a time which grows like a power of the size (N) of the problem. There may exist better algorithms than the enumeration of the 2^N allocations, but they all require a computer time growing exponentially with N.

What is the relationship of this sociological problem with our glasses? As one can guess from the choice of notations, this is just an example of a spin glass problem, the famous 'SK model' (Sherrington and Kirkpatrick 1975; Kirkpatrick and Sherrington 1978). Assigning person i to the uphill hotel is equivalent to having the Ising spin S_i pointing up $(S_i = +1)$, a person in the downhill hotel corresponds to the spin pointing down $(S_i = -1)$, and the aim of the organizer is to find a spin configuration which minimizes the interaction energy $E = -\sum_{1<i\leq j<N} J_{ij}S_iS_j$: he is seeking the ground state of the spin glass with exchange interaction constants J_{ij}. This is a special spin glass because every spin interacts with every other one: it has infinite range interactions. This actually simplifies the mathematical study because this infinite connectivity of interactions allows for an exact mean field solution. To be precise the solution of this problem, originally due to Parisi (1979, 1980; Mézard *et al.* 1987) has recently been shown to be exact by Talagrand (2003a), thanks to the beautiful mathemat-

ical developments of Guerra and Toninelli (2002), Guerra (2003), and Talagrand (2003a, 2003b). [Mean field spin glasses are the only cases for which we have such exact solutions; knowledge on spin glasses in finite dimension with short range interactions is very poor: nothing is known for sure, not even the existence of a phase transition, although the best numerical simulations point towards the existence of a spin glass phase, and this phase presents some similarities to what is found in mean field (Marinari *et al.* 1998; Krzakala and Martin 2000; Palassini and Young 2000)]. ¿From this solution (Parisi 1979, 1980) we can learn a few important facts on our original problem. The best assignments has a ('ground state') energy E_0 behaving for large N as $-.7633\ N^{3/2}$, which is very far above what would happen in the simple unfrustrated world where the energy scales as $-N^2$: despite all the efforts of our organizer, and his spending a lot of computer time, most people will be rather unhappy and he will not do a much better job than a random assignment of people into the two hotels! The physicist looks at this problem not only at zero temperature (where the problem reduces to finding a ground state), but also at finite temperature, where the various assignments are given a probability defined by the Boltzmann weight $\exp(-E/T)$. Then he can get some information on the structure of the assignments of low energy. It turns out that there are many such meta-stable states, which can be very different one from another: typically one can find an assignment which has an energy E_1 which is very close to E_0 (the difference between the two remaining finite when N becomes large), but which is very different, having half of the people changed hotel. On top of this, the set of meta-stable states has a fascinating hierarchical structure, building what is called an ultrametric space (Mézard *et al.* 1984a, 1984b).

A whole class of 'complex systems' can be studied similarly in the framework of equilibrium statistical mechanics. It contains many combinatorial optimization problems, in which one seeks a globally optimal configuration (a ground state) in a very large set of allowed ones (Mézard *et al.* 1987). One new idea brought in by physics is precisely this generalization of the problem to a finite temperature one: instead of asking for the ground state, one asks about the properties of the accessible configurations with a given energy, allowing for the introduction of useful notions such as entropy, free energy, phase transitions etc...

This turns out to be a fruitful strategy, both as an algorithmic device and as a theoretical tool. On the algorithmic side the idea gave rise to the simulated annealing algorithm which basically amounts to a Monte Carlo simulation of the problem in which one gradually reduces the temperature in order to try to find the ground state (Kirkpatrick *et al.* 1983). It is not a panacea and it can probably be outperformed by more specialized algorithms on any given problem. But it is a very versatile strategy, and one which can be very useful for practical problems because of its flexibility. In particular it allows to add new constraints as penalties in the energy functions with a rather small effort, where a more dedicated algorithm would just require a new development from scratch. Practical applications range from chip positioning to garbage collection scheduling, to routing and to financial market modeling!

Apart from trying to get an algorithm in order to find the optimal configuration, one aim could be to get some analytic prediction on this ground state, without necessarily constructing it. This is what happened to our conference organizer above: from spin glass theory he could get the optimal 'energy' of the best assignment of his colleagues into two hotels (or more precisely its large N limit), without knowing how to construct it, and he could learn about the distribution of meta-stable states. This type of knowledge is the first step towards the elaboration of a phenomenology of the problem, where one will aim for instance at understanding the importance of various type of correlations in the friendship distribution, etc... It also builds up an interesting class of problems in probability theory. These are the 'random' combinatorial problems in which one studies the properties of ground states of some random systems, given a certain probability distribution of samples. A famous example is the assignment problem: given N persons and N jobs, and a set of numbers giving the performance of each person for each of the possible jobs, find the best assignment of the jobs to the persons. The probabilist can ask the question of the performance of the best assignment for a given set of samples, for instance when the individual performances are independent identically distributed random variables taken from a given distribution. Very often the large N limit is 'self-averaging', meaning that this optimal length is the same for almost all samples in the set. The statistical mechanics approach has led to predictions concerning this optimal performance (Mézard and Parisi 1985), which have been confirmed recently by a rigorous approach (Aldous 2001).

1.1.4 *Networks of interacting individuals: dynamics*

Although the systems which we have just described already provide a large class of interesting problems, we are still very far from any real situation in sociology. Our use of equilibrium statistical mechanics is restrictive at least on two crucial points. One of them is the focus onto an equilibrium situation, the other one is the search of a *global* equilibrium. Keeping for another while to our toy conference problem, you have noticed that human activity is in general not organized in this totalitarian way of having an 'organizer' trying to optimize everybody's life (as we know such attempts are catastrophic, not only because of the practical impossibility of finding the optimal configuration). The more realistic situation of individual strategies where people have a large probability to change hotel if they are too unhappy leads to a dynamical problem, which could be described again as the relaxation towards some local equilibrium. We enter the world of dynamics, in a case which is still familiar in the sense that we can think of relaxational dynamics (the situation can be described by a heat bath). Familiar does not mean easy: at low temperatures (i.e. when each individual insists a lot in changing when this is favorable for him), this is the dynamics of a spin glass, and the relaxation time will be very large. What is found in spin glasses is that such a system, starting from initial conditions, will not find an equilibrium state, but will wander for ever (Bouchaud 1992). However the more time has elapsed, the longer the characteristic time scale for it to diffuse further away: such a system is aging, meaning that its response to an external stress depends on its age. This property has been observed for instance in polyvinylchloride, or in spin glasses, and its study has turned out to be an extremely valuable tool (Bouchaud *et al*. 1998).

One step further in complexity is the dynamical evolution when there is no energy. At zero temperature the energy is a Lyapunov function which keeps decreasing. Without such a Lyapunov function all kinds of behaviors become possible. We are going away from the physics of systems close to equilibrium, into much more complicated situations which are just beginning to be explored. Progress has been made in some cases (Challet *et al*. 2000a, 2000b; Dubois *et al* 2002), and I would particularly like to mention briefly one case, taken not from sociology, but rather from biology.

This is the study of neural networks, and particularly some attempts to build up a consistent theory of how memory can be organized in the brain (Amit 1989; Krogh *et al*. 1991). Elaborating on decades of experiments, it seems plausible that one important level of description of the brain, relevant for the treatment of information, is the level of activity of the neurons, measured as the number of spikes they emit per second (this is not obvious, and the information may be encoded in more subtle ways, such as for instance spike correlations). Focusing onto the spikes, one can take as the relevant elementary variables, either the spiking rate in each neuron, averaged over some time window of some tens of milliseconds, or its instantaneous version which is the Boolean variable: 0 if there is no spike, 1 if there is one. An active (spiking) neuron, through its synapses towards an other neuron, will either favor the spiking of this other one if the synapses are excitatory, or it may inhibit the other neuron's activity. At a caricatural level, the neural network might be considered as a highly interconnected network (there are of the order of 10^4 synapses per neuron) of variables, either continuous-if one models the activity through firing rates, or binary-if one uses spikes. The details of when the neuron decides to spike can be described by monitoring the membrane potential (the neuron fires when the potential exceeds some threshold), and in the end what such a network does is basically governed primarily by which are the excitatory synapses and which are the inhibitory ones.

Fifteen years ago, in a typical physicist's approach, John Hopfield tried to understand if such a caricatural network could be used as a memory (Hopfield 1982). He studied a network which was trained as follows: one shows it some external patterns and one reinforces a synapse whenever the two neurons it connects fire simultaneously. This process, known as Hebb's rule, builds a set of synapses which is such that the network memorizes the pattern: when presented an initial configuration which is a corrupted version of the pattern, it will spontaneously evolve towards the pattern. This way of fixing the synapses actually builds a set of *symmetric* synapses: the influence of neuron i onto neuron j is the same as that of j onto i. Because of this equality of action and reaction, there exists an energy function in this problem, and the evolution of the system, taking into account the stochastic nature of firing, is just that of a spin glass, where the exchange couplings between spins are the strengths of the synapses. A spin glass which has been tailored in such a way that its meta-stable states are the memorized patterns. It is no surprise that such a physical spin system, when evolving from an initial configuration which is not too far from a meta-stable state (one pattern), will flow towards it, and thus recover the full information on the pattern. This spin glass problem has been studied in great details: one can show that if too many patterns are memorized then the system can no longer memorize them, one can compute memory capacities, one can degrade the network, destroying a sizeable fraction of neurons and/or synapses, without altering its memory, etc... This was an extremely useful existence proof of the existence of associative memory effects in a very simplified neural network, and it allowed for many interesting quantitative studies. Its starting point was very remote from the reality on one crucial point: the assumption of symmetric synapses. Dropping this assumption forbids to introduce an energy function, and immediately drives one away from any equilibrium statistical mechanics studies. Yet it has been shown afterwards that many of the key properties of the network still persisted in the presence of some degree of asymmetry. Hopfield's daring assumption, which was once described by G. Toulouse as a "clever step backward," allowed to reduce the problem to a solvable one, which provided a solid background that one could elaborate upon in order to get a more realistic model. Several physicists started from this point and then added more realistic ingredients in order to get closer to biological reality. This is of course a very important elaboration, which is still moving ahead. One should remember that, even in presence of asymmetric interactions, the statistical mechanics approach may be useful in various ways, whether it will provide a solvable limiting case as in Hopfield's model, or whether one uses some of the purely dynamical approaches that will be described in the next section.

1.2 Tools and concepts

1.2.1 *Statistical description*

Let us also step backwards towards the 'easy' case of amorphous solid states: glasses. As soon as one tries to go beyond the crystal, or the crystal with defects, one faces the basic obstacle: how to describe an amorphous solid state? As we saw, it is out of question to try and describe the glass by listing the equilibrium positions of all the atoms. The point is that, in a given glass state, and even after averaging over the thermal fluctuations, the environment of each atom differs from that of all the other ones. Furthermore there is a very large number of long-lived glass states, a number which scales exponentially with the size of the system and therefore gives a contribution to the entropy, called the configurational entropy. In systems with quenched disorder, each sample is different from all the other ones. All these facts call for a statistical description of the properties of amorphous solid states. We have to give up the idea of describing in detail the equilibrium positions of the atoms in a glass state. Instead we shall give a statistical description of the relative equilibrium positions. The first step is to get rid of the thermal fluctuations, defining, in a given glass state, the density of particles at point x by the thermal average $\rho(x) = \sum_i \langle \delta(x - x_i) \rangle$. Here x_i is the position of particle i and the brackets stand for the average over thermal fluctuations in a given glass state, at a given temperature. While this would be just a constant in the liquid, it is a complicated function in the glass, with peaks at all the equilibrium positions of the atoms, a much too complicated object. Basically what one can hope to compute are some correlations such as the probability, given that ρ has a peak at a point x, that it will have another peak at some point $x + r$. This object in turn could depend on the glass state one is considering; in all cases studied so far it does not (a property of the large N limit called reproducibility), but if it would, one should again consider the probability distribution of the correlation when one changes the glass state. For systems with quenched disorder it could also depend on the sample and one would play the same game, but again this situation has not been encountered: most properties of a disordered system, including all thermodynamical properties, are said to be 'self-averaging' which means that they are the same for almost all samples (with probability one in the large N limit).

Giving up the idea of deciphering one particular sample and moving to the study of generic properties of all samples is a big shift of focus which has been described as a paradigmatic shift. It is comparable to what was done when people introduced statistical physics, giving up the idea of following the Newtonian trajectory of every particle, to concentrate on the probability distributions. In the study of glassy phases we have to take this step of a statistical modeling twice: first in order to deal with the thermal fluctuations (the usual statistical physics description), secondly in order to describe the fluctuations in the local environments, which exist even after thermal averaging (I shall call it the second statistical level). Some of the first successful implementations of this idea appear in the pioneering works of Sam Edwards and collaborators, both in spin glasses (Edwards and Anderson 1975), and in cross-linked macromolecules (Deam and Edwards 1976). The reason for the introduction of statistical physics finds its roots from the chaotic motion of particle, leading to sensitive dependence on initial conditions and forcing one to abandon the hope to follow a trajectory. In our case one reason of the statistical description is probably similar. In spin glasses it is well established that there exists some chaoticity, so that changing the sample slightly (e.g. changing a small fraction of the coupling constants) will lead to a system in which the metastable states are totally uncorrelated with the previous ones. In structural glasses the situation is less clear but it seems plausible that by changing slightly the number of particles from N to $N + \delta N$ with $1 << \delta N << N$ the (zero temperature) metastable states again become uncorrelated.

Chaoticity in the above sense is thus related to the property of self-averageness.

These are probably important ingredients allowing for the relevance of the statistical description. Again the case of proteins appears to be rather complicated from this point of view, partly because of their relatively small size, but mostly because the proper distribution of disorder in the sequence, and the corresponding chaoticity properties, have not been found. It is not known whether evolution has selected the proteins very specifically among all sets of heteropolymers or whether it has selected a class of sequences with some correlations, with some type of chaoticity property when one changes the sequence staying within the class. On the other hand a problem like brain modeling would seem to lend itself to the statistical description. Again it does not mean that the connections are random, but neither are they all preprogrammed (the information necessary to encode the 10^{14} synapses is much larger than that contained in DNA). There is an amount of randomness in the wiring, and there also exist generic properties common to most brains which one can hope to understand in this statistical sense, without having to care about all details of the wiring. In this respect the situation is very different from the study of a globally optimized device such as for instance a computer card.

1.2.2 *Physics without symmetry: equilibrium.*

The theoretical study of glassy phases is a notoriously difficult problem in physics, and one in which the progress has been relatively slow. One key reason is the absence of symmetry. All the simple computations on crystalline solid states which you find in the first pages of the textbooks, diffraction pattern, phonon spectrum, band structure, rely completely on the existence of a symmetry group. Even the simplest of these computations cannot be done in the glass phase. To face this situation, theorists have invented a number of methods which all amount to using the second statistical level, and introducing some kind of auxiliary symmetry, as we will explain below.

In usual problems it is relatively easy to understand the type of phase which can be found, using simple mean field arguments. The only more subtle questions which are not well captured by the mean field usually refer to some special points of the phase diagram, where the vicinity of a second order phase transition induces some long range correlations.

In glassy systems it turns out that understanding the gross features of the phase diagram is in itself a complicated task. The nature of the solid phase is much richer than usual. Mean field has naturally been applied to these problems, yielding a rather complicated but beautiful solution (Mézard *et al.* 1987). Again the basic ideas are simpler to express in the case of Ising spin glasses, with N spins taking values ± 1 and interacting with random exchange coupling. Detailed mean field computations have established the following picture. Above a critical temperature T_c the system is paramagnetic and the local magnetization vanishes in the absence of an external magnetic field: $< S_i >= 0$, where $< . >$ denotes an average over thermal fluctuations. Below T_c we enter the spin glass phase where an infinite spin glass will develop spontaneously a non-zero local magnetization:

$< S_i > \neq 0$. Compared to the more usual low temperature 'solid' phases, the spin glass phase possesses two distinctive properties:

- The spontaneous magnetization $< S_i >$ fluctuates widely from site to site; the global magnetization vanishes, and in fact all its Fourier components also vanish. Mathematically we face a breakdown of the lattice translational invariance to a random state, with no conserved symmetry subgroup of the translational group. A simple order parameter which characterizes the onset of the spin glass phase is the one introduced by Edwards and Anderson (1975): $q = (1/N) \sum_i < S_i >^2$.
- There exists an infinity of glass states. In the state α, the spontaneous magnetization on site i, $< S_i >_\alpha$, varies from state to state. The idea of several states is familiar from the usual case of ferromagnetism: in an Ising ferromagnet there are two states, in which the magnetization points either up or down. Here there exist many states, and they are not related one to the other by a symmetry. The order parameter should be written rather as $q_{\alpha\alpha} = (1/N) \sum_i < S_i >_\alpha^2$, but it turns out to be α independent.

Working within one given state is very difficult: the spins polarize into 'random' directions, which one does not know how to deduce from the original exchange couplings of the system; so one cannot use a conjugate magnetic field to polarize the spin glass into a given state. Even the definition of the states beyond mean field is an open mathematical problem. The best one can do so far is to postulate that the states exist and have properties similar to those found in mean field, and check if the simulation or experimental results can be analyzed in these terms. It turns out that this is the case. For instance a simple indicator consists in using two identical replicas of the system (with the same quenched disorder), weakly coupled through an infinitesimal attractive interactions, such as the product of the local bond energies in each system. One lets the system size go to infinity first, and the coupling between replicas go to zero afterwards. If there remains a non trivial correlation between the two replicas in this double limit, the system is in a glass phase. Basically in this game each system is playing the role of a small polarizing field for the other system.

The same method can be applied to identify the glass phase in structural glasses (Mézard 2001). Taking for notational simplicity a glass composed only of N identical atoms, the microscopic degrees of freedom are now the positions x_i of these N particles. One can introduce a second replica of the same system, composed of N particles at positions y_j. The x particles interact with each other, the y particles also. The x particles are nearly transparent to the y particles, except for a very small attraction, which is short range. The order parameter for the glass phase is then the cross correlation function between these two systems (i.e. the probability, given that there is an x particle at one point r_1 , that there be a y particle at a point $r_1 + r$), in the limit where the cross attraction vanishes. In the liquid phase the x and y particles just ignore each other in this limit, and there is no cross correlation. Instead, in the glass phase, the weak attraction ensures that the two systems polarize in the same glass state. They develop correlations because of the fact that they are in a solid phase, and these correlations still exist in the limit when the attraction vanishes. This provides a good mathematical definition of any solid phase.

1.2.3 *Replicas*

For the theorist a choice method is the replica method (Mézard *et al.* 1987). It uses the idea of having some identical replicas of the original problem, but their number is not limited to two, but can become any real number. The replica method is always presented as a trick to deal with quenched disorder: in disordered systems, the free energy is generally self-averaging in the thermodynamic limit, and therefore one can as well try to compute the average of the free energy over quenched disorder. This is rather difficult to compute, in general. A much easier task is to compute the average of the nth power, Z^n, of the partition function, which is nothing but the partition function of n non interacting replicas. Taking the $n \to 0$ limit one gets the quenched average of the logarithm of the partition function, which is proportional to the free energy. This trick is certainly very old (Giorgio Parisi dates it back to at least the fourteenth century when the bishop of Lisieux Nicolas d'Oresme used a similar trick in order to define non integral powers!) and has been used many times in the literature. Its first non-trivial application to the statistical physics of systems with quenched randomness is probably the seminal work of Edwards and Anderson (1975).

Going much beyond a simple mathematical trick, the replica method allows for a study of the free energy landscape, and principally of the regions of low free energy (the notion of a free energy landscape, in the very large dimensional space describing the configurations of a system in statistical mechanics, requires some thinking; however it is well defined in mean field, and it helps developing some intuitive picture, which is why I shall use it here for a simple presentation). The replicated partition function, after averaging over disorder, becomes a partition function for n systems, without disorder, but with an attractive interaction between the various replicas: the reason for this attraction is simple: Because they share the same Hamiltonian, with the same disorder, the various replicas will be attracted towards the same favorable regions of phase space, and repelled from the same unfavorable regions. Both effects tend to group the replicas together. If one has a simple phase space, with basically one large valley, then the replicas all fall into this valley, and the order parameter is a number, the typical distance between any two replicas, which gives directly the size of this valley. But in a system with several metastable states, the situation can be more complicated with some replicas choosing to fall into one valley, while others fall into other valleys. This effect has been called 'replica symmetry breaking'. Technically it appears as a standard spontaneous breaking of a symmetry. This symmetry is the permutation symmetry S_n of the n replicas. The problem is that this symmetry is broken only when one considers some number of replicas n which is non integer, and in fact smaller than one.

Based on some remarkable intuition about the permutation group with zero replicas, Parisi proposed at the end of the seventies a scheme of breaking the symmetry which is consistent, and has been applied successfully to many problems (Parisi 1979, 1980). Basically the order parameter turns out to be a function, which is the disorder averaged probability density, $P(q)$, picking up at random two thermalized non-interacting replicas of the system, that their distance will take a given value q. This order parameter could be computed at the mean field level in a variety of systems. In some cases it could be checked versus some other analytic computations, not involving the replica method, it could also be compared to simulations (a direct experimental measurement of $P(q)$ is not possible, but the recent developments on out of equilibrium dynamics, explained below, provide an indirect access to its measurement). So far it has always been found correct, although a rigorous mathematical status is still lacking.

The cavity method (Mézard *et al.* 1985; Mézard *et al.* 1987) has been developed in order to write down explicitly the assumptions underlying Parisi's replica symmetry breaking scheme, and develop a direct self-consistent probabilistic approach, equivalent to the replica method, based on these assumptions. The recent proof of the validity of Parisi's solution for the SK model basically follows this kind of cavity approach (Talagrand 2003a; Guerra and Toninelli 2002; Guerra 2003).

Fundamentally, three types of solid phases have been found at the moment with the replica method. Speaking in terms of an Ising spin glass system, with spins S_i, and defining the overlap between two spin configurations as $q = (1/N)\sum_{i=1}^{N} S_i S'_i$, we can characterize them from the shape of the overlap distribution $P(q)$. At high temperature the system is not in a solid phase and one has $P(q) = \delta(q)$: the thermal fluctuations win, there are no correlation between replicas. At low temperatures, in the presence of a small magnetic field which breaks the global spin reversal symmetry, one can find either:

- A replica symmetric phase with $P(q) = \delta(q - q_0)$. This happens for instance in a ferromagnet, where q_0 is the square of the magnetization.
- A situation called 'one step replica symmetry breaking' where $P(q) = x\delta(q - q_0) + (1 - x)\delta(q - q_1)$. This describes s system in which there are many free energy valleys, the width of each valley is measured by q_1, and the valleys are generically equidistant in phase space, their distance being measured by q_0. Very often $q_0 = 0$ and the valleys are located in random directions of the large dimensional configuration space. This situation thus occurs in a rather generical case where the low lying valleys are not correlated. Some mean field spin glasses are known to belong to this category, which is also thought to be the relevant one for the description of structural glasses of the fragile type.
- A situation called 'full replica symmetry breaking' where $P(q) = xp(q) + (1 - x)\delta(q - q_1)$, where p(q) is a continuous function normalized to one. In this case the low lying valleys become correlated. This is the category to which the standard spin glass systems belong.

The reader may find it surprising that, although the replica method was introduced to handle systems with quenched disorder (the whole story about approximating the free energy through Z^n is in order to be able to average on various realizations of quenched disorder), we mentioned the structural glasses, which have no quenched disorder, as physical systems displaying a one step replica symmetry breaking phenomenon. In fact I believe that the replica method is much more general than a trick for computing a logarithm. To illustrate this point, let me explain briefly how one can use a kind of replica method in the structural glass case. Let us assume that the free energy landscape of a structural glass is indeed made up of many valleys, such that the low lying valleys point in uncorrelated directions of phase space. Assume further that the number of valleys at a given free energy f is exponentially large, so that the entropy of the system is the sum of an internal entropy measuring the size of each valley, and of a configurational entropy $S_c(f)$ measuring their number. Proving these assumptions, purely from the microscopic Hamiltonian, is a task which seems totally hopeless at the moment, but one accessible method of approach is to postulate this structure, work out its consequences, and compare them to what is observed in experiments and simulations. How can one use replicas in such a case? The technique is a simple generalization of the two replicas used in the previous section to define the order parameter. Take m identical replicas of our glass, with a small short range attraction. In the glass phase this small attraction will polarize the system into the same valley. It is easy to see that the free energy of the replicated system $F(m)$, considered as a function of m, is the Legendre transform of $S_c(f)$. While it is very difficult to compute directly $S_c(f)$, one can easily develop simple approximation schemes for $F(m)$, and this gives access to the thermodynamic properties of the glass phase (Mézard and Parisi 1999).

1.2.4 *Physics without symmetry: dynamics*

The glass phase is very difficult to observe at equilibrium. Experimentally a glass is an out of equilibrium system, at least if the sample is large enough. The equilibrium properties which we have just discussed cannot be used in a direct quantitative comparison with the experiments. They can be of direct relevance for other amorphous solid states like optimization problems, or memory neural networks which are evolving from an initial configuration close to one of the memorized patterns. They can be useful to interpret some experimental findings, as is the case for the hierarchical structure of metastable states, but a direct comparison is difficult. The equilibrium studies provides the properties of the free energy landscape, focusing onto the low lying states. It is doubtful whether experimentalist will ever come up with a system prepared in one glass state α (the equivalent of a ferromagnetic crystal, uniformly polarized, without domain walls). Instead their systems age for ever.

The point may be illustrated from the dynamical definition of an order parameter, which we shall formulate again for simplicity in a spin glass language. In its original formulation by Edwards and Anderson (1975), the order parameter was defined as the long time limit of the spin autocorrelation: $q = \lim_{t\to\infty} \lim_{N\to\infty} < S_i(t)S_i(0) >$, where the brackets mean an average over the thermal noise (some underlying dynamics, for instance of a Langevin type, can be assumed for this classical spin system). This gives a correct definition only if the system is thermalized inside one glass state α at time $t = 0$. Then it is kind of tautological: the system remains inside the same state, the probability of the spin configurations decouple at large time and we obviously get back to the equilibrium definition $q = \lim_{N\to\infty}(1/N)\sum_i < S_i >_\alpha < S_i >_\alpha$. We are back to our problem: the system cannot be thermalized at time $t = 0$, so what should one do? Experiments provide the answer: the glass is aging. Somewhere it keeps a trace of the date at which it was born (Bouchaud *et al.* 1998). Let us call $t = 0$ this time, defined as the time at which the system was quenched below the glass transition temperature (if one cools slowly a structural glass, there are cooling rate effects, which may tell us a lot, but we won't discuss them here). The correlation function between times t_w and $t_w + \tau$ is $C(t_w + \tau, t_w) = \lim_{N\to\infty}(1/N)\sum_i < S_i(t_w)S_i(t_w + \tau) >$. As the relaxation time is infinite, or in any case much larger than any experimental time scale, the system is never thermalized at time t_w, whatever its age t_w is. One must study the dependence of the correlation as a function of the two times: the age t_w and the measurement time τ. The correct definition of the order parameter becomes $q = \lim_{\tau\to\infty} \lim_{t_w\to\infty} C(t_w + \tau, t_w)$. This turns out to give the same result as the equilibrium definition, showing that the system in this sense comes arbitrarily close to equilibrium, but now this order parameter can be measured. One can realize the subtlety of the approach to equilibrium by noticing that, in the reverse order of limits, $\lim_{\tau\to\infty} C(t_w + \tau, t_w) = 0$, for any t_w. This situation has been called weak ergodicity breaking (Bouchaud 1992), and seems to be present both in spin glasses and structural glasses. Experimental measurements, done on response functions rather than correlations, have found it for instance in systems such diverse as PVC (aging in the mechanical response: if I measure the response of your plastic ruler to a stress, I can deduce when the ruler was fabricated -provided I can perform a measurement on time scale of the order of its age!) and in spin glasses (aging in the relaxation of the thermoremanent magnetization).

Taking into account properly the aging effect implies thinking in the two time plane: the effects one can then study are not just the very complicated and system dependent transient effect, but they relate to what happens when both t_w and τ go to infinity, along various paths. It turns out that there seem to exist few universality classes for the behavior of the two times response and correlation functions in this limit. This have been first found by Cugliandolo and Kurchan in mean field spin glasses (Cugliandolo and Kurchan 1993). Based on these relatively simple models for which the dynamics can be solved explicitly, a generic scenario of glassy dynamics has been worked out, implying a well understood generalization of the fluctuation dissipation theorem, where an effective temperature, measurable but distinct from the bath temperature, characterizes the proportionality between the time derivative of the correlation and the instantaneous response, when these quantities are measured on time scales comparable to the age of the system. On these time scales the new relaxation processes which appear are 'thermalized' with an effective temperature which is close to that of the glass transition temperature, rather than to that of the room.

A proper account of these fascinating recent developments goes much beyond the scope of this paper. What I just want to point out here is that the measurement of this new effective temperature appearing in the generalized fluctuation dissipation theorem, which can be done by doing response and noise measurements, monitoring properly the age of the system, allows for an experimental determination of the type of glassy phase which one encounters, in the classification of section 1.2.3 (Cugliandolo and Kurchan 1993; Franz and Mézard 1994a, 1994b; Cugliandolo and Kurchan 1994; Franz *et al.* 1998).

Numerical simulations in spin glasses and structural glasses have confirmed that the $P(q)$ order parameter can be measured either from a well equilibrated small system, or from the generalized fluctuation dissipation theorem in the out of equilibrium dynamics of large systems (Parisi 1997; Kob and Barrat, 1997); the two procedures give results which agree with each other, although this does not imply that the asymptotic regime has been reached. The results point in the direction of a one step replica symmetry breaking in the structural glasses, and a full replica symmetry breaking in spin glasses.

On the experimental side, a recent beautiful experiment in a spin glass material has managed to measure the fluctuation dissipation ratio, and finds a rather good qualitative agreement with the predictions of the full replica symmetry breaking scenario (Herisson and Ocio 2002), although again it is not clear if the 'true' asymptotic regime can be measured. At present it seems that the mean field predictions provide at least good guidelines to the experimental systems at least on the time scales that can be obtained in the laboratory. Similar measurements have been attempted in structural glasses (Bellon and Ciliberto 2002) but the results seem to depend a lot on the observable and the situation is not yet clear.

1.2.5 *Simulations*

As we have seen, the theory of amorphous solid states has been developed in close connection with the progress in numerical simulations, and it will continue to do so. The collective behavior of strongly interacting systems can display very complicated, and sometimes surprising, behaviors, for which simulations help to provide some intuition, and to bridge the gap between theory and experiments. Reviewing the progress on the simulations goes beyond my abilities and beyond the scope of this paper, I shall rather refer the reader to Marinari *et al.* (1998). But one should be aware that in this field, the simulations play a very important role, on equal footing with theory and experiments, and this three-fold strategy is necessary for progress.

1.3 Directions

Predicting what will be the important developments in the future is bound to fail. I will not risk doing so, but just state a few topics which I find interesting at the moment. Their importance, the stage of their development and the time-scale of their study is totally uneven. The reader should just take them as some discussion topics such as they arise more or less randomly in a chat with colleagues, a winter evening, around the fireplace. As always the most interesting developments will be those that I cannot think of at this moment.

1.3.1 *Physical glasses*

The theory of glasses is still in its early infancy. The idea that glasses may be experimental realization of systems with one step replica symmetry breaking, although it is more than ten years old, has given shape to an actual microscopic model only very recently. The most obvious open questions concern the dynamics in the low temperature phase (we have no microscopic theory of aging in structural glasses so far), and the whole behavior in the temperature window above the glass transition temperature. The mean field models with one step replica symmetry breaking have two transition temperatures. The thermodynamic transition temperature, which should be the ideal glass transition temperature (that of a glass cooled infinitely slowly), and a dynamical transition temperature which is larger, at which the system becomes non ergodic, but where there is no thermodynamic singularity. This dynamical transition (which is also the one that is detected by mode coupling theory) is presumably a mean field artifact: the system gets trapped into metastable states which have an extensive free energy excitation with respect to the equilibrium state. One expects that this dynamical transition will be rounded in any real system by the 'activated processes', i.e. bubble nucleation. These are not understood at the moment, and their correct description is needed in order to understand the rapid increase of relaxation times upon cooling in glasses.

Letting aside for a moment all the unsolved mathematical questions which I shall discuss later, it is clear that the theory of spin glasses is more advanced. Yet we face two difficult problems concerning the extension of mean field theory to the spin glasses in dimensions smaller than six. On the technical side the standard field theory expansion around mean field is extremely difficult. The progress has been steady but slow, and indeed some of its first predictions have been confirmed numerically recently. Getting further along this direction will require some better understanding of the mathematical structures underlying replica algebra. The physical picture is not crystal clear either. We certainly would like to understand better how the many states are realized in real space. The physical discussion which can be given now is at the more abstract level of phase space, and it has shown its value in the design and discussion of experiments, but a fuller understanding requires going to the level of spins. In spite of many attempts at defining length scales in glasses, my feeling is that the situation is still rather unclear. Let me state a simple illustration: if one has only two states, the out of equilibrium dynamics is that of coarsening, and, after gauge transforming the spins one can think of it in terms of coarsening in an Ising ferromagnet. The generalization of the fluctuation dissipation theorem takes then a simple form, which has a very intuitive interpretation. After a large waiting time t_w, the system has developed some domains of each of the two phases, and the typical size of the domain is $\ell(t_w)$ (in a pure ferromagnet it would be $\ell = \sqrt{t_w}$, in presence of impurities, the growth of the domains will be slower). Then the dynamics after the time t_w is very different depending on whether one considers time $t_w + \tau$ with $\tau \ll t_w$, or with $\tau \gg t_w$. In the first case a given spin, which is generically far away from the domain walls, sees an environment which is at equilibrium. One thus expects the usual fluctuation dissipation theorem to be valid. On the contrary when $\tau \ll t_w$ a given spin sees some domain walls sweeping it all the time, and therefore its dynamics is that of a spin at infinite temperature. This is exactly what is predicted by the generalized fluctuation dissipation theorem for a replica symmetric system. As soon as we have replica symmetry breaking, whether it is one step or full replica symmetry breaking, we know the mathematical characterization of the generalized fluctuation dissipation theorem, it is a very nice structure which is confirmed by the simulations, but we cannot give yet a simple intuitive description of it, similar to the one I just presented.

1.3.2 *Random systems*

We seem to be on the way towards some general classification and characterization of the behavior of random systems, both in their equilibrium and non equilibrium behavior. The original fracture between the systems with and without disorder (roughly speaking: spin glasses and glasses) has been partially bridged (Bouchaud and Mézard 1994; Marinari *et al.* 1994a, 1994b; Chandra *et al.* 1995): if a system without disorder has a glassy phase, this phase may look very much like the one of a disordered system. This is kind of reminiscent of Wigner's successful step, when he substituted the complicated Hamiltonian of a nucleus by a random matrix with the same symmetries. In the framework of amorphous solid states such a step has been carried through in the case of a few specific examples, but we do not have yet any systematic equivalence, and the symmetry classes are not known.

Many of the ideas which I have presented here can have a resonance with other problems of physics. A better characterization of the low temperature thermodynamics of glasses involves the computation of spectrum and localization properties of vibrations in random structures, which is a problem appearing in many areas of physics. The interplay of the amorphous solid state ideas with the ones developed in electron localization could certainly also be a source of enrichment of both fields. Although I kept here within the scope of classical statistical mechanics, the quantum behavior of amorphous solid states is also very interesting: the quantum critical points appearing at zero temperature have very interesting properties which have just began to be worked out, but offer a wonderful playground for future developments.

On top of all the examples I have mentioned so far, from protein folding to brain theory, some of the most active areas of glassy physics outside of physics involve problems in computer science and information theory (Mézard 2003) such as error correcting codes (Nishimori 2001) and the satisfiability problem (Dubois *et al.* 2002), as well as its application to game theory and economic modeling (Challet *et al.* 2000a, 2000b; Bouchaud and Potters 2000). At a very basic level, the field which we have been studying in the last two decades is just that of collective behavior of interacting agents which are heterogeneous, whether this heterogeneity is here from the beginning or is generated by the system through its dynamical evolution. Obviously, this is a very general topic with many possible applications. I am thus confident that the spreading of this ideas will go on for a while.

1.3.3 *The unreasonable inefficiency of mathematics*

In some sense the equilibrium statistical mechanics of amorphous solid states is a branch of probability theory. A direct probabilistic solution of the mean field theory of spin glasses has been developed, at the mean field level, through the cavity method. After many years of study, and clever mathematical improvements, it now offers a rigorous solution for the SK model, and in optimization for 'simple' problems like the assignment or random link traveling salesman problem. Clearly this is a very active line of research and one can expect that new exact results will be obtained in this field in the forthcoming years.

But by far the easiest approach, the most compact as far as actual computation are concerned, the first one that one will use on any new random problem, is the replica one. It is very strange that nobody has yet come up with a mathematical framework to study the permutation group with a real number of elements and provide a justification to Parisi's replica symmetry breaking scheme, or maybe generalize it. This is a perfectly well defined scheme, where the computations , as well as the underlying probabilistic structure (which is exactly the contain of the cavity method) are completely understood.

The amorphous solid states are the low lying configurations of certain hamiltonians. It is no surprise that these will be related to the theory of extreme event statistics. If the configurations of a glassy system have independent random energies, then the extreme event theory tells us the statistics of these energies: they are given by Gumbel's law, which is the one relevant for us since we expect the energy distribution to be unbounded in the thermodynamic limit, but to fall off rapidly enough, faster than a power law. It turns out to be exactly the statistics which is found by the replica symmetry breaking method at one step replica symmetry breaking, as was found early on in the case of the random energy model. This provides some very encouraging connection between standard probability tools and physics. Of course in any physical system the energy of the configurations are correlated random variables. But one may hope that, after grouping together the configurations which are near to each other, one builds up some valleys for which the free energies are uncorrelated (keeping with the low lying valleys). These systems will form a universality class, containing the systems where the amorphous solid state is of the type 'one step replica symmetry breaking'. The present belief is that the glass phase of simple glasses (for example hard spheres or soft spheres) could be of this type. A better understanding of the random packings of spheres could help to confirm this conjecture. But the spin glass offer us some other universality classes, in which the low lying valleys are not uncorrelated, but possess a very specific type of hierarchical correlations: these are the problems where the amorphous solid state is described by the full replica symmetry breaking scheme. Putting them in the framework of extreme events statistics is an interesting mathematical problem. (In this respect one can draw an analogy with the universal behaviors of sums of random variables, rather than extremes, which is much easier. Everyone knows that if the variables are only weakly correlated the sum is universally distributed as a Gaussian variable; phantom polymer chains offer a physical example. Now if correlations are stronger, which means here that they can couple very distant variables, then physics offers the new universality class of self avoiding polymers, where the typical size of the sum is known to scale as the number to a power $\nu \neq 1/2$, but which is much harder to describe mathematically).

The field of spin glasses in particular offers many examples of facts that every physicist believes is true, but one cannot prove rigorously. This is not unusual in other branches of physics, and one should not be too worried about it. However it would be very welcome to have a proof of the existence of a spin glass phase in a finite dimensional model with short range interactions, to just mention the most obvious such fact.

I would not be surprised if the study of random solid states, and the various tools which have been developed in physics for that purpose, would lead in the future to interesting new mathematics, maybe with connections to probabilistic arithmetics.

1.3.4 *Consilience*

The statistical physics process of building a microscopic theory of amorphous solid states is a slow and difficult step of the development of physics. Many colleagues will just not want to make the intellectual investment of getting into it and will argue that a phenomenological description is enough. While I understand that the investment is hard, and for most people it may be better to wait until the theory has been understood better so that it can be simplified, I do believe that the microscopic modeling is an absolutely necessary step. We need phenomenological descriptions, trying to find out some description in terms of the smallest number of parameters. But we need to be able to relate them to the microscopic structure, and show the consistency of both. In this respect I think for instance that an elaboration of the scaling picture of spin glasses (McMillan 1984; Bray and Moore 1986; Fisher and Huse 1987, 1988), which would take into account the existence of many states, would be a very interesting achievement.

As we saw, the field of amorphous solid states is full of connections with many other branches of science. This is because of the richness of these amorphous phases, and their ability to have many different states coexist. In this respect its theory is a part of the development of a theory of complex systems (in the very broad sense of many interacting agents exhibiting complex collective behaviors). This field is not well defined enough for there to be a unique theory of complex systems. There are various approaches to it, applying to various levels, and each will be judged both on its own results, and on its consistency with the other ones. Of course statistical physics is just about finding out the collective behavior, starting from the microscopic description of the atoms. In this vague sense one could say it is central to the field. On the other hand if one looks at what statistical physics is able to achieve, one will rather say that it is not (yet) central. The available techniques can be judged as rather efficient to deal with the systems in which the dynamical evolution has a property of detailed balance, which means that they can be described by an energy function, and the evolution is just relaxation in some (free) energy landscape. This is a very strong restriction, and, as we saw on the example of neural networks, most of the interesting problems in complex systems will not obey it. Although some attempts have been made to develop some statistical mechanics study of the dynamics of systems without detailed balance, (in particular in asymmetric neural networks, or in random mappings of phase space), this is a very vast field which is much less understood. The virtue of the theory of amorphous solid state is that it can provide some very detailed information on some specific and oversimplified problems, which can then serve as solid starting points for further elaboration.

It might also be that some interesting problems, particularly in biology, have been so well selected by evolution that every single detail of the microscopic description is relevant: they are not generic at all, and the statistical description will have nothing to say about them. I feel reluctant to accept this as a general principle, mainly for philosophical reasons which I will not bother the reader with. Basically I feel that some level of statistical description, and therefore some degree of genericity, is unavoidable in order to build up a theory of many interacting elements, whatever they are (a simulation of tens of thousands of coupled differential equations reproducing some experimental behavior is not what I would call a theory, although it may be a very useful step in the elaboration of a theory). Physics has a long tradition of oversimplifying the real world in order to achieve a correct description, and then reincorporating the left-out details (think of the theory of gases for instance). This strategy, which is also the one that was followed for instance in the physical theory of neural networks, is probably the best one that can be followed in order to elaborate a theory. I understand that it may seem odd to our colleagues in other fields, particularly the fields which are very experimental ones, but I believe that one day or another their science will also benefit from such a strategy. Which field the statistical physics of amorphous solid states is able to help now, I leave the reader to decide, hoping that the above can provide a few guidelines.

Acknowledgments

Randomness may have some strange consequences. One of them is that I have never had a chance to discuss with Sam Edwards. Nevertheless I feel I know him well because of a long familiarity to his work. As this text shows, his ideas have had an enormous impact on the field of amorphous solid states, dealing with very fundamental issues like the order parameter or the second level of probabilistic descriptions. After nearly thirty years, these concepts are totally central to the whole domain. It is thus a great pleasure for me to contribute to this volume in his honor.

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