Quantum plasmas with or without a uniform magnetic field. II. Exact low-density free energy

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The exact analytical expression of the free energy f of a quantum Coulomb plasma in the presence of a uniform magnetic field $\mathbf{B}_0$ is produced at low density ρ. This regime corresponds to low degeneracy, weak Coulomb coupling but any strength of the magnetic field and fully quantum dynamics. Thus f is expanded around its value for an ideal gas in the Maxwell-Boltzmann (MB) approximation which provides a description of orbital diamagnetism and Pauli paramagnetism. The ρ expansion for f is derived from an adequate Mayer diagrammatic representation of the ratio between the plasma density and the density of the ideal gas with the same chemical potentials and MB statistics. A systematic scaling analysis of the dependence of Mayer bonds upon density is devised. This provides a natural truncation of the trace of the two-body Gibbs factor as well as diffraction contributions specific to the long range of the Coulomb potential. The $\rho^{3/2}$ term in f is the purely classical Debye contribution. From order ρ^2 on, $\mathbf{B}_0$ is involved through quantum dynamical and statistical effects which are the root of ferromagnetism. (Moreover, we retrieve the purely classical contributions at order $\rho^{5/2}$ in a very compact form.) Our results are compared with semiclassical expressions in the case of the one-component plasma. [S1063-651X(98)02710-X]

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I. INTRODUCTION

The present paper is devoted to the principles of the derivation of exact analytical low-density expansions in the framework of the loop formalism of paper I, which takes exchange effects systematically into account. The method is applied to the calculation of the free energy of multi- or one-component plasmas in the presence—as well as in the absence—of a uniform magnetic field $\mathbf{B}_0$. [A one-component plasma (OCP) is a system where only one species of charges is moving in a rigid neutralizing background.] The Hamiltonian of a multicomponent plasma reads

$$H_{\{N_\alpha\}}(\boldsymbol{B}_0)=\sum_i \frac{1}{2m_{\alpha_i}}\left(\mathbf{p}_i-\frac{e_{\alpha_i}}{2c}\mathbf{B}_0\wedge\mathbf{r}_i\right)^2-\sum_i g_{\alpha_i}\mu_{B\alpha_i}\mathbf{S}_i\cdot\mathbf{B}_0 + \frac{1}{2}\sum_{i\neq j} e_{\alpha_i}e_{\alpha_j}v_C(\mathbf{r}_i-\mathbf{r}_j), \tag{1}$$

with the same notations as in Paper I. The low-density limit corresponds to a regime of low degeneracy and weak Coulomb coupling for any strength of the uniform magnetic field. Thus the volume density of free energy f is expanded around its value $f_{\mathrm{id}}^{\mathrm{MB}}$ for a quantum ideal gas in the Maxwell-Boltzmann (MB) approximation and with the same densities in the presence of $\mathbf{B}_0$. We stress that the results contain all quantum effects at any order in $\hbar$. We get the analytical expression of f up to order $\rho^{5/2}$, where ρ is a generic notation for the particle densities. We use the convention that a term of order ρ^n may include powers of $\ln\rho$. (In other words, the possible logarithmic terms are considered to be of order $\rho^0 \equiv 1$.) These $\ln\rho$ terms as well as half-integer powers of the density appear because of collective screening effects due to the long range of the Coulomb potential. We retrieve the results up to order $\rho^{5/2}$ for the case $\mathbf{B}_0=\mathbf{0}$ produced in Refs. [1] and derived in Refs. [2–4], though our method starts from a different thermodynamic expression for the free energy and treats exchange effects systematically from the beginning instead of perturbatively. (When $\mathbf{B}_0=\mathbf{0}$ the low-density equation of state properly describes the core of the Sun, where dynamics proves to be controlled by Debye and exchange effects [5].) In the path integral formalism, the magnetic field shows up only in phase factors and the structures of both the derivation and results are similar whether $\mathbf{B}_0$ is switched on or not. A brief discussion of the effects arising from the presence of the magnetic field has already been given in Ref. [6].

We point out that this exact calculation starts from the first principles of quantum mechanics. All contributions that can be interpreted as being purely classical do not involve the magnetic field, in agreement with the Bohr–van Leeuwen theorem: magnetism is intrinsically quantum in its statistical origin. The MB free energy $f_{\mathrm{id}}^{\mathrm{MB}}$ for the ideal gas already incorporates the orbital diamagnetism arising from quantum dynamics as well as the Pauli paramagnetism due to the coupling between $\mathbf{B}_0$ and the spin quantum degree of freedom. These one-body phenomena appear at the first order in density, namely at order ρ. A correction of order $\rho^{3/2}$ comes from the exponential screening of the monopole potential created by a charge and its polarization cloud at large distances. The latter screening is valid at both classical and quantum levels and this first correction to $f_{\mathrm{id}}^{\mathrm{MB}}$ is independent from $\mathbf{B}_0$. The combined effect of the one-body spinorial coupling with the external magnetic field, Coulomb interactions, and quantum statistics emerges only from order ρ^2 on. It both renormalizes and mixes diamagnetism and paramagnetism. As a consequence, an effective coupling between spins shows up, though there is no fundamental magnetic dipolar interaction between spins in the Hamiltonian. In the sense that the ρ^2 term in the free energy can be related to an ef-

fective two-body potential, the root of ferromagnetism is already present at the scale of the two-body exchange effect in the presence of Coulomb interaction.

The rest of the paper is organized as follows. In Sec. II we present the main results. The dimensionless coupling parameters of the problem at finite temperature $1/\beta$ are discussed in Sec. II A. The reference free energy is recalled in Sec. II B and our results for the free energy are given in Sec. II C. The limiting case of the OCP free energy and its semiclassical value are discussed in Sec. II D. The main features of the method are discussed in Sec. II E and comparison is made with another formalism that allows one to derive the exact low-density free energy in the absence of $\mathbf{B}_0$.

In Sec. III the scheme for low-density expansions is displayed. In Sec. III A we give the thermodynamic formula that relates f to $f_{\mathrm{id}}^{\mathrm{MB}}$ through the primitive of $\ln(\rho_\alpha/\rho_\alpha^{\mathrm{id}\star,\mathrm{MB}})$ where ρ_α is the density in the plasma and $\rho_\alpha^{\mathrm{id}\star,\mathrm{MB}}$ is the density in an ideal gas $\mathcal{S}^{\mathrm{id}\star}$ in the MB approximation and with the same chemical potentials. We select the relations between the particle and loop densities together with a useful diagrammatic representation of the latter one (Sec. III B). The first terms in the ρ-expansion of various intermediate objects are obtained readily in Sec. III C. The explicit expressions for the ideal gas which plays the role of the reference system are given; in particular, the covariance of paths associated with a particle only submitted to a uniform magnetic field is derived in Appendix A by using three different methods. In Sec. III D we investigate the strategy of calculations more precisely. First, the expansion of the loop density around its noninteracting value for the same chemical potential is performed in powers of κ and of the loop densities. At the same time, the inverse screening length κ of the resummed interaction between total loop charges is expanded around its Debye value at the first order in density. Then, by a recurrence scheme, we expand the ratio $\rho_\alpha/\rho_\alpha^{\mathrm{id}\star,\mathrm{MB}}$, κ, and the loop density in powers of the particle densities. The general recurrence scheme is exemplified by the performance of its first step. In Sec. III E, by anticipation of the result of the diagrammatic survey, we give the formal structure of the low-density expansions for the free energy and for the density in terms of the first terms in the ρ expansion of the loop densities with degeneracy indices $p=1$ and $p=2$.

In Sec. IV we discuss the first part of the procedure, namely, the scaling analysis in loop density of Mayer diagrams. For that purpose we introduce a formal decomposition of the bonds in powers of κ (Sec. IV A). For bonds entirely scaled by κ, and which are integrable at finite density, a mere Taylor expansion at large distances is used. The dressed bond, which involves not only κ but also lengths that do not depend on the density, is not absolutely integrable at large distances, and a more delicate and systematic expansion in Fourier space is devised. The useful explicit values used in the following are derived in Sec. IV B and a procedure to determine the minimal order of any diagram in loop density is developed in Sec. IV C.

In Sec. V explicit contributions from diagrams involved in the loop density $\rho_{\alpha,p}(\mathbf{X})$ for $p=1$ are calculated and collected in order to exhibit partial derivatives with respect to some density ρ_α. The purely classical terms are derived in Appendix B. Thus we get the final formula exhibited in Sec. II C.

In Sec. VI we turn to the case of the one-component plasma. In Sec. VI A the free energy of the OCP is derived from the formula valid for a two-component plasma by sending the mass of one species to infinity and its charge to zero while keeping charge neutrality. An ingredient of the limiting process is the derivation of the small-x behavior of the generalized direct function $Q(x,u_C)$ introduced in Ref. [7] for $u_C=0$. The OCP has a well-behaved classical limit in the MB approximation, because all moving charges are of the same sign and quantum statistics is not needed to avoid any macroscopic collapse. In Sec. VI B we analyze the regimes of parameters in which the system goes to a semiclassical limit for any strength of Coulomb or orbital magnetic couplings. In Sec. VI C, we check that the low-density expansions are coherent with the semiclassical ones given in Ref. [8] for any value of $u_C\equiv\beta\mu_B B_0$. (A semiclassical investigation in the limiting case $u_C\ll 1$ was made in Ref. [9].)

II. MAIN RESULTS

A. Dimensionless coupling parameters

As announced above, from now on, we only consider systems for which statistics is weakly quantum. Let a be the mean interparticle distance. As long as all particle densities are of the same order, we do not distinguish the average distance between any kind of particles and the distance a_α between two particles of the same species α. The de Broglie thermal wavelength $\lambda_\alpha\equiv\sqrt{\beta\hbar^2/m_\alpha}$ is the amplitude of the quantum position fluctuations of a free particle with energy of order $1/\beta$ and so λ_α/a measures the average overlap of wave functions at temperature T. A weak degeneracy of quantum statistics is characterized by

$$\lambda_\alpha\ll a. \tag{2}$$

Then the average thermal energy per particle at equilibrium is given by Maxwell-Boltzmann statistics and is of order $1/\beta$.

Therefore dimensionless coupling parameters are defined as the ratios of average interaction energies and the order of magnitude $1/\beta$. Inspection of the Hamiltonian shows that there are three dynamical parameters. For the Coulomb interaction the coupling parameter is

$$\Gamma_{\alpha\gamma}\equiv\frac{\beta|e_\alpha e_\gamma|}{a}=\frac{b_{\alpha\gamma}}{a}, \tag{3}$$

where $b_{\alpha\gamma}\equiv\beta|e_\alpha e_\gamma|$ is the two-body average classical distance of closest approach for species α and γ (also called Landau length) for a relative trajectory governed by the Coulomb interaction and with energy of order $1/\beta$. In the following, we will use the notation Γ instead of $\Gamma_{\alpha\gamma}$ when all charges e_α are of the same order of magnitude as well as all densities ρ_α. (Such a situation is compatible with the local neutrality equation.) The dimensionless coupling parameters $u_{C\alpha}$ and $u_{S\alpha}$ for the magnetic interactions are equal to β times the quantum energies associated with the orbital motion and the spinorial precession in the quantum level with lowest energy, respectively:

$$u_{C\alpha}=\beta\mu_{B\alpha}B_0=\beta\frac{1}{2}\hbar\omega_{C\alpha}, \tag{4}$$

where $\omega_{C\alpha}=e_\alpha B_0/m_\alpha c$ is the cyclotronic frequency ($B_0\equiv|\mathbf{B}_0|$), while

$$u_{S\alpha}=\frac{g_\alpha}{2}u_{C\alpha}, \tag{5}$$

because the spinorial frequency is half of the cyclotronic one times the Landé factor g_α.

The coupling parameters depend on the fundamental constants and on the thermodynamic parameters T and ρ for the Coulomb interaction, T and B_0 for the magnetic case. When the density is varied, the length scale that measures the collective Coulomb effects, namely, the Debye length $\xi_D\equiv[4\pi\beta\Sigma_\alpha\rho_\alpha e_\alpha^2]^{-1/2}$, must be introduced. Up to a numerical factor,

$$\Gamma_{\alpha\gamma}\propto\left(\frac{a}{\xi_D}\right)^2. \tag{6}$$

Thus, according to Eq. (3),

$$\Gamma\ll 1\Leftrightarrow b_{\alpha\gamma}\ll a\ll\xi_D. \tag{7}$$

On the other hand, $u_{C\alpha}$ is linked to a length $l_{C\alpha}$ that depends only on B_0 (and not on the density),

$$u_{C\alpha}=\frac{1}{2}\left(\frac{\lambda_\alpha}{l_{C\alpha}}\right)^2=\frac{1}{2}\frac{\lambda_\alpha}{R_{C\alpha}}. \tag{8}$$

In Eq. (8) $l_{C\alpha}=\sqrt{\hbar c/e_\alpha B_0}$ is the characteristic quantum length (radius of ''orbits'') associated with the first quantum Landau level and $R_{C\alpha}\equiv\sqrt{mc^2/\beta e^2B_0^2}$ is the radius of the classical cyclotronic orbit of a particle with energy $1/\beta$. As a consequence

$$u_{C\alpha}\ll 1\Leftrightarrow\lambda_\alpha\ll l_{C\alpha}\ll R_{C\alpha}. \tag{9}$$

B. Low-density regime

In the present and following papers, we are interested in the low-density limit that corresponds to a regime of low degeneracy $\lambda_\alpha\ll a$ and weak Coulomb coupling,

$$\Gamma\ll 1, \tag{10}$$

whereas the interaction with the magnetic field is of any intensity.

In the absence of interactions, the Maxwell-Boltzmann approximation is well defined. The Hamiltonian is reduced to that of Pauli's theory. The volume free energy for an ideal gas with MB statistics reads

$$\begin{aligned}\beta f_{\rm id}^{\rm MB}(\beta,\{\rho_\alpha\},B_0)&=\sum_\alpha\rho_\alpha\{\ln[(2\pi\lambda_\alpha^2)^{3/2}\rho_\alpha]-1\}+\beta f_{\rm para}^{\rm MB}(\{\rho_\alpha,u_{S\alpha}\})\\&\quad+\beta f_{\rm dia}^{\rm MB}(\{\rho_\alpha,u_{C\alpha}\}).\end{aligned} \tag{11}$$

The first term is the free-energy density of an ideal gas in the absence of magnetic field and the second and third terms are the paramagnetic and diamagnetic ideal-gas contributions in the MB approximation, respectively.

$$\beta f_{\rm para}^{\rm MB}(\{\rho_\alpha,u_{S\alpha}\})=\sum_\alpha\rho_\alpha\ln\left(\frac{\sinh u_{S\alpha}}{\sinh[(2S_\alpha+1)u_{S\alpha}]}\right) \tag{12}$$

and

$$\beta f_{\rm dia}^{\rm MB}(\{\rho_\alpha,u_{C\alpha}\})=\sum_\alpha\rho_\alpha\ln\left(\frac{\sinh u_{C\alpha}}{u_{C\alpha}}\right). \tag{13}$$

The volume magnetization is derived from the density of free energy by the formula $\mathcal{M}=-\partial f/\partial B_0$ at fixed densities and fixed β. For the ideal gas in MB approximation

$$\mathcal{M}_{\rm id}^{\rm MB}=-\sum_\alpha\rho_\alpha\mu_{B\alpha}[g_\alpha S_\alpha\mathcal{B}_{S_\alpha}(2S_\alpha u_{S\alpha})+L(u_{C\alpha})], \tag{14}$$

where $\mathcal{B}_{S_\alpha}$ is the Brillouin function of order S_α,

$$\mathcal{B}_{S_\alpha}(x)\equiv\frac{1}{2S_\alpha}\left[(2S_\alpha+1)\coth\left((2S_\alpha+1)\frac{x}{2S_\alpha}\right)-\coth\left(\frac{x}{2S_\alpha}\right)\right] \tag{15}$$

and L is the Langevin function $L(x)\equiv\coth x-(1/x)$.

C. Free energy of a multicomponent plasma

For sets of densities that satisfy the local neutrality relation $\Sigma_\alpha e_\alpha\rho_\alpha=0$, we get

$$\beta f(\beta,\{\rho_\alpha\},B_0)=\beta f_{\rm id}^{\rm MB}+\beta f^{\{3/2\}}+\beta f^{\{2,5/2\}}+o(\rho^{5/2}), \tag{16}$$

where $o(\rho^n)$ denotes a term of order greater than ρ^n. In Eq. (16), as in the following, the orders in density will be denoted by braces, whereas the orders in $|\mathbf{k}|$ will be referred to in parentheses. At order ρ (and $\rho\ln\rho$), all effects are contained in the contribution $\beta f_{\rm id}^{\rm MB}$ [see Eq. (11)] from the gas of independent particles in the MB approximation. In the weak-coupling and low-degeneracy regime, the next contribution is of order $\rho^{3/2}$ and it coincides with the classical excess free energy in the linearized Debye-Hückel approximation,

$$\beta f^{\{3/2\}}=\beta f_D=-\frac{\kappa_D^3}{12\pi}. \tag{17}$$

Indeed, according to Paper I, the bare two-body Coulomb potential is partially screened by collective effects over a length scale κ^{-1}. κ depends on the density and tends to the Debye value κ_D when exchange effects are negligible. We stress again that the purely classical Debye free energy f_D does not involve $\mathbf{B}_0$, in agreement with the Bohr–van Leeu-

wen theorem. Effects from many-body interactions beyond the linearized mean-field classical Debye approximation together with short-ranged exchange mechanisms appear only from order ρ^2 on. At orders ρ^2 and $\rho^{5/2}$, the exact contributions arising from quantum dynamics and quantum statistics with interactions are

$$\beta f^{\{2,5/2\}} = -\frac{1}{2}\sum_{\alpha}(-1)^{2S_\alpha}[1+\beta\kappa_D e_\alpha^2]\frac{\tanh u_{S\alpha}}{\tanh[(2S_\alpha+1)u_{S\alpha}]}\rho_\alpha^2\frac{\sinh u_{C\alpha}}{u_{C\alpha}}(4\pi\lambda_\alpha^2)^{3/2}\int d\mathbf{r}\langle -\mathbf{r}|e^{-\beta h_{\mathrm{rel},\alpha}}|\mathbf{r}\rangle \tag{18a}$$

$$-\frac{1}{2}\sum_{\alpha,\gamma}[1+\beta\kappa_D e_\alpha e_\gamma]\rho_\alpha\rho_\gamma\lim_{R\to\infty}\left\{\int_{r<R}d\mathbf{r}\left[\frac{\sinh u_{C\alpha}}{u_{C\alpha}}\frac{\sinh u_{C\gamma}}{u_{C\gamma}}(2\pi\lambda_\alpha\lambda_\gamma)^3\langle\mathbf{0},\mathbf{r}|e^{-\beta H_{\alpha\gamma}}|\mathbf{0},\mathbf{r}\rangle\right.\right.$$
$$\left.\left.-1+\frac{\beta e_\alpha e_\gamma}{r}-\frac{(\beta e_\alpha e_\gamma)^2}{2r^2}\right]+\frac{2\pi}{3}(\beta e_\alpha e_\gamma)^3\ln(\kappa_D R)\right\}-\frac{\pi}{3}[C+\ln 3]\beta^3\left(\sum_\alpha\rho_\alpha e_\alpha^3\right)^2$$
$$-\frac{\pi}{3}[-1+C+2\ln 2]\beta^4\kappa_D\left(\sum_\alpha\rho_\alpha e_\alpha^4\right)^2 \tag{18b}$$

$$+\frac{2}{3}C_1\beta^5\frac{1}{\kappa_D}\left(\sum_\alpha\rho_\alpha e_\alpha^3\right)^2\left(\sum_\gamma\rho_\gamma e_\gamma^4\right)+\frac{2}{3}C_2\beta^6\frac{1}{\kappa_D^3}\left(\sum_\alpha\rho_\alpha e_\alpha^3\right)^4 \tag{18c}$$

$$+\frac{1}{24}\hbar^2\beta^2\kappa_D^3\sum_\alpha\rho_\alpha\frac{e_\alpha^2}{m_\alpha}+\frac{1}{6}\frac{\beta\hbar c}{B_0}\kappa_D^3\sum_\alpha\rho_\alpha e_\alpha L^{[3]}(u_{C\alpha}), \tag{18d}$$

where $C=0.577\,215\ldots$ is the Euler-Mascheroni constant. In the contributions from order ρ^2 on, the magnetic field appears through normalization factors involving $u_{C\alpha}$ and $u_{S\alpha}$, through Hamiltonian operators, and through ''diffraction'' contributions which are functions of the $u_{C\alpha}$'s . $H_{\alpha\gamma}$ is the two-body Hamiltonian without the spin contribution,

$$H_{\alpha\gamma}(1,2)\equiv\frac{1}{2m_1}\left[\mathbf{p}_1-\frac{e_\alpha}{2c}\mathbf{B}_0\wedge\mathbf{r}_1\right]^2+\frac{1}{2m_2}\left[\mathbf{p}_2-\frac{e_\gamma}{2c}\mathbf{B}_0\wedge\mathbf{r}_2\right]^2+\frac{e_\alpha e_\gamma}{|\mathbf{r}_1-\mathbf{r}_2|}. \tag{19}$$

For two particles of the same species, the position of the center of mass, with mass $2m_\alpha$ and charge $2e_\alpha$, and that of the relative particle, with mass $m_\alpha/2$ and charge $e_\alpha/2$, are separable variables even when $\mathbf{B}_0\neq\mathbf{0}$. The Hamiltonian corresponding to the latter fictitious particle in the Coulomb potential created by a charge $2e_\alpha$ is $h_{\mathrm{rel},\alpha}$,

$$h_{\mathrm{rel},\alpha}(\mathbf{B}_0)\equiv\frac{1}{m_\alpha}\left(\mathbf{p}-\frac{e_\alpha}{4c}\mathbf{B}_0\wedge\mathbf{r}\right)^2+\frac{e_\alpha^2}{r}. \tag{20}$$

For the center of mass the de Broglie wavelength is equal to $\sqrt{2}\lambda_\alpha$ and $u_{C\alpha}$ has the same value as for each particle of species α. Thus (see Sec. III C 2)

$$\int d\mathbf{r}(2\pi\lambda_\alpha^2)^3\left(\frac{\sinh u_{C\alpha}}{u_{C\alpha}}\right)^2\langle\mathbf{r},\mathbf{0}|e^{-\beta H_{\alpha\alpha}}|\mathbf{0},\mathbf{r}\rangle$$
$$=\int d\mathbf{r}(4\pi\lambda_\alpha^2)^{3/2}\frac{\sinh u_{C\alpha}}{u_{C\alpha}}\langle -\mathbf{r}|e^{-\beta h_{\mathrm{rel},\alpha}}|\mathbf{r}\rangle. \tag{21}$$

The bound and diffusion states are contained in the quantum density-matrix elements.

The exchange effects, which are short ranged whether there are interactions or not, are not perturbed by any collective effect at order ρ^2, while, at order $\rho^{5/2}$, the bare contribution is only renormalized by a multiplicative factor arising from classical Debye screening. On the contrary, the direct term involves screening in an essential way from order ρ^2. Indeed, the truncation of the matrix element in the direct term (18b) ensures that the integral only diverges as a $\ln R$ term which is exactly compensated by the $\ln(\kappa_D R)$ inside the braces; this truncation arises from the low-density limit of the screened bonds in a natural way.

The constants in Eq. (18c) read

$$C_1=6\pi\int_0^\infty du\,\frac{[\arctan(u/2)]^2}{1+u^2} \tag{22}$$

and

$$C_2=-12\pi^2\int_0^\infty du\,\frac{[\arctan(u/2)]^2}{(1+u^2)^2}-\frac{1}{16}\tilde{I}_{\mathrm{bridge\,6}}, \tag{23}$$

where $\tilde{I}_{\mathrm{bridge\,6}}$ is an integral corresponding to a so-called ''bridge'' diagram with six Debye bonds, as defined in Sec. V D. The expressions for C_1 and C_2 are more compact than those given in Ref. [3]. We notice that the analytical expression for $\tilde{I}_{\mathrm{bridge\,6}}$ may also be written as the sum of formulas (4.2) and (4.3) in Ref. [3] with a global multiplicative factor 1/2 which was omitted in these equations. Moreover, the most recent numerical values for these constants can be found in Ref. [10], $2C_1/3\equiv a_1=10.134\,779\,10\ldots$, and

$2C_2/3=-a_{2\alpha}-a_{2,\omega}$ with $a_{2,\omega}$ corresponding to the term proportional to $\tilde{I}_{\text{bridge 6}}$ and $a_{2\alpha}$ to the other term in C_2; $a_{2\alpha}=8.052\,814\pm0.000\,001$ and $a_{2\omega}=1.7699\pm0.0001$. If powers of κ_D could be forgotten, the terms in Eq. (18c) might be interpreted in terms of effective interactions between three or four bodies according to their powers in ρ, as in the case of short-ranged forces. However, the powers of κ_D come from (linear or nonlinear) collective effects, and the interpretation is not so simple. (In particular, there is a dressing of nonlinear effective interactions by linear Debye bonds.)

The term (18d), also called the diffraction term in the absence of $\mathbf{B}_0$, is a quantum dynamical effect due to the fact that the long-ranged Coulomb potential is only algebraically screened. This term vanishes at order ρ^2 because of the local neutrality relation. It can be decomposed into a part independent from $\mathbf{B}_0$ plus a correction which involves a generalization $L^{[3]}(x)$ of the Langevin function that appears in the orbital magnetization (14) of a gas of independent charges. $L^{[3]}(x)\equiv L(x)-x/3$ behaves as $-x^3/45$ when x goes to zero. Thus the correction to the diffraction term that is due to the magnetic field is proportional to B_0^2 when B_0 goes to zero, whereas it goes to a constant in the limit of strong fields. The diffraction term may be expressed in terms of the plasma frequencies $\omega_{p\alpha}=[4\pi e_\alpha^2\rho_\alpha/m_\alpha]^{1/2}$, which are related to the dynamics of the center of mass.

All thermodynamic quantities can be obtained from the free-energy density. For instance, the pressure $P=-f+\Sigma_\alpha\rho_\alpha\partial f/\partial\rho_\alpha$ has an expression similar to that of f up to order $\rho^{5/2}$. On the contrary, the expression of the volume magnetization $\mathcal{M}=-\partial f/\partial B_0$ requires a detailed spectral analysis, which is far beyond the scope of the present paper. The diamagnetic and paramagnetic magnetizations of the MB quantum ideal gas are renormalized and coupled by interactions and quantum statistics. In Eq. (158) the term $\rho_\alpha^2\tanh u_{S\alpha}/\tanh[(2S_\alpha+1)u_{S\alpha}]$ is the sum of the squared densities of particles α in the $2S_\alpha+1$ spin states in the absence of Coulomb interactions, and the combination of exchange and direct density-matrix elements in position space is linked to the origin of ferromagnetism.

D. OCP free energy

The formulas for the OCP are derived in Sec. VI from those established for a two-component plasma by using the following procedure. First, the mass of one given species goes to infinity; then its charge vanishes as its density becomes infinite so that their product is kept constant and ensures global neutrality.

We introduce a generalization of the standard notations Q and E used in Ref. [7],

$$Q\left(-\frac{\beta e^2}{\lambda},u_C\right)=\frac{1}{4\pi\lambda^3}\lim_{R\to\infty}\left\{\int_{r<R}d\mathbf{r}\left[(4\pi\lambda^2)^{3/2}\frac{\sinh u_C}{u_C}\langle\mathbf{r}|e^{-\beta h_{\text{rel}}}|\mathbf{r}\rangle-1+\frac{\beta e^2}{r}-\frac{\beta^2e^4}{2r^2}\right]+\frac{2\pi}{3}\beta^3e^6\left[\ln\left(\frac{3R}{\lambda}\right)+C\right]\right\} \quad (24)$$

and

$$E\left(-\frac{\beta e^2}{\lambda},u_C\right)=2\sqrt{\pi}\frac{\sinh u_C}{u_C}\int d\mathbf{r}\langle-\mathbf{r}|e^{-\beta h_{\text{rel}}}|\mathbf{r}\rangle. \quad (25)$$

The difference with the formulas for $\mathbf{B}_0=\mathbf{0}$ is that there appear two dimensionless variables, $-\beta e^2/\lambda$ and u_C, instead of one. With these notations, we get

$$\beta f^{\text{OCP}}(\beta,\rho,B_0)=\rho\{\ln[(2\pi\lambda^2)^{3/2}\rho]-1\}+\rho\ln\left(\frac{\sinh u_C}{u_C}\right)$$

$$+\rho\ln\left(\frac{\sinh u_S}{\sinh[(2S+1)u_S]}\right)-\frac{1}{12\pi}\kappa_D^3 \quad (26a)$$

$$-2\pi(-1)^{2S}\rho^2[1+\beta\kappa_De^2]\frac{\tanh u_S}{\tanh[(2S+1)u_S]}\times\lambda^3E(-\beta e^2/\lambda,u_C) \quad (26b)$$

$$-2\pi\rho^2[1+\beta\kappa_De^2]\lambda^3Q(-\beta e^2/\lambda,u_C) \quad (26c)$$

$$-\frac{\pi}{3}\beta^3\rho^2e^6[1+\beta\kappa_De^2]\ln(\lambda\kappa_D)+\beta^4\kappa_D\rho^2e^8\left\{\frac{\pi}{3}(1+\ln3-2\ln2)+\frac{C_1}{6\pi}+\frac{C_2}{24\pi^2}\right\} \quad (26d)$$

$$+\frac{\pi}{3}\frac{\beta^2\hbar^2}{m}\rho^2e^2\left[1+\frac{1}{2}\beta\kappa_De^2\right]\left[1+\frac{2}{u_C}L^{[3]}(u_C)\right]+o(\rho^{5/2}). \quad (26e)$$

The expression for the pressure has an analogous structure, as in the case of the multicomponent plasma.

Up to order $\rho^{5/2}$, the result for the OCP is similar to Eq. (16), apart from the diffraction term, which does not vanish at order ρ^2 and reads $(4\pi/3)(\beta\hbar ec/B_0)\rho^2L^{[3]}(\beta\mu_BB_0)$. The origin of the diffraction terms for the OCP may be viewed as the sum of two contributions. First, there is a contribution from moving particles as for a multicomponent plasma, but in the latter system the term of order ρ^2 is canceled, because all species move and obey the neutrality relation. Second, there is an extra contribution at order ρ^2 that comes from the expansion of the direct quantum term Q_{12} between species 1 and 2 when particles of species 2 are turned into a rigid background. (At order $\rho^{5/2}$ the coupling in e_2 is of higher orders and the contribution from Q_{12} disappears in the limit where m_2 goes to infinity and e_2 vanishes.)

Besides, in regimes of low degeneracy and weak quantum dynamical effects at $u_C\equiv\beta\mu_BB_0$ fixed, the expression of the OCP free energy can be expanded with respect to $\hbar$, because the OCP has a well-defined thermodynamic limit even with MB statistics. The exchange density-matrix element in position space vanishes exponentially fast when $\hbar$ goes to zero [8,9,11]. According to Ref. [8], in the semiclassical limit, valid for any strength of Coulomb and magnetic couplings in a regime of low degeneracy, the quantum term of lowest

order in $\hbar$ in the free-energy density is the contribution from the MB gas of independent charges, which is of order ρ. The interactions are involved only in the next-order $\hbar^2$ term, which is exactly proportional to ρ^2 for any density [8,9]. We have checked that the first two terms in the $\hbar$ expansion of the exact low-density free energy up to order $\rho^{5/2}$ (valid for weak Coulomb coupling and derived in the present paper) coincide with the first four terms in the ρ expansion of the exact semiclassical free energy up to order $\hbar^2$ (valid for any Coulomb coupling and given in Ref. [8]). In particular, the contributions of order $\rho^{5/2}$ in the low-density expression cancel each other at order $\hbar^2$, as they should.

E. Comments about the method

A few comments may be made about our method. In the absence of magnetic field, pioneering work about the derivation of the free energy was achieved by the method of effective potentials [7], and an exact analytic expression up to order $\rho^{5/2}$ is given in Ref. [1] and derived in Refs. [2–4]. Our method, which has various similarities with that used in the latter references, allows us first to retrieve the previous results and to study very straightforwardly, and to our knowledge for the first time, the differences that originate from the magnetic field. Indeed, as already stressed in Paper I, the presence of the magnetic field is entirely contained in a phase factor incorporated in the generalized fugacity of each loop in the path integral formalism. Moreover, in the low-density limit, calculations can be performed explicitly (in terms of matrix elements of a two-body Hamiltonian), because they involve the covariance of Brownian paths of independent particles in a magnetic field: the latter problem is solvable and the covariance can be exactly expressed in terms of products of hyperbolic functions.

Two advantages of Mayer-diagram methods derived from the path integral representation, and which are also used in Refs. [2–4], are the following. First, the origin of effects at stake is clearly exhibited. Classical (and quantum) screening of monopole-monopole interactions is described by the bond F^{cc}, diffraction effects resulting from the combination of the long range of the Coulomb potential and the wave nature of quantum dynamics are described by the bond F^{cm}, while short-distance properties generated by quantum dynamics, such as the absence of collapse of two opposite charges together with the existence of bound and scattering states, are contained in the bond F_R.

Another interest of the Mayer-bond method is that a scaling analysis allows us to select very quickly from which order in density a diagram contributes. Moreover, half-integer powers of the density appear in the low-density expansions in a quite natural way; they come through a length scale arising from screening collective effects which is the only length depending on the density that is involved in the three kinds of bonds.

The differences between our method and that used in Refs. [2–4] are essentially of two kinds. First, in the loop formalism exchange effects are not treated perturbatively from the start but they are handled systematically. For instance, up to order $\rho^{5/2}$, the exchange contribution comes from the loop density with exchange degeneracies p equal to 2 and from the expansion of κ around its low-degeneracy value κ_D in the contribution from interaction bonds for loops with $p=1$.

Second, our basic formula for the derivation of the free energy reduces the problem to the identification of partial derivatives with respect to the particle density in the expression of the diagrammatic expansion of the density around its MB value for an ideal gas. In the other Mayer-diagram method [2–4], the free energy is related to the integral of the internal energy for a coupling parameter g when g varies from 0 to 1. This integral is expressed in terms of the diagrammatic expansion of the particle correlations (times a Coulomb interaction). The difference in the starting formula has three consequences. First, the diagrams to be considered in the diagrammatic expansion of ρ are less numerous than in the expansion of the correlation. Second, the identification of partial derivatives proves to be more elementary than the subtle integration over the coupling parameter g and the devices needed to obtain an explicit result only in terms of matrix elements at the considered value of the coupling, namely, $g=1$. Third, from a practical point of view, the necessity of collecting various contributions as the sum of the different terms in the derivative of a product of functions is a good guide to avoid numerical mistakes. Indeed, the global coefficient 1/2 that is missing in Eqs. (4.2) and (4.3) of Ref. [3] and which comes from a symmetry factor must be taken into account in the analogous part of our calculation in order to recognize a sum of classical ''bridge'' contributions as the derivative of a bridge function times a function of densities and κ_D. Nevertheless, we stress that the existence of the two methods is a good means for checking analytical results derived from rather long procedures.

III. SCHEME FOR LOW-DENSITY EXPANSIONS

A. Thermodynamic formula for the free energy

In this subsection we derive an integral thermodynamic relation between the free-energy density and the densities. This relation provides a starting point for the calculation of the free energy that is different from the procedure used in Refs. [2–4]. We start from the relation

$$\left.\frac{\partial(\beta f)}{\partial \rho_\alpha}\right|_{\beta,\{\rho_\gamma\}\gamma\neq\alpha,B_0} = \beta\mu_\alpha . \tag{27}$$

Let $\rho_\alpha^{\mathrm{id}\star,\mathrm{MB}}(\beta,\{\mu_\alpha\},B_0)$ be the density of particles of species α in an ideal gas $\mathcal{S}^{\mathrm{id}\star}$ in the MB approximation at the same inverse temperature β and with the same chemical potentials μ_α. According to the well-known expression of $\rho_\alpha^{\mathrm{id}\star,\mathrm{MB}}$, which will be rederived in the following:

$$\beta\mu_\alpha = \ln\left((2\pi\lambda_\alpha^2)^{3/2}\rho_\alpha \frac{\sinh u_{S\alpha}}{\sinh[(2S_\alpha+1)u_{S\alpha}]} \frac{\sinh u_{C\alpha}}{u_{C\alpha}} \right) - \ln\left(\frac{\rho_\alpha}{\rho_\alpha^{\mathrm{id}\star,\mathrm{MB}}} \right). \tag{28}$$

In Eq. (28) the density ρ_α of the interacting system has been artificially introduced.

Now let us consider another ideal gas $\mathcal{S}^{\mathrm{id}}$ in the MB approximation at the same inverse temperature β and with the

same densities $\{\rho_\alpha\}$. Inspection of the expression (11) shows one that the first term in the right-hand side of Eq. (28) is just equal to $\partial(\beta f_{\rm id}^{\rm MB})/\partial\rho_\alpha$. Subsequently,

$$\beta f(\beta,\{\rho_\gamma\},B_0)=\beta f_{\rm id}^{\rm MB}(\beta,\{\rho_\gamma\},B_0)-\int \ln\left(\frac{\rho_\alpha}{\rho_\alpha^{\rm id\star,MB}}\right), \tag{29}$$

where $\int\ln(\rho_\alpha/\rho_\alpha^{\rm id\star,MB})$ denotes the primitive of $\ln(\rho_\alpha/\rho_\alpha^{\rm id\star,MB})=g_\alpha(\beta,\{\rho_\gamma\},B_0)$ that reduces to the ideal-gas exchange terms when there is no interaction. The derivation of the low-density expansion of the free energy is thus replaced by the calculation of the low-density expansion of ρ_α around its value for an ideal gas $\mathcal{S}^{\rm id\star}$ in the MB approximation and with the same chemical potentials $\{\mu_\gamma\}$.

B. Basic relations valid at any density

The low-density expansions will be derived from the three following equations. First, the relation between the particle density ρ_α and the loop densities $\rho_{\alpha,p}(\mathbf{X}_p)$ of species α with various exchange degeneracies p reads

$$\rho_\alpha=\sum_{p=1}^{\infty} p\int D(\mathbf{X}_p)\rho_{\alpha,p}(\mathbf{X}_p). \tag{30}$$

Contrarily to what was done in Paper I, in the present paper we add a subscript p to the loop-shape variable $\mathbf{X}$ in order to keep track of the exchange degeneracy.

Second, the representation of the loop density $\rho(\mathcal{L})=\rho_{\alpha,p}(\mathbf{X}_p)$ in terms of diagrams where each internal point has a weight $\rho(\mathcal{L})$ and where bonds depend on the loop density only through $\kappa\equiv4\pi\beta\Sigma_{\alpha,p}(pe_\alpha)^2\int D(\mathbf{X}_p)\rho_{\alpha,p}(\mathbf{X}_p)$ takes the form

$$\rho_{\alpha,p}(\mathbf{X}_p)=z_{\alpha,p}(\mathbf{X}_p)\exp[J(\mathcal{L}_a)], \tag{31}$$

where

$$z_{\alpha,p}(\mathbf{X}_p)=\left(\frac{e^{\beta\mu_\alpha}}{(2\pi\lambda_\alpha^2)^{3/2}}\right)^p\frac{\eta_\alpha^{p-1}}{p}\times\frac{\sinh([2S_\alpha+1]pu_{S\alpha})}{\sinh(pu_{S\alpha})}\times e^{(ie/2\hbar c)\mathbf{B}_0\cdot\int_0^p\mathbf{X}_p(\tau)\wedge d\mathbf{X}_p(\tau)}e^{-\beta E_\beta^{\rm int}(\mathbf{X}_p)}, \tag{32}$$

with the definitions given in Sec. III C of Paper I. $J(\mathcal{L})$ arises from the Mayer diagrams in the presence of interactions and involves powers of $\rho(\mathcal{L})^n\kappa^m$ with $2n+m>0$. When $p=1$, $z_{\alpha,1}(\boldsymbol{\xi})$ involves no interaction. Thus, in the absence of interactions, according to Eq. (31), the loop density for $p=1$ in the quantum ideal gas with the same chemical potentials $\{\mu_\gamma\}$ is $\rho_{\alpha,1}^{\rm id\star}(\boldsymbol{\xi})=z_{\alpha,1}(\boldsymbol{\xi})$. Since the value $\rho_\alpha^{\rm id\star,MB}$ of the ideal-gas density in the MB approximation reduces to the contribution from the loops with an exchange degeneracy p equal to 1, according to Eq. (30), it reads

$$\rho_\alpha^{\rm id\star,MB}=\int D(\boldsymbol{\xi})z_{\alpha,1}(\boldsymbol{\xi}). \tag{33}$$

Third, the expression of κ itself has a low-density expansion, because, as shown in Ref. [12],

$$\kappa=\left\{\kappa_D^2+4\pi\beta\sum_\alpha e_\alpha^2\int d\mathbf{r}\rho_{\alpha\alpha}^{(2)\,T}|_{\rm exch}(\mathbf{r})\right\}^{1/2}, \tag{34}$$

where $\kappa_D^2\equiv4\pi\beta\Sigma_\alpha e_\alpha^2\rho_\alpha$ is the squared inverse Debye length. In Eq. (34)

$$\rho_{\alpha\alpha}^{(2)\,T}|_{\rm exch}(\mathbf{r})=\sum_{p=2}^{\infty}p\int D(\mathbf{X}_p)\sum_{l^*=2}^{p}\delta(\mathbf{x}_{l^*}-\mathbf{x}_1-\mathbf{r})\rho_{\alpha,p}(\mathbf{X}_p) \tag{35}$$

is the part of the particle-particle distribution function arising from the configurations where the two particles separated by the distance r belong to the same exchange cyclic permutation.

C. First simple results at low density

1. Reference quantities

In a low degeneracy and weak Coulomb coupling regime, ρ_α is of order $\rho_\alpha^{\rm id,MB}$, with $\rho_\alpha\sim a^{-3}$, $\rho_\alpha^{\rm id,MB}\sim\exp(\beta\mu_\alpha)/(2\pi\lambda_\alpha^2)^{3/2}$, and $\lambda\ll a$. Thus the small dimensionless parameter that measures the order in the expansion is $\exp(\beta\mu_\alpha)\sim(\lambda/a)^3$ and $z_{\alpha,p}(\mathbf{X}_p)$ is of order ρ_α^p, $z_{\alpha,p}(\mathbf{X}_p)=O(\rho^p)$. $O(\rho^p)$ denotes a term which is of order ρ^p, namely, whose density expansion starts at order ρ^p. According to Eqs. (31) and (32), the term in $\rho_{\alpha,p}(\mathbf{X}_p)$ that is of lowest order in density ρ_α coincides with $z_{\alpha,p}(\mathbf{X}_p)$ and

$$\rho_{\alpha,p}(\mathbf{X}_p)=O(\rho_\alpha^p). \tag{36}$$

Henceforth, the part (35) of the correlation that comes from exchange effects involves has a low-density expansion which starts at order ρ^2. (Indeed, exchange involves at least two particles.) Thus, according to Eq. (34),

$$\kappa=\kappa_D+O(\rho^{3/2}), \tag{37}$$

where κ_D is of order $\rho^{1/2}$ by definition. Another consequence of Eq. (36) is that, according to Eq. (30), $\rho_\alpha=\int D(\boldsymbol{\xi})z_{\alpha,1}(\boldsymbol{\xi})+o(\rho)$, where $o(\rho)$ denotes a term which is of greater order than ρ. Subsequently, according to Eq. (33), we retrieve that

$$\rho_\alpha=\rho_\alpha^{\rm id\star,MB}+o(\rho). \tag{38}$$

We notice that if the neutrality relation

$$\sum_\alpha e_\alpha\rho_\alpha=0 \tag{39}$$

were not satisfied, then, according to the explicit low-density expressions derived below in Sec. IV, the expression of ρ_α calculated from the Mayer diagrams would not be equal to $\rho_\alpha^{\rm id\star,MB}$ at first order in density ρ. Since this coincidence must happen in the weak Coulomb coupling and low-degeneracy limit, the Mayer diagrams must be calculated with the constraint (39). (Subsequently, the ideal gas $\mathcal{S}^{\rm id}$ defined in Sec. III A also satisfies the neutrality relation.)

2. Explicit results for the ideal MB gas

The explicit analytical results in the case of the ideal gas that will be used are the following. If $\mathbf{B}_0=\mathbf{0}$, $u_{S\alpha}=0$, and $z_{\alpha,1}(\boldsymbol{\xi})$ is independent from $\boldsymbol{\xi}$. Thus, according to Eqs. (32) and (33), $\rho_\alpha^{\mathrm{id}\star,\mathrm{MB}}$ reduces to

$$\rho_\alpha^{\mathrm{id}\star,\mathrm{MB}}(\mathbf{B}_0=\mathbf{0})=(2S_\alpha+1)\frac{e^{\beta\mu_\alpha}}{(2\pi\lambda_\alpha^2)^{3/2}}. \tag{40}$$

When $\mathbf{B}_0\neq\mathbf{0}$, $u_{S\alpha}\neq 0$, and $z_{\alpha,1}(\boldsymbol{\xi})$ involves a phase factor arising from the magnetic field and the spin degeneracy factor $2S_\alpha+1$ is changed into a paramagnetic expression. Thus

$$\rho_\alpha^{\mathrm{id}\star,\mathrm{MB}}(\mathbf{B}_0\neq\mathbf{0})=\frac{\sinh([2S_\alpha+1]u_{S\alpha})}{\sinh u_{S\alpha}}\frac{u_{C\alpha}}{\sinh u_{C\alpha}}\frac{e^{\beta\mu_\alpha}}{(2\pi\lambda_\alpha^2)^{3/2}}. \tag{41}$$

In Eq. (41) the well-known diamagnetic contribution to $\rho_\alpha^{\mathrm{id}\star,\mathrm{MB}}$ is derived by using the Feynman-Kac-Itô formula,

$$\int D(\boldsymbol{\xi})\exp\left[\frac{ie\lambda_\alpha^2}{2\hbar c}\mathbf{B}_0\cdot\int_0^1\boldsymbol{\xi}(s)\wedge d\boldsymbol{\xi}(s)\right]$$
$$=(2\pi\lambda_\alpha^2)^{3/2}\langle\mathbf{r}|\exp[-\beta h_{B_0,\alpha}^{(0)}|\mathbf{r}\rangle=\frac{u_{C\alpha}}{\sinh u_{C\alpha}}, \tag{42}$$

where $h_{B_0,\alpha}^{(0)}$ is the position-dependent part of the Hamiltonian of one particle of species α in the magnetic field $\mathbf{B}_0$,

$$h_{B_0,\alpha}^{(0)}=\frac{1}{2m_\alpha}\left[\mathbf{p}-\frac{e_\alpha}{2c}\mathbf{B}_0\wedge\mathbf{r}\right]^2. \tag{43}$$

(See Appendix A for further details.) For convenience's sake, in the following we shall use the normalized measure $D_{\mathbf{B}_0}(\boldsymbol{\xi})$, such that $\int D_{\mathbf{B}_0}(\boldsymbol{\xi})=1$. According to Eq. (42)

$$D_{\mathbf{B}_0}(\boldsymbol{\xi})\equiv\frac{\sinh u_{C\alpha}}{u_{C\alpha}}D(\boldsymbol{\xi})e^{(ie_\alpha\lambda_\alpha^2/2\hbar c)\mathbf{B}_0\cdot\int_0^1\boldsymbol{\xi}(s)\wedge d\boldsymbol{\xi}}. \tag{44}$$

Contrarily to the case of free motion, the covariance in the presence of $\mathbf{B}_0$,

$$\mathrm{cov}_{\mu\nu}^\alpha(s,s';\mathbf{B}_0)\equiv\int D_{\mathbf{B}_0}(\boldsymbol{\xi})[\boldsymbol{\xi}(s)]_\mu[\boldsymbol{\xi}(s')]_\nu$$
$$=\mathrm{cov}_{\mu\nu}(s,s';u_{C\alpha}), \tag{45}$$

depends on the considered species when μ and ν are indices of coordinates in the plane perpendicular to the direction of $\mathbf{B}_0$, because the coupling with the magnetic field depends on e_α/m_α. Properties derived from symmetry arguments have been displayed in Sec. V C of Paper I. The values of the various nonvanishing covariances are calculated by three different methods in Appendix A with the following results. In the z-axis direction, the motion is still free, and

$$\mathrm{cov}_{zz}(s,s')=\inf(s,s')[1-\sup(s,s')]. \tag{46}$$

On the contrary

$$\mathrm{cov}_{xx}^\alpha(s,s';\mathbf{B}_0)=\mathrm{cov}_{yy}^\alpha(s,s';\mathbf{B}_0)$$
$$=\frac{1}{u_{C\alpha}\sinh u_{C\alpha}}\cosh[(s-s')u_{C\alpha}]$$
$$\times\sinh[\inf(s,s')u_{C\alpha}]$$
$$\times\sinh\{[1-\sup(s,s')]u_{C\alpha}\} \tag{47}$$

while

$$\mathrm{cov}_{xy}^\alpha(s,s';\mathbf{B}_0)=-\mathrm{cov}_{yx}^\alpha(s,s';\mathbf{B}_0)$$
$$=-i\,\mathrm{sgn}(s-s')\frac{1}{u_{C\alpha}\sinh u_{C\alpha}}$$
$$\times\sinh[|s-s'|u_{C\alpha}]\sinh[\inf(s,s')u_{C\alpha}]$$
$$\times\sinh\{[1-\sup(s,s')]u_{C\alpha}\}. \tag{48}$$

[In fact Eq. (48) will not be used in the following.] In the limit of weak coupling with the magnetic field, Eqs. (47) does tend to the free motion expression (46) and Eq. (48) becomes $\mathrm{cov}_{xy}^\alpha(s,s';\mathbf{B}_0=0)=0$, as it should.

According to Eqs. (32), (38), and (41), the first term in the ρ expansion of $\rho_{\alpha,p}^{\{p\}}(\mathbf{X}_p)$ reads

$$\rho_{\alpha,p}^{\{p\}}(\mathbf{X}_p)=z_{\alpha,p}^{\{p\}}(\mathbf{X}_p) \tag{49}$$

with

$$z_{\alpha,p}^{\{p\}}(\mathbf{X}_p)=\rho_\alpha^p\frac{\eta_\alpha^{p-1}}{p}\frac{\sinh(p[2S_\alpha+1]u_{S\alpha})}{\{\sinh([2S_\alpha+1]u_{S\alpha})\}^p}\frac{(\sinh u_{S\alpha})^p}{\sinh(pu_{S\alpha})}$$
$$\times\left(\frac{\sinh u_{C\alpha}}{u_{C\alpha}}\right)^p\exp\left[\frac{ie}{2\hbar c}\mathbf{B}_0\cdot\int_0^p\mathbf{X}_p(\tau)\right.$$
$$\left.\wedge d\mathbf{X}_p(\tau)\right]\exp[-\beta E_\beta^{\mathrm{int}}(\mathbf{X}_p)]. \tag{50}$$

D. Effects of exchange and interactions

1. Double-stepped scheme

The low-density expansions are performed in two steps. The first step will be called loop-density expansion and denoted by ρ_{loop} expansion. The integrals corresponding to the Mayer diagrams with weight $\rho(\mathcal{L})$ are expanded in terms of powers of $\rho(\mathcal{L})$ and κ. Indeed, κ is the only length scale through which the Mayer bonds depend on the densities and κ vanishes with the densities. The order in loop density will be denoted by ρ_{loop}^n and we use the convention that each length scale κ defined in Eq. (34) gives a contribution that starts at order $\rho_{\mathrm{loop}}^{1/2}$. Thus n may take half-integer values. The diagrammatic survey and the scaling analysis performed in Sec. III will show that

$$\rho(\mathcal{L})=z_{\alpha,p}(\mathbf{X}_p;\beta\mu_\alpha)\exp[J_{\mathrm{loop}}^{\{1/2\}}(pe_\alpha)$$
$$+J_{\alpha,\mathrm{loop}}^{\{1\}}(\mathbf{X}_p)+J_{\alpha,\mathrm{loop}}^{\{3/2\}}(\mathbf{X}_p)], \tag{51}$$

where $J_{\alpha,\mathrm{loop}}^{\{n\}}(\mathbf{X}_p)$ is of order ρ_{loop}^n. In the following, the quantities denoted by $f_{\mathrm{loop}}^{\{n\}}(\mathbf{X}_p)$ will always refer to quanti-

ties that are exactly of order $\rho_{\rm loop}^n$ and which are not integrated over the shape $\mathbf{X}_p$ of the loop.

In the second step, we turn to the expansions in terms of κ_D and of the quantum particle densities ρ_α. For that purpose, $\rho_\alpha/\rho_\alpha^{\rm id\star,MB}$ and κ are expanded around their low-degeneracy values 1 and κ_D, respectively, in terms of the ρ_γ's; simultaneously the loop densities, namely the $z_{\alpha,p}(\mathbf{X}_p)$'s and the $J_{\alpha,\rm loop}^{\{n\}}(\mathbf{X}_p)$'s, are expanded in powers of the particle densities ρ_γ. When the term of order $[\exp(\beta\mu_\alpha)]^p$ in ρ is known, we can calculate $\rho_{\alpha,p'}(\mathbf{X}_{p'})$, with $1\leqslant p'\leqslant p+1$ up to order ρ^{p+1}. Indeed, $\rho_{\alpha,p'}(\mathbf{X}_p)$ contains the multiplicative factor $z_{\alpha,p'}(\mathbf{X}_p)$ that starts at order $\rho^{p'}$ by a term $\rho_{\alpha,p'}^{\{p'\}}(\mathbf{X}_{p'})$ that is exactly known and the $J(\mathbf{X}_{p'})$ have to be calculated only up to order $\rho^{p+1-p'}$, which is lower than or equal to ρ^p. As shown below, we get

$$\sum_{n=1}^{\infty} J_{\alpha,\rm loop}^{\{n/2\}}(\mathbf{X}_p)=\sum_{n=1}^{\infty} J_\alpha^{\{n/2\}}(\mathbf{X}_p), \tag{52}$$

where $f^{\{n/2\}}$ is a term of order $\rho^{n/2}$, and

$$z_{\alpha,p}(\mathbf{X}_p;\beta\mu_\alpha)=\rho_{\alpha,p}^{\{p\}}(\mathbf{X}_p)+\sum_{n=0}^{\infty} z_{\alpha,p}^{\{p+1+n/2\}}(\mathbf{X}_p). \tag{53}$$

$\rho_{\alpha,p}^{\{p\}}(\mathbf{X}_p)$ is given by Eq. (50). In Eq. (53) the jump in powers from ρ^p to ρ^{p+1} is determined by the fact that $\rho_\alpha=\rho_\alpha^{\rm id\star,MB}+O(\rho^2)$, as proved just below.

We notice that the summation over the species indices and the exchange degeneracy p do not increase the order in $\rho(\mathcal{L})$ and κ of a given diagram. These summations may at most cancel some contributions, in which case they may increase the order in the loop density (as already discussed in Sec. III B of Ref. [3]).

2. Useful property

A property that simplifies explicit calculations of ρ expansions is the following. As shown in Sec. V B, the only contribution of order $\rho_{\rm loop}^{1/2}$ in $J(\mathcal{L}_a)$ reads

$$J_{\alpha,\rm loop}^{\{1/2\}}(pe_\alpha)=\frac{1}{2}\beta(pe_\alpha)^2\kappa \tag{54}$$

and is independent from $\mathbf{X}_a$. According to Eq. (37),

$$J^{\{1/2\}}(pe_\alpha)=\frac{1}{2}\beta(pe_\alpha)^2\kappa_D, \tag{55}$$

which is the value of $J_{\alpha,\rm loop}^{\{1/2\}}(pe_\alpha)$ when κ is replaced by κ_D.

3. Starting point of the recurrence scheme

The recurrence scheme may be started as follows. The fact that $J^{\{1/2\}}(e_\alpha)$ is independent from $\boldsymbol{\xi}$ implies that

$$D(\boldsymbol{\xi})\rho_{\alpha,1}(\boldsymbol{\xi})=\rho_\alpha D_{\mathbf{B}_0}(\boldsymbol{\xi})+O(\rho^2). \tag{56}$$

Indeed, we first notice that Eqs. (32) and (41) and the normalization (44) imply the relation

$$\frac{D(\boldsymbol{\xi})z_{\alpha,1}(\boldsymbol{\xi})}{\rho_\alpha^{\rm id\star,MB}}=D_{\mathbf{B}_0}(\boldsymbol{\xi}). \tag{57}$$

Thus, according to Eq. (30), the contribution to $\rho_\alpha/\rho_\alpha^{\rm id\star,MB}$ from the loops with $p=1$, for which $E_\beta^{\rm int}(\boldsymbol{\xi})=0$, is the integral of

$$\frac{D(\boldsymbol{\xi})\rho_{\alpha,1}(\boldsymbol{\xi})}{\rho_\alpha^{\rm id\star,MB}}=e^{J^{\{1/2\}}(e_\alpha)}D_{\mathbf{B}_0}(\boldsymbol{\xi})e^{[J_\alpha^{\{1\}}(\boldsymbol{\xi})+J_\alpha^{\{3/2\}}(\boldsymbol{\xi})]}+O(\rho^2), \tag{58}$$

where we have used Eqs. (36) and (37). Since the low-density expansion of $\rho_{\alpha,p}(\mathbf{X})$ starts at order ρ_α^p, Eqs. (30) and (58) lead to

$$\rho_\alpha=\rho_\alpha^{\rm id\star,MB}e^{J^{\{1/2\}}(e_\alpha)}+O(\rho^2). \tag{59}$$

Equation (56) is derived from Eqs. (58) and (59), while Eqs. (55) and (59) imply that

$$\rho_\alpha=\rho_\alpha^{\rm id\star,MB}\left[1+\frac{\beta}{2}e_\alpha^2\kappa_D\right]+O(\rho^2). \tag{60}$$

On one hand, $\rho_{\alpha,2}(\mathbf{X}_2)$ may be calculated readily up to order $\rho^{5/2}$ according to Eqs. (50) and (51) and to the simple form of $J^{\{1/2\}}(pe_\alpha)$, and we get

$$\rho_{\alpha,2}(\mathbf{X}_2)=\rho_\alpha^2\frac{1}{2}[1+\beta e_\alpha^2\kappa_D]E_{\rm exch,\alpha}^*(\mathbf{X}_2)+O(\rho^3), \tag{61}$$

with

$$E_{\rm exch,\alpha}^*(\mathbf{X}_2)\equiv\eta_\alpha\frac{\tanh u_{S\alpha}}{\tanh([2S_\alpha+1]u_{S\alpha})}\left(\frac{\sinh u_{C\alpha}}{u_{C\alpha}}\right)^2 \times e^{(ie/2\hbar c)\mathbf{B}_0\cdot\int_0^2\mathbf{X}_2(\tau)\wedge d\mathbf{X}_2(\tau)}e^{-\beta E_\beta^{\rm int}(\mathbf{X}_2)}. \tag{62}$$

Indeed, by combining of Eqs. (50), (51), and (59), we get

$$\frac{\rho_{\alpha,2}(\mathbf{X}_2)}{\rho_\alpha^{\rm id\star,MB}}=\frac{1}{2}\rho_\alpha e^{J^{\{1/2\}}(2e_\alpha)-J^{\{1/2\}}(e_\alpha)}E_{\rm exch,\alpha}^*(\mathbf{X}_2)+O(\rho^2). \tag{63}$$

According to Eqs. (30), (55) and (63), the contribution from loops with $p=2$ to $\rho_\alpha/\rho_\alpha^{\rm id\star,MB}$ reads

$$\frac{2\int D(\mathbf{X}_2)\rho_{\alpha,2}(\mathbf{X}_2)}{\rho_\alpha^{\rm id\star,MB}}=\rho_\alpha\left[1+\frac{3}{2}\beta e_\alpha^2\kappa_D\right]E_\alpha^*+O(\rho^2). \tag{64}$$

We have set $E_\alpha^*\equiv\int D(\mathbf{X}_2)E_{\rm exch,\alpha}^*(\mathbf{X}_2)$ where $\mathbf{r}$ is the relative position of the particles in the loop. E_α^* involves

$$\int D(\mathbf{X}_2)e^{(ie_\alpha/2\hbar c)\mathbf{B}_0\cdot\int_0^2\mathbf{X}_2(\tau)\wedge d\mathbf{X}_2(\tau)}e^{-\beta E_\beta^{\rm int}(\mathbf{X}_2)}$$

$$=\int d\mathbf{r}(2\pi\lambda_\alpha^2)^3\langle\mathbf{r},\mathbf{0}|e^{-\beta H_{\alpha\alpha}}|\mathbf{0},\mathbf{r}\rangle, \tag{65}$$

where $H_{\alpha\gamma}$ is the two-body Hamiltonian without the spin contribution defined in Eq. (19). In Eq. (65) we could choose $\mathbf{r}_1=\mathbf{0}$ and $\mathbf{r}_2=\mathbf{r}$, because the left-hand side of Eq. (65) is invariant under translation of the loop position. The right-hand side of Eq. (65) may be reexpressed in terms of a one-body Hamiltonian thanks to Eq. (21).

On the other hand, $\rho_{\alpha,1}(\boldsymbol{\xi})$ may be calculated up to order $\rho^{5/2}$. Indeed, $J^{\{1\}}_{\mathrm{loop},\alpha}(\boldsymbol{\xi})+J^{\{3/2\}}_{\alpha}(\boldsymbol{\xi})$ can be expressed explicitly in terms of the densities ρ_γ by using Eq. (56) and the equation

$$\kappa=\kappa_D+2\pi\beta\sum_\alpha e_\alpha^2\frac{\rho_\alpha^2}{\kappa_D}E^*_\alpha+O(\rho^2). \tag{66}$$

Equation (66) is derived from Eqs. (34) and (61).

E. Formal results up to order $\rho^{5/2}$

1. Free energy

A straightforward low-density expansion implies that, after expansion of the exponential in the generalization of Eqs. (58) and (63), an integration over the loop shapes gives

$$\frac{\rho_\alpha}{\rho_\alpha^{\mathrm{id}\star,\mathrm{MB}}}=\sum_{p=1}^{\infty}\sum_{n=0}^{\infty}A_p^{\{p-1+n/2\}}(e_\alpha), \tag{67}$$

where the index p refers to the integrals $\int D(\mathbf{X}_p)\ldots$ where $A_p^{\{q\}}$ comes from; for instance, $A_1^{\{1/2\}}=J^{\{1/2\}}(e_\alpha)$, $A_1^{\{1\}}=\int D_{\mathbf{B}_0}(\boldsymbol{\xi})J^{(1)}(\boldsymbol{\xi})$, $A_1^{\{3/2\}}=\int D_{\mathbf{B}_0}(\boldsymbol{\xi})J_\alpha^{(3/2)}(\boldsymbol{\xi})$, $A_2^{\{1\}}=\rho_\alpha E^*_\alpha$, and $A_2^{\{3/2\}}=[J^{\{1/2\}}(2e_\alpha)-J^{\{1/2\}}(e_\alpha)]\rho_\alpha E^*_\alpha$ where E^*_α is defined in Eq. (64). A reexponentiation of the expansion (67) leads to

$$\ln\left(\frac{\rho_\alpha}{\rho_\alpha^{\mathrm{id}\star,\mathrm{MB}}}\right)=\sum_{n=1}^{\infty}B_\alpha^{\{n/2\}}, \tag{68}$$

where $B_\alpha^{\{1/2\}}=A_1^{\{1/2\}}$, $B_\alpha^{\{1\}}=A_1^{\{1\}}+A_2^{\{1\}}$, and $B_\alpha^{\{3/2\}}=A_1^{\{3/2\}}+A_2^{\{3/2\}}-A_1^{\{1/2\}}A_2^{\{1\}}$. More explicitly,

$$B_\alpha^{\{1/2\}}=J^{\{1/2\}}(e_\alpha), \tag{69a}$$

$$B_\alpha^{\{1\}}=\int D_{\mathbf{B}_0}(\boldsymbol{\xi})J^{\{1\}}(\boldsymbol{\xi})+\rho_\alpha E^*_\alpha, \tag{69b}$$

$$B_\alpha^{\{3/2\}}=\int D_{\mathbf{B}_0}(\boldsymbol{\xi})J_\alpha^{\{3/2\}}(\boldsymbol{\xi})+\rho_\alpha[J^{\{1/2\}}(2e_\alpha)-2J^{\{1/2\}}(e_\alpha)]E^*_\alpha. \tag{69c}$$

As a consequence, in order to calculate the free-energy density up to order $\rho^{5/2}$, we have to compute the loop density only up to order $\rho^{3/2}$ for $p=1$, while the contribution from loops with $p=2$ has already been taken into account up to order $\rho^{3/2}$. The scaling analysis of diagrams in terms of the loop density is presented in Sec. IV and the low-density expansions themselves are given in Sec. V.

We already notice that the terms of order $\rho^{1/2}$ and ρ contain obvious partial derivatives with respect to ρ_α. According to Eqs. (69a) and (55) the term

$$B_\alpha^{\{1/2\}}=\frac{1}{2}\beta e_\alpha^2\kappa_D=\frac{\partial}{\partial\rho_\alpha}\left(\frac{\kappa_D^3}{12\pi}\right) \tag{70}$$

is the opposite of the derivative of the Debye free energy. Moreover, at order ρ the contribution to $\ln(\rho_\alpha/\rho_\alpha^{\mathrm{id}\star,\mathrm{MB}})$ from exchange effects given by Eq. (69b) is also a partial derivative by itself,

$$B^{\{1\}}_{\mathrm{exch},\alpha}=\rho_\alpha E^*_\alpha=\frac{\partial}{\partial\rho_\alpha}\left(\frac{1}{2}\sum_\gamma\rho_\gamma^2E^*_\gamma\right). \tag{71}$$

2. Loop density

The ρ expansion of the loop densities $\rho_{\alpha,p}(\mathbf{X}_p)$ will be useful in the discussion of Paper III. By inversion of Eq. (68), $\rho_\alpha^{\mathrm{id}\star,\mathrm{MB}}$ may be expressed in terms of the ρ_γ's, and insertion of this expansion in Eqs. (58) and (63) leads to

$$\begin{aligned}D(\boldsymbol{\xi})\rho_{\alpha,1}(\boldsymbol{\xi})=\rho_\alpha D_{\mathbf{B}_0}(\boldsymbol{\xi})\Big\{&1+J_\alpha^{(1)}(\boldsymbol{\xi})-\int D_{\mathbf{B}_0}(\boldsymbol{\xi}')J_\alpha^{(1)}(\boldsymbol{\xi}')\\&-\rho_\alpha E^*_\alpha+J_\alpha^{(3/2)}(\boldsymbol{\xi})-\int D_{\mathbf{B}_0}(\boldsymbol{\xi}')J_\alpha^{(3/2)}(\boldsymbol{\xi}')\\&-\rho_\alpha\beta e_\alpha^2\kappa_DE^*_\alpha\Big\}+O(\rho^3)\end{aligned} \tag{72}$$

while $\rho_{\alpha,2}(\mathbf{X}_2)$ is given up to order $\rho^{5/2}$ by Eq. (61).

IV. SCALING ANALYSIS IN LOOP DENSITY

A. Formal scale decomposition

Since the bonds introduced in Paper I depend on the density, a scale decomposition is introduced in order to determine to which orders in $\rho(\mathcal{L})$ and κ a given diagram contributes. Similar principles are used in the decomposition chosen in Sec. V D of Ref. [2].

We first notice that in the simple diagrams which we will have to consider, all terms that involve an odd number of derivatives with respect to $\mathbf{r}$ disappear after integration over $\mathbf{X}$. Indeed, every such term originates from a large-distance expansion of a function f, and it takes the form $g(\mathbf{X})[\mathbf{X}]_{\mu_1}\cdots[\mathbf{X}]_{\mu_n}\partial_{\mu_1,\ldots,\mu_n}f(r)$ where $g(\mathbf{X})$ is invariant under rotations of $\mathbf{X}$ (while $\partial_{\mu_1\mu_2}$ denotes a derivative with respect to the components $[\mathbf{r}]_{\mu_1}$ and $[\mathbf{r}]_{\mu_2}$). Since the weight $D(\mathbf{X})\rho(\mathbf{X})$ is invariant under inversion of $\mathbf{X}$ and is short ranged with respect to the extent of $|\mathbf{X}|$, the integration over $\mathbf{X}$ may be performed first and terms with an odd number of components of $\mathbf{X}$ are canceled. Thus terms with an odd number of derivatives vanish in the absence as well as in the presence of the magnetic field.

1. Bonds entirely scaled by κ

For bonds F, such as F^{cc} and F^{cm}, that are entirely scaled by κ, the scale decomposition takes the very simple form

$$F(\mathbf{r};\kappa)=\sum_{n=0}^{\infty}F^{\langle n\rangle}(\mathbf{r};\kappa) \tag{73}$$

with

$$F^{\langle n\rangle}(\mathbf{r};\kappa)\equiv\kappa^n\widetilde{F}^{\langle n\rangle}(\mathbf{r}/\kappa). \tag{74}$$

The dependence on the charges e_i and e_j will not be mentioned in the argument of the bonds. In the case of F^{cc}, only one power of κ is involved,

$$F^{cc}(\mathbf{r}=\mathbf{x}/\kappa;\kappa)=\kappa\widetilde{F^{cc}}(\mathbf{x}), \tag{75}$$

with $\widetilde{F^{cc}}(\mathbf{x})=-\beta_{ij}\exp(-x)/x$ and $\beta_{ij}\equiv\beta e_i e_j$.

In the case of F^{cm}, there appears a series in powers of κ. This series is derived from the Taylor expansion $E_T[F^{cm}]$ of F^{cm} at large distances, though it does not coincide with this expansion, as explained in the following. A term in the Taylor expansion of $F^{cm}(\mathbf{r},\mathbf{X}_j)$ is denoted by

$$T^{cm\langle n\rangle}(\mathbf{r},\mathbf{X}_j)=\beta_{ij}\int_0^p d\tau[\mathbf{X}_j(\tau)\cdot\boldsymbol{\nabla}]^{n-1}\phi(\mathbf{r};\kappa), \tag{76}$$

with $n\geqslant 2$ and

$$\phi(r;\kappa)=\frac{e^{-\kappa r}}{r}. \tag{77}$$

$T^{cm\langle n\rangle}$ is entirely scaled by the nth power of κ and

$$E_T[F^{cm}(\mathbf{r}=\mathbf{x}/\kappa,\mathbf{X}_j;\kappa)]=\sum_{n=2}^{\infty}\kappa^n\widetilde{T}^{cm\langle n\rangle}(\mathbf{x},\mathbf{X}_j). \tag{78}$$

For instance,

$$\widetilde{T}^{cm\langle 2\rangle}(\mathbf{x},\mathbf{X}_j)\equiv\beta_{ij}\int_0^p d\tau\,\mathbf{X}_j(\tau)\cdot\boldsymbol{\nabla}_{\mathbf{x}}\left(\frac{e^{-x}}{x}\right)$$
$$=\beta_{ij}\int_0^p d\tau\,\mathbf{X}_j(\tau)\cdot\hat{\mathbf{x}}\left[-\frac{e^{-x}}{x}-\frac{e^{-x}}{x^2}\right], \tag{79}$$

where $\hat{\mathbf{x}}$ is the unit vector $\hat{\mathbf{x}}\equiv\mathbf{x}/x$. More generally the term $T^{cm\langle n\rangle}$ is a sum of n contributions each of which decays as $\exp(-A_p\kappa r)/r^p$ where A_p is a constant and $1\leqslant p\leqslant n$. In the Taylor expansion the $T^{cm\langle n\rangle}$'s with $n\geqslant 3$ are not integrable at the origin. Henceforth, the term $\widetilde{F}^{cm\langle n\rangle}(\mathbf{x},\mathbf{X}_j)$ in the scale decomposition does not coincide with the term $T^{cm\langle n\rangle}$ in the Taylor expansion of F^{cm},

$$F^{cm}(\mathbf{r}=\mathbf{x}/\kappa,\mathbf{X}_j;\kappa)=\sum_{n=2}^{\infty}\kappa^n\widetilde{F}^{cm\langle n\rangle}(\mathbf{x},\mathbf{X}_j), \tag{80}$$

with $\widetilde{F}^{cm\langle n\rangle}\neq\widetilde{T}^{cm\langle n\rangle}$. The problems arising from the nonintegrability of $T^{cm\langle n\rangle}$ at short distances can be solved by a regularization procedure in Fourier space, but the latter is far beyond the scope and need of the present paper.

Indeed, in the following, we will use the property that, if $g(\mathbf{r})$ is a function invariant under rotations and is regular at the origin

$$\int d\mathbf{x}\,g(|\mathbf{x}|)\widetilde{F}^{cm\langle n\rangle}(\mathbf{x},\mathbf{X}_j)=\int d\mathbf{x}\,g(|\mathbf{x}|)\widetilde{T}^{cm\langle n\rangle}(\mathbf{x},\mathbf{X}_j). \tag{81}$$

Equation (81) holds though $\widetilde{F}^{cm\langle n\rangle}(\mathbf{x},\mathbf{X}_j)\neq\widetilde{T}^{cm\langle n\rangle}(\mathbf{x},\mathbf{X}_j)$ because $\int d\mathbf{x}\,g(|\mathbf{x}|)\widetilde{T}^{cm\langle n\rangle}(\mathbf{x},\mathbf{X}_j)$ is integrable at the origin. The reason relies on two facts: each $T^{cm\langle n\rangle}$ involves derivatives of order $n-1$ of the function $\phi(r;\kappa)$, and it is multiplied by functions that are invariant under rotations of $\mathbf{r}$. Let us consider the integral of g times any partial derivative of a function f, where both f and g are invariant under rotations. After integration over the orientation of $\mathbf{r}$, any partial derivative of f with an odd number of coordinates gives a vanishing contribution, whereas the derivative of order $2p$ gives a term which is proportional to $\Delta^p f$ times a tensor of rank $2p$. (This result can be easily derived in Fourier space.) For instance, a fundamental relation used in the following is

$$\int d\mathbf{r}\,g(|\mathbf{r}|)\partial_{\mu\nu}f(|\mathbf{r}|)=\delta_{\mu,\nu}\frac{1}{3}\int d\mathbf{r}\,g(|\mathbf{r}|)\Delta f(|\mathbf{r}|). \tag{82}$$

Moreover, ϕ obeys the equation

$$\Delta\phi-\kappa^2\phi=-4\pi\delta(\mathbf{r}) \tag{83}$$

and, subsequently, it can be shown by recurrence that $\Delta^p\phi$ is integrable at the origin. Thus $\int d\mathbf{x}\,g(|\mathbf{x}|)T^{cm\langle 2p\rangle}(\mathbf{x},\mathbf{X}_j)$ involves in fact only $g\Delta^p\phi$, which is integrable at the origin even in the presence of the magnetic field.

2. Bond F_R

In the case of the bond F_R three kinds of scale lengths are involved: the lengths $\beta|e_\alpha e_\gamma|$ that measure the coupling with the Coulomb potential and, for each species, the radius of the orbits in the first Landau level $l_{C\alpha}=\sqrt{2\hbar c/e_\alpha B_0}$ and the thermal de Broglie wavelength λ_α. Thus, after integration over $\mathbf{r}$, the corresponding truncated two-body density-matrix element depends on at most three kinds of dimensionless parameters $\beta e_\alpha e_\gamma/\sqrt{\lambda_\alpha\lambda_\gamma}$, $\lambda_\alpha^2/l_{C\alpha}^2=\beta\hbar\omega_{C\alpha}/2=u_{C\alpha}$, $\lambda_\gamma^2/l_{C\gamma}^2=u_{C\gamma}$, and $\lambda_\alpha/\lambda_\gamma=\sqrt{m_\gamma/m_\alpha}$. We notice that, when $e_\alpha/m_\alpha=e_\gamma/m_\gamma$, the motion equations can be decoupled in two independent equations for the motions of the center of mass and a relative particle, respectively; then $u_{C\alpha}=u_{C\gamma}$.

On the other hand, the large-distance Taylor expansion of F_R reads

$$F_R(\mathbf{r};\kappa)=\sum_{n=2}^{\infty}T_R^{\langle n\rangle}(\mathbf{r};\kappa). \tag{84}$$

The first term

$$T_R^{\langle 2\rangle}(\mathbf{r};\kappa)=\frac{1}{2}[F^{cc}(\mathbf{r};\kappa)]^2=\frac{\beta_{ij}^2}{2}\phi^2(r;\kappa) \tag{85}$$

is integrable at short distances, whereas the higher-order terms in the Taylor expansion are not. In the following we will consider separately $[F^{cc}]^2/2$, because $[F^{cc}]^2/2$ decays only as $1/r^2$ at $\kappa=0$ and is integrable at large distances for any finite κ, while it is integrable at short distances for any κ. We define the truncated resummed bond

$$F_{RT}\equiv F_R-T_R^{(2)}. \tag{86}$$

The diagrams Π are then replaced by diagrams Π_T which are built as the diagrams Π (namely, with the same topological rules but) with the only difference that there are now five bonds F^{cc}, F^{cm}, F^{mc}, F_{RT}, and $[F^{cc}]^2/2$.

F_{RT} decays as $1/r^3$ at $\kappa=0$ and is conditionally convergent for any finite κ when integrations over angles are performed in the first place. As a consequence, the low-density expansion of $\int d\mathbf{r} F_{RT}(\mathbf{r},\mathbf{X}_i,\mathbf{X}_j)$ starts by a singular $\ln\kappa$ term when κ goes to zero, as detailed in the next section. Let us denote the low-density expansion of a function F by $E_{\rm LD}[F]$. The fundamental formula that allows one to produce a scale decomposition of the Fourier transform of F_{RT}, $F_{RT}(\mathbf{k};\kappa)\equiv\int d\mathbf{r}\exp[i\mathbf{k}\cdot\mathbf{r}]F_{RT}(\mathbf{r};\kappa)$, reads

$$E_{\rm LD}\left[\int d\mathbf{r}\, e^{i\mathbf{k}\cdot\mathbf{r}}F(\mathbf{r})\right]=\lim_{R\to\infty}\left\{\int_{r<R}d\mathbf{r}E_{\rm LD}[F(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}]+E_{\rm LD}\left[\int_{r>R}d\mathbf{r}F_{\rm as}(\mathbf{r})\, e^{i\mathbf{k}\cdot\mathbf{r}}\right]\right\}, \tag{87}$$

where $F_{\rm as}$ is the part of the asymptotic behavior of F that gives nonvanishing contributions to

$$\int_{r>R}d\mathbf{r}F_{\rm as}(\mathbf{r})\exp(i\mathbf{k}\cdot\mathbf{r})$$

when R goes to infinity. (We notice that the notation R for the parameter that goes to infinity has nothing to do with the subscript R in F_{RT}.) Since $T_R^{\langle n\rangle}$ denotes the term with $(n-1)$ derivatives with respect to $\mathbf{r}$ in the Taylor expansion of F_{RT} at large distances r [see Eq. (76)], $T_R^{\langle n\rangle}$ is a sum of contributions entirely scaled by κ^n. According to Eq. (87),

$$E_{\rm LD}[F_{RT}(\kappa\mathbf{q};\kappa)]=\lim_{R\to\infty}\left\{\int_{r<R}d\mathbf{r}\,E_{\rm LD}[F_{RT}(\mathbf{r})e^{i\kappa\mathbf{q}\cdot\mathbf{r}}]+\int_{x>\kappa R}d\mathbf{x}\,\tilde{T}_R^{\langle 3\rangle{\rm alg}}(\mathbf{x})e^{i\mathbf{q}\cdot\mathbf{x}}+\sum_{n=3}^{\infty}\kappa^{n-3}E_{\rm LD}\left[\int_{x>\kappa R}d\mathbf{x}\,\tilde{T}_R^{\langle n\rangle{\rm exp}}(\mathbf{x})e^{i\mathbf{q}\cdot\mathbf{x}}\right]\right\}, \tag{88}$$

where we have omitted the dependence on the loop shapes, because it does not play any role in the formula. In Eq. (88) $\tilde{T}_R^{\langle 3\rangle{\rm alg}}$ denotes the purely algebraic part of $\tilde{T}_R^{\langle 3\rangle}$, $\tilde{T}_R^{\langle 3\rangle{\rm alg}}=W_3$, while $\tilde{T}_R^{\langle n\rangle{\rm exp}}$ denotes the part of $T_R^{\langle n\rangle}$ that decays exponentially at large distances. The $\tilde{T}_R^{\langle n\rangle{\rm exp}}$ are not integrable at the origin, but it does not matter because they appear only at distances $r>R$. $\tilde{T}_R^{\langle n\rangle{\rm alg}}$ with $n\geqslant 4$ do not appear in the right-hand side of Eq. (88), because their contribution vanishes when R goes to infinity. On the contrary the contribution from the $\tilde{T}_R^{\langle n\rangle{\rm exp}}$ lead to expansions in powers of $\kappa^m(\kappa R)^{m'}$.

For instance, $\int_{x>\kappa R}d\mathbf{x}\exp(i\mathbf{q}\cdot\mathbf{x})\exp(-3x)/x^3$ arises in the contribution of $\tilde{T}_R^{\langle 3\rangle{\rm exp}}$; this integral can be expanded in positive powers of κR and includes a $\ln(\kappa R)$ term. Indeed,

$$\int_{x>\kappa R}d\mathbf{x}\frac{e^{-3x}}{x^3}e^{i\mathbf{q}\cdot\mathbf{x}}=A(3\kappa R)+\int d\mathbf{x}\frac{e^{-3x}}{x^3}[e^{i\mathbf{q}\cdot\mathbf{x}}-1]-\int_{r<R}d\mathbf{r}\frac{e^{-3\kappa r}}{r^3}[e^{i\kappa\mathbf{q}\cdot\mathbf{r}}-1], \tag{89}$$

where

$$A(n\kappa R)\equiv\int_{x>\kappa R}d\mathbf{x}\frac{e^{-nx}}{x^3} \tag{90}$$

and (see, for instance, page 956 of Ref. [13])

$$E_{\rm LD}[A(n\kappa R)]=A^{(0)}(n\kappa R)+\kappa n\int_{r<R}d\mathbf{r}\frac{1}{r^2}-\kappa^2\frac{n^2}{2}\int_{r<R}d\mathbf{r}\frac{1}{r}+O((\kappa R)^3), \tag{91}$$

where $A^{(0)}(n\kappa R)=-4\pi[C+\ln(n\kappa R)]$ and C is the Euler constant.The $\ln\kappa$ terms in the low-density expansion of $\int d\mathbf{r}F_{RT}$ arise from the $A^{(0)}(n\kappa R)$'s. Moreover, the second term in the right-hand side of Eq. (89) reads

$$\int d\mathbf{x}\frac{e^{-3x}}{x^3}[e^{i\mathbf{q}\cdot\mathbf{x}}-1]=-4\pi\left[\ln\left(\sqrt{\frac{9+q^2}{3}}\right)+\frac{3}{q}\arctan\left(\frac{q}{3}\right)-1\right]. \tag{92}$$

The integral in the third term in the right-hand side term of Eq. (89) is convergent at the origin and its low-density expansion is merely obtained by expanding the integrand in powers of κ. This procedure generates a series in powers of $q\kappa R$ that starts at order $O(q^2\kappa^2R^2)$. Another example of contributions arising from $\tilde{T}_R^{\langle 4\rangle{\rm exp}}$ is the contribution from ϕ^4. After an integration by parts,

$$\kappa\int_{x>\kappa R}d\mathbf{x}\frac{e^{-4x}}{x^4}=4\pi\frac{e^{-4\kappa R}}{R}-4\kappa A(4\kappa R). \tag{93}$$

The corresponding low-density expansion reads

$$E_{\rm LD}\left[\kappa\int_{x>\kappa R}d\mathbf{x}\frac{e^{-4x}}{x^4}\right]=\frac{4\pi}{R}-16\pi\kappa-4\kappa A^{(0)}(4\kappa R)+O(\kappa^2R), \tag{94}$$

where the first term vanishes when R goes to infinity.

B. Explicit value of the scale decomposition of F_{RT}

In the following, after determination of the diagrams that contribute to the first three orders in density, F_{RT} will only appear in convolutions. Thus the calculations will be per-

formed in Fourier space, and we give only the explicit value of the low-density expansions involving the Fourier transform of F_{RT}.

1. Short-ranged contributions

The expressions of $[F^{cc}]^2/2$ and F_R given in Sec. IV A of Paper I are expanded in powers of κ for a fixed $\mathbf{r}$ with the result

$$E_{\mathrm{LD}}[F_{RT}](\mathbf{r},\mathbf{X}_i,\mathbf{X}_j)=f_T+\beta_{ij}[v^{mc}+v^{cm}] + \kappa p_i p_j \beta_{ij}\left[f_T+\frac{(\beta_{ij}p_ip_j)^2}{2r^2}\right] + O(\kappa^2), \tag{95}$$

with f_T defined as

$$f_T(\mathbf{r},\mathbf{X}_i,\mathbf{X}_j)=e^{-\beta_{ij}v(\mathbf{r},\mathbf{X}_i,\mathbf{X}_j)}-1+\frac{\beta_{ij}p_ip_j}{r}-\frac{(\beta_{ij}p_ip_j)^2}{2r^2}. \tag{96}$$

f_T is integrable at the origin and decays as $1/r^3$ at large distances. So its contribution to $\int_{r<R}\cdots$ behaves as $\ln R$ and is compensated by the $\ln(\kappa R)$ that comes from $\int d\mathbf{r}\, T_R^{\langle 3\rangle \exp}$. The second term in Eq. (95) is the so-called diffraction term which is specific to the long range of the Coulomb potential,

$$\int_{r<R} d\mathbf{r}\,\beta_{ij}v^{cm}(\mathbf{r},\mathbf{X}_j) = \beta_{ij}\frac{1}{2}\int_0^p d\tau\,[\mathbf{X}_j(\tau)]_\mu^2\int_{r<R} d\mathbf{r}\,\partial_{\mu\mu}\left(\frac{1}{r}\right). \tag{97}$$

In Eq. (97) we have only written the first term coming from the large-distance Taylor expansion of v^{cm}. Indeed, after integration over the orientation of $\mathbf{r}$, the next terms involve only the functions $\Delta^p(1/r)$ or $[\partial^{2n}/(\partial z)^{2n}]\Delta^{p-n}(1/r)$, with $p>n$, or $[\partial^{2p}/(\partial z)^{2p}](1/r)$, with $p\geq 2$. The first two expressions are equal to derivatives of the Dirac distribution, and their integrals vanish, while the third function is proportional to the Legendre polynomial $P_{2p}(\cos\theta)$ and gives a zero contribution after integration over the angle θ between the z axis and $\mathbf{r}$ (see Sec. V C of Paper I).

For loops with $p=1$, $\mathbf{X}_1=\lambda_\alpha\boldsymbol{\xi}$ and Eq. (97) involves the covariance defined in Eq. (45) with the result

$$\int_{r<R} d\mathbf{r}\left\{\left(\int_0^1 ds\,\mathrm{cov}_{xx}^{\alpha}(s,s;\mathbf{B}_0)\right)[\partial_{xx}+\partial_{yy}]\left(\frac{1}{r}\right) + \left(\int_0^1 ds\,\mathrm{cov}_{zz}(s,s)\right)\partial_{zz}\left(\frac{1}{r}\right)\right\} = -4\pi\left[C_0+\frac{2}{3}\delta C(u_{C_\alpha})\right], \tag{98}$$

where we have used Eq. (82) with $g(r)=1$ and $f(\mathbf{r})=1/r$ together with the identity $\int_{r<R}d\mathbf{r}\Delta(1/r)=-4\pi$. In Eq. (98) C_0 denotes the integrated covariance in the absence of magnetic field,

$$C_0\equiv\int_0^1 ds\,\mathrm{cov}_{xx}(s,s;\mathbf{B}_0=\mathbf{0})=\frac{1}{6}. \tag{99}$$

Since $\mathrm{cov}_{zz}(s,s)=\mathrm{cov}_{xx}^{\alpha}(s,s;\mathbf{B}_0=0)$

$$\delta C(u_{C_\alpha})\equiv\int_0^1 ds\,[\mathrm{cov}_{xx}^{\alpha}(s,s;\mathbf{B}_0)-\mathrm{cov}_{xx}^{\alpha}(s,s;\mathbf{B}_0=0)] = \frac{1}{2u_{C_\alpha}}L^{(3)}(u_{C_\alpha}), \tag{100}$$

where $L^{[3]}(x)=\coth x-(1/x)-(x/3)$ is a generalization of the Langevin function $L(x)$ defined in Eq. (14). Finally, the diffraction contribution from $\int d\mathbf{r}F_{RT}$ given by Eq. (97) reads

$$\int_{r<R}d\mathbf{r}\int D_{\mathbf{B}_0}(\boldsymbol{\xi}_j)\beta_{ij}v^{cm}(\mathbf{r},\boldsymbol{\xi}_j) = -\beta_{ij}2\pi\lambda_{\alpha_j}^2\left[C_0+\frac{2}{3}\delta C(u_{C\alpha_j})\right], \tag{101}$$

where

$$C_0+\frac{2}{3}\delta C(u_{C_\alpha})=\frac{1}{6}\left[1+\frac{2}{u_{C_\alpha}}L^{[3]}(u_{C_\alpha})\right]. \tag{102}$$

2. Long-ranged contributions

At the orders of interest we have to consider only the contributions from $\tilde{T}_R^{\langle 3\rangle \mathrm{alg}}=W_3$, $T_R^{\langle 3\rangle\exp}$, and $T_R^{\langle 4\rangle\exp}$ to integrals of the form

$$\int_{x>\kappa R}d\mathbf{x}\int D(\mathbf{X}_i)D(\mathbf{X}_j)g_i(\mathbf{X}_i)g_j(\mathbf{X}_j)F(\mathbf{x},\mathbf{X}_i,\mathbf{X}_j)e^{i\mathbf{q}\cdot\mathbf{x}}, \tag{103}$$

where the weights $D(\mathbf{X}_i)g_i(\mathbf{X}_i)$ are invariant under inversion of $\mathbf{X}_j$.

Since $W_3(\mathbf{r},\mathbf{X}_i,\mathbf{X}_j)$ is odd under inversion of each loop shape, the contribution from W_3 to an integral (103) vanishes according to parity arguments. When we consider the case $\mathbf{q}=\mathbf{0}$, another argument can be used. Indeed, after integration over the orientations of $\mathbf{r}$, W_3 gives a term proportional to $\Delta(1/r)$, which is short ranged, so that $\int_{r>R}d\mathbf{r}\,W_3=0$.

The exponential part of the large-distance Taylor expansion of F_{RT} can be written for the $n=3$ and 4 terms as

$$T_R^{\langle 3\rangle\exp}=-\beta_{ij}[\phi_{\mathrm{elect}}^{\langle 3\rangle}-\phi_{\mathrm{elect}}^{cm\langle 3\rangle}-\phi_{\mathrm{elect}}^{mc\langle 3\rangle}]+\beta_{ij}^2\phi\,\phi_{\mathrm{elect}}^{\langle 2\rangle}-\frac{\beta_{ij}^3}{3!}\phi^3 \tag{104}$$

and

$$T_R^{\langle 4\rangle\exp}=-\beta_{ij}[\phi_{\mathrm{elect}}^{\langle 4\rangle}-\phi_{\mathrm{elect}}^{cm\langle 4\rangle}-\phi_{\mathrm{elect}}^{mc\langle 4\rangle}] + \beta_{ij}^2\left[\frac{1}{2}(\phi_{\mathrm{elect}}^{\langle 2\rangle})^2+\phi\phi_{\mathrm{elect}}^{\langle 3\rangle}\right]-\frac{\beta_{ij}^3}{2}\phi^2\phi_{\mathrm{elect}}^{\langle 2\rangle}+\frac{\beta_{ij}^4}{4!}\phi^4. \tag{105}$$

$-\beta_{ij}\phi^{\langle n\rangle}_{\rm elect}$ and $-\beta_{ij}\phi^{cm\langle n\rangle}_{\rm elect}$ are the nth-order terms in the large-distance Taylor expansion of $-\beta_{ij}\phi_{\rm elect}$ and F^{cm} with the notations of Sec. IV A in Paper I. $\phi^{\langle 3\rangle}_{\rm elect}-\phi^{cm\langle 3\rangle}_{\rm elect}-\phi^{mc\langle 3\rangle}_{\rm elect}$ involves a product $[\mathbf{X}_i]_\mu[\mathbf{X}_j]_\nu$, while $\phi^{\langle 2\rangle}_{\rm elect}$ and the $\phi^{\langle 4\rangle}_{\rm elect}$'s contain odd numbers of components of either $\mathbf{X}_i$ or $\mathbf{X}_j$. Thus, after integration over the loop shapes with weights that are invariant under inversion of $\mathbf{X}$, only the term proportional to ϕ^3 in $T_R^{\langle 3\rangle\exp}(\mathbf{x})$ does contribute to Eq. (103) (and it has already been computed in Sec. IV A), while only $(\phi^{\langle 2\rangle}_{\rm elect})^2$, $\phi\phi^{\langle 3\rangle}_{\rm elect}$, and ϕ^4 in $T_R^{\langle 4\rangle\exp}$ give nonvanishing contributions to Eq. (103). After integration over the loop shapes, the contribution from the term $(\phi^{\langle 2\rangle}_{\rm elect})^2$ is proportional to

$$\int_{x>\kappa R} d\mathbf{x}\left[\boldsymbol{\nabla}\left(\frac{e^{-x}}{x}\right)\right]^2 = 2\pi e^{-2\kappa R}+4\pi\frac{e^{-2\kappa R}}{\kappa R} = \frac{4\pi}{\kappa R}-6\pi+O(\kappa R) \tag{106}$$

while the contribution from the term $\phi\phi^{\langle 3\rangle}_{\rm elect}$ involves

$$\int_{x>\kappa R} d\mathbf{x}\frac{e^{-x}}{x}\Delta\left(\frac{e^{-x}}{x}\right)=2\pi e^{-2\kappa R}=2\pi+O(\kappa R). \tag{107}$$

The contribution from ϕ^4 has already been given in Eq. (94).

3. Relevant results

By collecting the previous results we get at order κ^0

$$\int D_{\mathbf{B}_0}(\boldsymbol{\xi}_i)\int D_{\mathbf{B}_0}(\boldsymbol{\xi}_j)F_{RT}(\kappa\mathbf{q};\boldsymbol{\xi}_i,\boldsymbol{\xi}_j) = Q^*_{\alpha_i\alpha_j}(3\kappa)-2\pi\beta_{ij}\left\{\lambda^2_{\alpha_i}\left[C_0+\frac{2}{3}\delta C(u_{C\alpha_i})\right]+\lambda^2_{\alpha_j}\left[C_0+\frac{2}{3}\delta C(u_{C\alpha_j})\right]\right\} - \frac{\beta_{ij}^3}{3!}\int d\mathbf{x}\frac{e^{-3x}}{x^3}[e^{i\mathbf{q}\cdot\mathbf{x}}-1]+O(\kappa), \tag{108}$$

where the last integral is given in Eq. (92) and

$$Q^*_{\alpha_i\alpha_j}(n\kappa)\equiv\lim_{R\to\infty}\left\{\int_{r<R}d\mathbf{r}\int D_{\mathbf{B}_0}(\boldsymbol{\xi}_i)\int D_{\mathbf{B}_0}(\boldsymbol{\xi}_j)f_T - \frac{\beta_{ij}^3}{3!}A^{(0)}(n\kappa R)\right\}. \tag{109}$$

$Q^*_{\alpha_i\alpha_j}(n\kappa)$ may be written in terms of matrix elements by using the Feynman-Kac-Itô formula given in Sec. III of Paper I. According to the value of $A^{(0)}(n\kappa R)$ given in Eq. (91),

$$Q^*_{\alpha_i\alpha_j}(n\kappa)=\lim_{R\to\infty}\left\{\int_{r<R}d\mathbf{r}\left[\frac{\sinh u_{C\alpha_i}}{u_{C\alpha_i}}\frac{\sinh u_{C\gamma_j}}{u_{C\gamma_j}}\times(2\pi\lambda_{\alpha_i}\lambda_{\alpha_j})^3\langle\mathbf{0},\mathbf{r}|e^{-\beta H_{\alpha_i\alpha_j}}|\mathbf{0},\mathbf{r}\rangle-1+\frac{\beta e_{\alpha_i}e_{\alpha_j}}{r}-\frac{(\beta e_{\alpha_i}e_{\alpha_j})^2}{2r^2}\right]+\frac{2\pi}{3}(\beta e_{\alpha_i}e_{\alpha_j})^3[C+\ln(n\kappa R)]\right\}, \tag{110}$$

where $H_{\alpha_i\alpha_j}$ has been defined in Eq. (19).

Moreover, we shall need the following expression up to order κ:

$$\int d\mathbf{r}\int D_{\mathbf{B}_0}(\boldsymbol{\xi}_i)\int D_{\mathbf{B}_0}(\boldsymbol{\xi}_j)F_{RT}(\mathbf{r},\boldsymbol{\xi}_i,\boldsymbol{\xi}_j;\kappa) = Q^*_{\alpha_i\alpha_j}(3\kappa)+\kappa\beta_{ij}Q^*_{\alpha_i\alpha_j}(4\kappa)-\left(2\pi\beta_{ij}+\kappa\frac{\pi}{2}\beta_{ij}^2\right)\times\left\{\lambda^2_{\alpha_i}\left[C_0+\frac{2}{3}\delta C(u_{C\alpha_i})\right]+\lambda^2_{\alpha_j}\left[C_0+\frac{2}{3}\delta C(u_{C\alpha_j})\right]\right\} - \kappa\frac{2}{3}\pi\beta_{ij}^4+O(\kappa^2). \tag{111}$$

When we expand κ in powers of κ_D, expressions (108) and (111) remain unchanged apart from the replacement of κ by κ_D at the considered orders. Thus according to Eq. (111) the low-density expansion of

$$\int d\mathbf{r}\int D_{\mathbf{B}_0}(\boldsymbol{\xi}_i)\int D_{\mathbf{B}_0}(\boldsymbol{\xi}_j)F_{RT}(\mathbf{r},\boldsymbol{\xi}_i,\boldsymbol{\xi}_j;\kappa)$$

does start by a logarithmic term which is equal to $(2\pi/3)(\beta e_\alpha e_{\alpha_j})^3\ln(3\kappa)$. The next term is a constant plus a rest of order κ.

C. Minimal order of diagrams in loop density

The minimal order in density to which a diagram Π_T introduced in Sec. IV A contributes can be determined by the following procedure. We recall that the powers κ^n are counted as powers $\rho_{\rm loop}^{n/2}$. The problem to be handled is that the bonds F^{cc}, F^{cm}, and $[F^{cc}]^2/2$ are entirely scaled by κ, whereas the bond F_{RT} is at the border of integrability and decays as $1/r^3$ over a length scale which does not depend on the density [see Eq. (88)]. For instance, $\lim_{\kappa\to 0}\int d\mathbf{r}F_{RT}(\mathbf{r},\boldsymbol{\xi}_i,\boldsymbol{\xi}_j)$ is independent from κ apart from a $\ln\kappa$ term. Let N be the number of internal points in the diagram. Let $M_{F_{RT}}$ ($M_{F^{cm}}$) be the number of bonds F_{RT} (F^{cm} or F^{mc}) and $M_{F^{cc}}$ be the number of functions F^{cc} contained in the bonds F^{cc} and $[F^{cc}]^2/2$. The integral corresponding to a given diagram is proportional to

$$\int\left[\prod_{i=1}^N d\mathbf{r}_i d\chi_i\rho(\chi_i)\right]\prod_{\rm bonds}F(\mathcal{L}_i,\mathcal{L}_j). \tag{112}$$

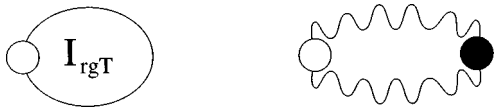

FIG. 1. Diagrams that contribute from order $\rho^{1/2}$ to $\ln(\rho_\alpha/\rho_\alpha^{\mathrm{id}\star,\mathrm{MB}})$. In Fig. 1, as in the following figures, a white disk represents the root point $\mathcal{L}_a$ and a black disk denotes an internal point whose loop coordinates are integrated over. I_{rgT} is the contribution of order $\rho^{1/2}$ from the ''ring'' diagrams which is given in Sec. V B. A wavy line corresponds to a bond F^{cc} and the symmetry factor of diagrams is not recalled in the figures; neither is the weight $\rho(\chi)$ of every internal point. Thus the second diagram in Fig. 1 stands for $(1/2)\int d\mathbf{r}\int d\chi\rho(\chi)[F^{cc}(\mathbf{r},\chi_a,\chi)]^2$.

For the sake of pedagogy, let us first consider the case $M_{F_{RT}}=0$. Then all bonds are entirely scaled by κ and the scaling change $\mathbf{r}=\mathbf{x}/\kappa$ is performed for the position of every internal point of the diagram. Each integration volume $d\mathbf{r}_i$ gives a factor κ^{-3} and is associated with a weight $\rho(\mathcal{L})$ that starts at the order κ^2. Each bond F^{cc} ($[F^{cc}]^2/2$) leads to a factor κ (κ^2) and, according to Eq. (80), each bond F^{cm} gives rise to a series in powers of κ, the first term of which is of order κ^2 and may vanish after integration over the orientation of $\mathbf{r}$ or over the loop shapes. Thus, since κ is considered as a term that starts at order $\rho_{\mathrm{loop}}^{1/2}$, the first term in the ρ_{loop} expansion of Eq. (112) is of order ρ_{loop}^n with

$$n(M_{F_{RT}}=0)\geqslant -\frac{N}{2}+\frac{1}{2}M_{F^{cc}}+M_{F^{mc}}. \tag{113}$$

We recall that the summations over the species or the exchange degeneracies or the integration over the loop shapes may only increase the order in ρ_{loop}. We notice that, since any diagram in Eq. (112) is connected and without any (internal or root) articulation point (see Sec. III C of Paper I) the number of bonds is greater than $N+1$ the total number of points in the diagram. Since the number of bonds is also lower than $M_{F^{cc}}+M_{F^{mc}}$, $M_{F^{cc}}+M_{F^{mc}}\geqslant N+1$, and $n(M_{F_{RT}}=0)\geqslant 1/2$.

In the generic case $M_{F_{RT}}\neq 0$. The integration over the distance between two points that are directly linked by a bond F_{RT} (and possibly indirectly linked by other paths of bonds) leads to a finite value whose limit when κ goes to zero is independent from the density, up to $\ln\kappa$ terms, because the distances that mainly contribute to the integrals involving F_{RT} are within a range l_0 which is independent from the density. So, as a first step, we integrate over all relative distances between pairs of points directly linked by a bond F_{RT}. We call $\mathbf{r}_j^{(1)}$ the positions of the internal points $\mathcal{L}_j^{(1)}$ that are left over after this first integration. Since only regions contained in a ''contraction disk'' with a radius l_0 and centered around either the position $\mathbf{r}_a$ of $\mathcal{L}_a$ or around the $\mathbf{r}_j^{(1)}$'s ($j=1,\ldots,N_{\mathrm{disk}}^{(1)}-1$) do contribute to the integral (112), the first integration step can be represented by a diagrammatical process. In this process, similar to that used in Sec. III of Ref. [3], every pair of points that are linked by a bond F_{RT} is replaced by a single point. (The latter is any one of the two points in the pair when they both are internal points, whereas the root point is chosen as the single point resulting from the collapse when it is involved in the bond

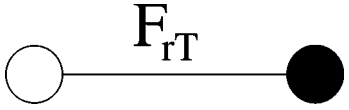

FIG. 2. Diagram with a single bond F_{RT}. This diagram contributes from order ρ to $\ln(\rho_\alpha/\rho_\alpha^{\mathrm{id}\star,\mathrm{MB}})$.

F_{RT}.) The contraction process of the first step ends when there is no more F_{RT} bond. A given contraction disk may originate from the fusion of several points that are linked together in the original diagrams by bonds F_{RT} and other kinds of bonds. Since l_0 does not depend on the density, κ^{-1} is far greater than l_0 in the low-density limit, so that, in the integral (112), the bonds F^{cc} and F^{cm} inside every contraction disk can be replaced by their values when κ vanishes and the result from the integration over all variables inside the contraction disks, except their centers $\mathcal{L}_j^{(1)}$, is independent from the density. Consequently, at the lowest order in density, the integrations over the $N-(N_{\mathrm{disk}}^{(1)}-1)$ internal points of the original diagram that are not centers $\mathbf{r}_j^{(1)}$ of contraction disks lead to a contribution of order $\rho_{\mathrm{loop}}^{N-(N_{\mathrm{disk}}^{(1)}-1)}$. Moreover, at the first order in loop density, each argument in any bond is replaced by the variable $\mathcal{L}_j^{(1)}$ which is in the same contraction disk (or by $\mathcal{L}_a$ in the case of an argument which is in the contraction disk centered on the root point). Thus the ρ_{loop} expansion of the integral (112) starts by a contribution of order $\rho_{\mathrm{loop}}^{N-(N_{\mathrm{disk}}^{(1)}-1)}$ times,

$$\int\left[\prod_{j=1}^{N_{\mathrm{disk}}^{(1)}-1}d\mathbf{r}_j^{(1)}d\chi_j\rho(\chi_j)\right]\times\prod_{\mathrm{bonds}}\prod_{\mathrm{disks/disks}}F_{\mathrm{disk/disk}}^{(1)}(\mathcal{L}_i^{(1)},\mathcal{L}_j^{(1)}). \tag{114}$$

In Eq. (114) the bonds $F_{\mathrm{disk/disk}}^{(1)}$ between the centers of the contraction disks are products of bonds F^{cc} and F^{cm}. Some of these bonds decay at least as $1/r^3$ over a length $l_0\ll\kappa^{-1}$ when κ vanishes.

Consequently, a second step is needed in the contraction procedure. Indeed, if we made the scaling transformation

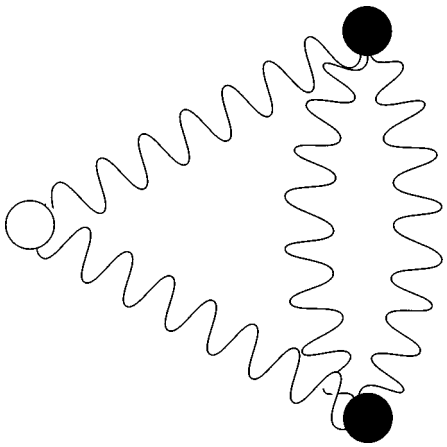

FIG. 3. Diagram which completes the direct contribution of order ρ from Fig. 2. in order to write it as a single partial derivative with ρ_α.

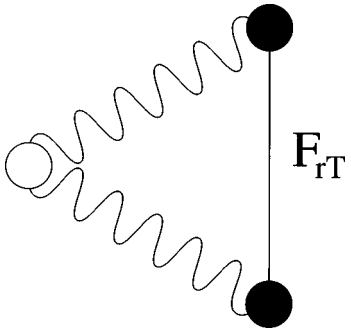

FIG. 4. Diagram which completes the direct contribution of order $\rho^{3/2}$ from Fig. 2. in order to write it as a single partial derivative with ρ_α.

$\mathbf{r}_j^{(1)}=\kappa\mathbf{x}_j^{(1)}$ at once, then the integration over $\mathbf{r}$ of any bond that decays as $\exp[-A\kappa r]/r^{m_j}$ with $m_j\geqslant 3$ would lead to a contribution κ^{m_j-3} with $m_j-3\geqslant 0$, whereas its low-density limit is in fact of order κ^0 because it decays at least as $1/r^3$ when κ vanishes. So, in order to obtain the lowest order in density, we have to perform a second step in which we integrate over the bonds $F^{(1)}_{\text{disk/disk}}$ that decay at least as $1/r^3$ at $\kappa=0$. This procedure is equivalent to introducing $N^{(2)}_{\text{disk}}$ new contraction disks. The integration over the $[N^{(1)}_{\text{disk}}-1]-[N^{(2)}_{\text{disk}}-1]$ internal points of these disks except their centers gives a contribution that starts at order $\rho_{\text{loop}}^{N^{(1)}_{\text{disk}}-N^{(2)}_{\text{disk}}}$. Thus the ρ_{loop} expansion of the integral (114) starts by the latter contribution times an integral that can be written as Eq. (114) with a superscript (2) in place of (1). The contraction process is repeated until there remain only bonds between disks that decay as $1/r$ or $1/r^2$ at $\kappa=0$. Let $N_{\text{disk,irr}}$ be the number of irreducible disks at the end of the contraction procedure. The ρ_{loop} expansion of the integral (112) starts by a term of order $\rho_{\text{loop}}^{N-(N_{\text{disk,irr}}-1)}$ times an integral similar to Eq. (114) where the bonds $F^{(\text{irr})}_{\text{disk/disk}}$ are only single bonds F^{cc}, $[F^{cc}]^2$ or F^{cm}. As discussed above in the case $M_{F_{RT}}=0$, the change of variable $\mathbf{r}=\mathbf{x}/\kappa$ for the $(N_{\text{disk, irr}}-1)$ positions of the centers which are to be integrated over shows that the latter integral is of order $\rho_{\text{loop}}^{n_\star}$ with $n_\star=-(1/2)(N_{\text{disk,irr}}-1)+(1/2)M_{F^{cc},\text{irr}}+M_{F^{mc},\text{irr}}$, where $M_{F^{cc},\text{irr}}$ ($M_{F^{cm},\text{irr}}$) is the number of bonds F^{cc} (F^{cm} or F^{mc}) between the irreducible disks. Eventually the term of lowest order in the ρ_{loop} expansion of the contribution of the diagram is of order

$$n(M_{F_{RT}})\geqslant N-\frac{3}{2}(N_{\text{disk,irr}}-1)+\frac{1}{2}M_{F^{cc},\text{irr}}+M_{F^{cm},\text{irr}}. \tag{115}$$

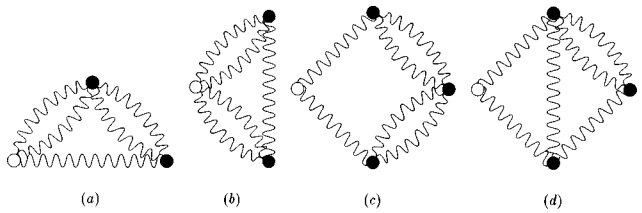

FIG. 5. Classical diagrams whose sum gives a contribution of order $\rho^{3/2}$ exactly and is a partial derivative with respect to ρ_α of a term proportional to $(1/\kappa_D)(\Sigma_\gamma\rho_\gamma e_\gamma^3)^2(\Sigma_{\gamma'}\rho_{\gamma'}e_{\gamma'}^4)$.

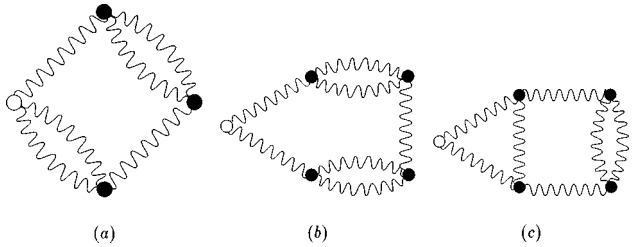

FIG. 6. Classical diagrams whose sum gives a contribution of order $\rho^{3/2}$ exactly and is a partial derivative with respect to ρ_α of a term proportional to $(1/\kappa_D^3)(\Sigma_\gamma\rho_\gamma e_\gamma^3)^4$.

We notice that, if $M_{F_{RT}}=0$, there is no contraction process, $N_{\text{disk,irr}}=N+1$, and we retrieve Eq. (113). If $M_{F_{RT}}\geqslant 1$, the number of disks is lower than the total number of points $N+1$ in the original diagram minus one, $N\geqslant N_{\text{disk,irr}}$. Moreover, since the diagram made with irreducible disks is still connected, the number of bonds is greater than $N_{\text{disk,irr}}-1$; since it is lower than $M_{F^{cc},\text{irr}}+M_{F^{cm},\text{irr}}$, we get $M_{F^{cc},\text{irr}}+M_{F^{cm},\text{irr}}\geqslant N_{\text{disk,irr}}-1$. Subsequently, the lower bound in Eq. (115) is greater than or equal to $1+(M_{F^{cm}}/2)\geqslant 1$. If the diagram is sufficiently connected with F_{RT} bonds—and there is no criterion about $M_{F_{RT}}$ for this phenomenon, contrary to what was said in Sec. III C of Ref. [3]—then all disks collapse into a single one and, according to Eq. (115) with $N_{\text{disk,irr}}=1$ and $M_{F^{cc}}=M_{F^{cm}}=0$, the order in density may be equal to $\rho(\mathcal{L})^N$ as in the case of short-ranged interactions.

V. EXPLICIT CONTRIBUTIONS FROM DIAGRAMS

We recall that, according to Sec. III C of Paper I, $J(\mathcal{L}_a)$ is the sum of a constant I_{rgT} (coming from some truncated contribution of Coulomb rings) and of all unlabeled topologically different connected diagrams $\mathbb{P}^*$ with one root point $\mathcal{L}_a$ and at least one internal point, and which are built with bonds F^{cc}, F^{cm}, F^{mc}, and F_R. The topological rules for these diagrams are the following. They contain no articulation point, they remain as a single piece when all bonds involving the root point are cut, and they obey the following excluded-convolution rule: there can be no convolution $F^{cc}*F^{cc}$, $F^{cc}*F^{cm}$, or $F^{mc}*F^{cc}$.

Diagrams that should contribute from one given order in density according to the scaling analysis but that prove to vanish after integration over the shape of the root point $\mathbf{X}_a$ will not be drawn. In Figs. 1–7 diagrams are put together according to the minimal order in density to which they contribute and according to the nature of the effects they describe. Moreover, diagrams whose sum gives one derivative with respect to the density are collected in a single figure. The species of the root point $\mathcal{L}_a$ will be called α in order to simplify the notations.

A. Single bonds with no contribution

The contribution from F^{cc} to $J(\mathbf{X}_a)$ disappears by virtue of the neutrality relation. Indeed, it reads

$$\int d\mathbf{r}\int d\chi\rho(\chi)F^{cc}(\mathbf{r},p_a e_\alpha,pe_\gamma)=-p_a e_\alpha\frac{4\pi\beta}{\kappa^2}\sum_\gamma e_\gamma\rho_\gamma. \tag{116}$$

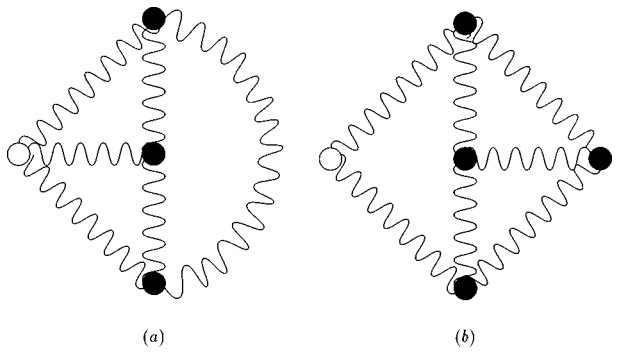

FIG. 7. The sum of these diagrams is analogous to that of diagrams in Fig. 6, apart from the fact that in the case of Fig. 6 the numerical coefficient involves a one-dimensional integral with elementary functions, whereas in the case of Fig. 7, the coefficient is a three-dimensional ''bridge'' integral.

We notice that if the neutrality relation (39) did not cancel the contribution from F^{cc}, this contribution would be responsible for the existence of a term of order ρ^0 that would not be a partial derivative with respect to ρ_α. In fact, such a term is not allowed, because, otherwise, ρ_α would not be equal to $\rho_\alpha^{\mathrm{id}\star,\mathrm{MB}}$ in the strict zero-density limit, as explained in Sec. III C.

The contribution from $F^{cm}(\mathbf{r},\chi_a,\chi)$ or $F^{mc}(\mathbf{r},\chi_a,\chi)$ vanishes after integration over the relative position $\mathbf{r}$ of the two loops,

$$\int d\mathbf{r} F^{cm}(\mathbf{r},\mathbf{X})=0. \tag{117}$$

A first argument can be given in Fourier space. Since the Fourier transform of F^{cm} reads

$$F^{cm}(\mathbf{k},\mathbf{X})=-\beta p_a e_\alpha e_\gamma \int_0^p d\tau (e^{-i\mathbf{k}\cdot\mathbf{X}(\tau)}-1)\frac{4\pi}{\mathbf{k}^2+\kappa^2} \tag{118}$$

and is multiplied by a function $\rho(\mathbf{X})$ every moment of which is integrable, its contribution vanishes when $|\mathbf{k}|$ goes to zero. Another reason may be provided in position space. According to Sec. IV A,

$$\int d\mathbf{r} F^{cm}(\mathbf{r},\mathbf{X})=\sum_{p=1}^{\infty}\int d\mathbf{r} T^{cm\langle 2p\rangle}(\mathbf{r},\mathbf{X}) \tag{119}$$

and the contribution from each $T^{cm\langle 2p\rangle}$ is proportional to that from $\Delta^p(\exp[-\kappa r]/r)$. Equation (83) and the identity

$$\int d\mathbf{r}\frac{\exp(-\kappa r)}{r}=\frac{4\pi}{\kappa^2} \tag{120}$$

imply that $\int d\mathbf{r}\Delta\phi(\mathbf{r})=0$. A recurrence allows one to show easily by using Eq. (83) again that, for any $p\geqslant 1$,

$$\int d\mathbf{r}\Delta^p\left(\frac{\exp(-\kappa r)}{r}\right)=0. \tag{121}$$

B. Diagrams contributing from the order $\rho^{1/2}$ to $\ln(\rho_\alpha/\rho_\alpha^{\mathrm{id}\star,\mathrm{MB}})$

Now we turn to the contributions from order $\rho_{\mathrm{loop}}^{1/2}$ to $\ln(\rho_\alpha/\rho_\alpha^{\mathrm{id}\star,\mathrm{MB}})$. We calculate $J_{(1)\mathrm{loop}}^{\{1/2\}}$ for any p because we need its value for $p=1$ and $p=2$, according to Eq. (69a) and (69c). After inspection, every diagram, but Fig. 1, proves to be at least of order ρ_{loop}. In other words, the term of order κ that comes from Fig. 1, $J_{(1)\mathrm{loop}}^{\{1/2\}}(p_a e_\alpha)$, is the only contribution of order $\rho_{\mathrm{loop}}^{1/2}$ to $\ln[\rho_{\alpha,p}(\mathbf{X}_p)/z_{\alpha,p}(\mathbf{X}_p)]$.

With the same notations as in Eq. (51), the contribution $J_{(1)\mathrm{loop}}^{\{1/2\}}$ of order $\rho_{\mathrm{loop}}^{1/2}$ from Fig. 1 is the sum of two contributions. First, after splitting F_R into F_{RT} and $[F^{cc}]^2/2$, the contribution from the diagram with only one bond F_R gives a term

$$\int d\mathbf{r}\int d\chi\rho(\chi)\frac{1}{2}[F^{cc}]^2(\mathbf{r},p_a e_\alpha,p e_\gamma)=\frac{1}{4}\beta(p_a e_\alpha)^2\kappa \tag{122}$$

according to the definition of κ recalled just before Eq. (31). The other contribution to $J_{\mathrm{loop}}^{\{1/2\}}$ comes from I_{rgT}. As explained in Sec. IV B of Paper I, I_{rgT} contains the sum of rings of Coulomb bonds plus the value that must be subtracted from F_{RT} in order to avoid any double counting. The value of I_{rgT} is given in Sec. IV B of Paper I. The contributions from $(1/2)[F^{cc}]^2$ and I_{rgT} are entirely scaled by κ. The ρ_{loop} expansion of their sum starts at order $\rho_{\mathrm{loop}}^{1/2}$ by

$$J_{(1)\mathrm{loop}}^{\{1/2\}}(p_a e_\alpha)=\frac{1}{2}\beta(p_a e_\alpha)^2\kappa. \tag{123}$$

I_{rgT} also gives a contribution proportional to κ^3,

$$-\kappa^3\frac{1}{4}\beta(p_a e_\alpha)^2\left\{\int_0^{p_a}\frac{d\tau}{p_a}\int_0^{p_a}\frac{d\tau'}{p_a}\frac{1}{3}\mathbf{X}_a(\tau)\cdot\mathbf{X}_a(\tau')-\int_0^{p_a}\frac{d\tau}{p_a}\frac{1}{3}[\mathbf{X}_a(\tau)]^2\right\} \tag{124}$$

plus higher-order terms in κ^{2n+1} with $n\geqslant 2$. The expression (124) is derived from the property (82) and from

$$\int d\mathbf{x}\left(\frac{1-e^{-x}}{x}\right)\left(\frac{1}{x}\right)=2\pi, \tag{125}$$

$$\int d\mathbf{x}\,\partial_\mu\left(\frac{1-e^{-x}}{x}\right)\partial_\mu\left(\frac{1}{x}\right)=2\pi \tag{126}$$

(with implicit summation over μ)

$$\int d\mathbf{x}\left\{\left(\frac{1}{x}\right)\Delta\left(\frac{1-e^{-x}}{x}\right)+\left(\frac{1-e^{-x}}{x}\right)\Delta\left(\frac{1}{x}\right)\right\}=-4\pi. \tag{127}$$

Since Fig. 1 provides the only contribution of order $\rho_{\mathrm{loop}}^{1/2}$, the only term of order $\rho^{1/2}$ in $J(\mathbf{X}_a)$, $J^{\{1/2\}}(p_a e_\alpha)$, is obtained by inserting the ρ expansion (66) of κ around κ_D in

$J^{\{1/2\}}_{\mathrm{loop}(1)}(e_\alpha)$ [namely, by replacing κ by κ_D in the value (123) of $J^{\{1/2\}}_{(1)\mathrm{loop}}(e_\alpha)$] with the result

$$J^{\{1/2\}}(p_a e_\alpha)=J^{\{1/2\}}_{(1)}(p_a e_\alpha)=\frac{1}{2}\beta(p_a e_\alpha)^2\kappa_D . \quad (128)$$

By using the fundamental formula that will be used several times in the following:

$$\frac{\partial \kappa_D^n}{\partial \rho_\alpha}=n2\pi\beta e_\alpha^2\kappa_D^{n-2}, \quad (129)$$

we get

$$J^{\{1/2\}}(p_a e_\alpha)=p_a^2\frac{\partial}{\partial\rho_\alpha}\left(\frac{\kappa_D^3}{12\pi}\right). \quad (130)$$

For $p_a=1$ $J^{\{1/2\}}$ is the classical Debye contribution. On the other hand, according to the formal study in Sec. III E, $J^{\{1/2\}}(p_a e_\alpha)$ for $p_a=1$ is in fact the only contribution at order $\rho^{1/2}$ in $\ln(\rho_\alpha/\rho_\alpha^{\mathrm{id}\star,\mathrm{MB}})$, namely, $B_\alpha^{\{1/2\}}=J^{\{1/2\}}(e_\alpha)$. Thus the Debye term is the only contribution of order $\rho^{3/2}$ in the free energy density and quantum corrections appear only at larger orders.

According to the results of Sec. III E, the next terms in the ρ expansion of $J(\mathcal{L}_a)$ contribute to the ρ expansion of $\ln(\rho_\alpha/\rho_\alpha^{\mathrm{id}\star,\mathrm{MB}})$ up to order $\rho^{3/2}$ only through $\int D(\boldsymbol{\xi})J^{\{n\}}(\boldsymbol{\xi})$ with $n=1$ or $n=3/2$. For the sake of conciseness, we introduce

$$I^{\{n\}}_{\mathrm{II}_T}\equiv\int D(\boldsymbol{\xi})J^{\{n\}}_{\mathrm{II}_T}(\boldsymbol{\xi}). \quad (131)$$

Figure 1 gives no contribution of order ρ, $I^{\{1\}}_{(1)}=0$, while $I^{\{3/2\}}_{(1)}$ is the sum of two terms.

One term in $I^{\{3/2\}}_{(1)}$ is an exchange term that comes from the $\rho^{3/2}$ term in the expansion of κ around κ_D when it is inserted in the value (123) of $J^{\{1/2\}}_{(1)\mathrm{loop}}(p_a e_\alpha)$. It reads

$$I^{\{3/2\}}_{(1)\mathrm{exch}}=\pi\beta^2 e_\alpha^2\frac{1}{\kappa_D}\sum_\gamma\ e_\gamma^2\rho_\gamma^2 E_\gamma^*=\frac{\beta}{2}\frac{\partial\kappa_D}{\partial\rho_\alpha}\sum_\gamma\ e_\gamma^2\rho_\gamma^2 E_\gamma^* . \quad (132)$$

After inspection of diagrams, there proves to be no other diagram that would give exchange contributions at order $\rho^{3/2}$. The term of order ρ has already been identified as a derivative with respect to ρ_α in Sec. III E. According to Eq. (69c), the exchange term $B_\alpha^{\{3/2\}}$ in $\ln(\rho_\alpha/\rho_\alpha^{\mathrm{id}\star,\mathrm{MB}})$ at order $\rho^{3/2}$ is the sum of two contributions. The first one, $\int D_{\mathbf{B}_0}(\boldsymbol{\xi})J^{\{3/2\}}_{\mathrm{exch}}(\boldsymbol{\xi})$, originates from the term $I^{\{3/2\}}_{(1)\mathrm{exch}}$ and may be expressed as $f\partial g/\partial\rho_\alpha$. The other one comes from the loops $\mathcal{L}_a$ with $p_a=2$ and is written in Eq. (69c). After insertion of the value of $J^{\{1/2\}}(pe_\alpha)$ in Eq. (69c), the latter contribution may be written as $g\partial f/\partial\rho_\alpha$. More precisely,

$$\int D(\boldsymbol{\xi})J^{\{3/2\}}_{\mathrm{exch}}(\boldsymbol{\xi})+\rho_\alpha[J^{\{1/2\}}(2e_\alpha)-2J^{\{1/2\}}(e_\alpha)]E_\alpha^*=\frac{\partial}{\partial\rho_\alpha}\left(\frac{1}{2}\beta\kappa_D\sum_\gamma\ \rho_\gamma^2 e_\gamma^2 E_\gamma^*\right). \quad (133)$$

Equation (133) is the first example of the adequate combination of different kinds of contributions in order to produce derivatives of products of functions that all depend on the density.

The other term in $I^{\{3/2\}}_{(1)}$ arises from the replacement of κ^3 in Eq. (124) for $p_a=1$ by its leading low-density value κ_D^3. It reads

$$I^{\{3/2\}}_{(1)\mathrm{diff}}=-\kappa_D^3\frac{1}{8}\beta e_\alpha^2\lambda_\alpha^2\left[C_0+\frac{2}{3}\delta C(u_{C\alpha})\right]=-\frac{\beta}{8}\kappa_D^3\frac{\partial}{\partial\rho_\alpha}\left(\sum_\gamma\ \left[C_0+\frac{2}{3}\delta C(u_{C\alpha})\right]\rho_\gamma e_\gamma^2\lambda_\gamma^2\right). \quad (134)$$

In Eq. (134) we have used the index diff because the latter term comes from ρ expansions of integrals involving the bond F^{cm} and the existence of the latter bond originates from the combination of quantum fluctuations with the resummation of collective effects arising from the long range of the Coulomb potential. We have already called ''diffraction'' contributions the terms in Eq. (111) that have a similar origin in the ρ expansion of $\int d\mathbf{r}F_R(\mathbf{r},\boldsymbol{\xi}_i,\boldsymbol{\xi}_j)$.

C. Diagrams contributing from order ρ to $\ln(\rho_\alpha/\rho_\alpha^{\mathrm{id}\star,\mathrm{MB}})$

1. Single bond F_{RT} and first ''direct'' contributions

The diagram in Fig. 2, which reduces to a single bond F_{RT}, is responsible for another kind of contributions, called ''direct'' terms in the following, because they involve the diagonal matrix element of the two-body Gibbs factor. These terms contain both short-ranged quantum effects, such as the existence of bound states, and a proper truncation that makes the contributions from Rydberg and diffusive quantum states finite, as a consequence of the screening of monopoles which is valid both at the classical and quantum levels. The direct terms appear from order ρ. The diagram in Fig. 3 (Fig. 4) will allow the completion of the direct contributions of F_{RT} (in Fig. 2) in order to obtain derivatives with respect to ρ_α of a term of order ρ ($\rho^{3/2}$).

The contribution from Fig. 2 to $\int D_{\mathbf{B}_0}(\boldsymbol{\xi}_a)J(\boldsymbol{\xi}_a)$,

$$I_{(2)}=\int d\mathbf{r}\sum_\gamma\ \rho_\gamma\int D_{\mathbf{B}_0}(\boldsymbol{\xi}_a)\int D_{\mathbf{B}_0}(\boldsymbol{\xi})F_{RT}(\mathbf{r},\boldsymbol{\xi}_a,\boldsymbol{\xi}), \quad (135)$$

is given by formula (111). $I_{(2)}$ may be decomposed as

$$I_{(2)}=I^{\{1\}}_{(2)\mathrm{dir}}+I^{\{1\}}_{(2)\mathrm{diff}}+I^{\{3/2\}}_{(2)\mathrm{dir}}+I^{\{3/2\}}_{(2)\mathrm{diff}}+o(\rho^{3/2}). \quad (136)$$

Terms of two kinds emerge at order ρ. One is a direct term,

$$I^{\{1\}}_{(2)\mathrm{dir}}=\sum_\gamma \rho_\gamma Q^*_{\alpha\gamma}(3\kappa_D)=\frac{\partial}{\partial\rho_\alpha}\left(\frac{1}{2}\sum_{\gamma,\gamma'}\rho_\gamma\rho_{\gamma'}Q^*_{\gamma\gamma'}(3\kappa_D)\right)-\frac{2}{3}\pi^2\beta^4e^2_\alpha\frac{1}{\kappa_D^2}\left[\sum_\gamma \rho_\gamma e^3_\gamma\right]^2, \tag{137}$$

where the second equality comes from the fact that $Q^*_{\gamma\gamma'}(3\kappa_D)$ depends on ρ by a $\ln\kappa_D$ term. The other term of order ρ is a diffraction contribution and may be viewed as a partial derivative,

$$\begin{aligned}I^{\{1\}}_{(2)\mathrm{diff}}&=-2\pi\beta\left\{e_\alpha\lambda^2_\alpha\left[C_0+\frac{2}{3}\delta C(u_{C\alpha})\right]\left(\sum_\gamma \rho_\gamma e_\gamma\right)\right.\\&\left.+e_\alpha\left(\sum_\gamma\left[C_0+\frac{2}{3}\delta C(u_{C\gamma})\right]\rho_\gamma e_\gamma\lambda^2_\gamma\right)\right\}\\&=\frac{\partial}{\partial\rho_\alpha}\left[-2\pi\beta\left(\sum_\gamma\left[C_0+\frac{2}{3}\delta C(u_{C\gamma})\right]\rho_\gamma e_\gamma\lambda^2_\gamma\right)\right.\\&\left.\times\left(\sum_{\gamma'}\rho_{\gamma'}e_{\gamma'}\right)\right].\end{aligned} \tag{138}$$

At order $\rho^{3/2}$ the bond F_{RT} also provides a direct as well as a diffraction contribution,

$$I^{\{3/2\}}_{(2)\mathrm{dir}}=-\frac{2}{3}\pi\beta^4e^4_\alpha\kappa_D\sum_\gamma \rho_\gamma e^4_\gamma+\beta e_\alpha\kappa_D\sum_\gamma \rho_\gamma e_\gamma Q^*_{\alpha\gamma}(4\kappa_D), \tag{139}$$

$$I^{\{3/2\}}_{(2)\mathrm{diff}}=-\frac{1}{8}\kappa_D^3\beta e^2_\alpha\lambda^2_\alpha\left[C_0+\frac{2}{3}\delta C(u_{C\alpha})\right]-\frac{\pi}{2}\beta^2\kappa_De^2_\alpha\left(\sum_\gamma\left[C_0+\frac{2}{3}\delta C(u_{C\gamma})\right]\rho_\gamma e^2_\gamma\lambda^2_\gamma\right). \tag{140}$$

2. Diagram of Fig. 3 and completion of the direct term at order ρ

The diagram in Fig. 3, namely,

$$\int d\mathbf{r}\{F^{cc}*(1/2)[F^{cc}]^2*F^{cc}\}(\mathbf{r}),$$

contributes from order ρ_{loop}. More precisely, Fig. 3 has a symmetry factor equal to 2 and

$$J_{(3)}(\boldsymbol{\xi}_a)=\frac{1}{2}\beta^4e^2_\alpha\frac{\left[\sum_\gamma e^3_\gamma\sum_{p=1}^{\infty}p^3\int D(\mathbf{X})\rho_{\gamma,p}(\mathbf{X})\right]^2}{\kappa^2}A, \tag{141}$$

where $A\equiv(1/2)\int[d\mathbf{q}/(2\pi)^3][\tilde\phi(\mathbf{q})]^2[\widetilde{\phi^2}](\mathbf{q})$ where $\tilde\phi(\mathbf{x})=\phi(\mathbf{x}/\kappa;\kappa)$ is defined in Eq. (77). Since

$$\tilde\phi(\mathbf{q})=\frac{4\pi}{1+\mathbf{q}^2} \tag{142}$$

and

$$\frac{1}{2}[\widetilde{\phi^2}](\mathbf{q})=\frac{2\pi}{q}\arctan\left(\frac{q}{2}\right) \tag{143}$$

we find $A=4\pi^2/3$.

After ρ expansion, the first term in

$$I_{(3)}\equiv\int D_{\mathbf{B}_0}(\boldsymbol{\xi}_a)J_{(3)}(\boldsymbol{\xi}_a)$$

is of order ρ. $I^{\{1\}}_{\{3\}}$ completes $I^{\{1\}}_{(2)\mathrm{dir}}$ in order to form a partial derivative, and the next term in $I_{(3)}$ is only of order ρ^2 according to Eqs. (36) and (56). More precisely,

$$I^{\{1\}}_3=I^{\{1\}}_{(3)\mathrm{dir}}=\frac{2}{3}\pi^2\beta^4e^2_\alpha\frac{1}{\kappa_D^2}\left[\sum_\gamma \rho_\gamma e^3_\gamma\right]^2 \tag{144}$$

and comparison with Eq. (137) shows that, since there is no other direct contribution at order ρ,

$$\int D_{\mathbf{B}_0}(\boldsymbol{\xi})J^{\{1\}}_{\mathrm{dir}}(\boldsymbol{\xi})=I^{\{1\}}_{(2)\mathrm{dir}}+I^{\{1\}}_{(3)\mathrm{dir}}=\frac{\partial}{\partial\rho_\alpha}\left(\frac{1}{2}\sum_{\gamma,\gamma'}\rho_\gamma\rho_{\gamma'}Q^*_{\gamma\gamma'}(3\kappa_D)\right). \tag{145}$$

We notice that the diagram $F^{mc}*(1/2)[F^{cc}]^2*F^{cc}$ gives a contribution whose ρ_{loop} expansion should *a priori* start at order $\rho^{3/2}_{\mathrm{loop}}$. However, the $\rho^{3/2}_{\mathrm{loop}}$ term vanishes after integration with the measure $D_{\mathbf{B}_0}(\boldsymbol{\xi}_a)$ and its contribution to $\int D_{\mathbf{B}_0}(\boldsymbol{\xi}_a)J(\boldsymbol{\xi}_a)$ starts only from order ρ^2_{loop}.

3. Diagram of Fig. 4 and completion of the direct term at order $\rho^{3/2}$

The contribution from Fig. 4, namely, $F^{cc}*F_{RT}*F^{cc}$, starts at order $\rho^{3/2}_{\mathrm{loop}}$ and, at this order, it only involves $\widetilde{F_{RT}}(\mathbf{q})$ at order ρ^0_{loop}. Figure 4 has a symmetry factor equal to 2 and

$$J_{(4)}(\boldsymbol{\xi}_a)=\frac{1}{2}\beta^2e^2_\alpha\frac{1}{\kappa}\int\frac{d\mathbf{q}}{(2\pi)^3}[\tilde\phi(q)]^2\times\sum_{\gamma,\gamma'}\rho_\gamma e_\gamma\rho_{\gamma'}e_{\gamma'}\int D_{\mathbf{B}_0}(\boldsymbol{\xi})\int D_{\mathbf{B}_0}(\boldsymbol{\xi}')\times\tilde F_{RT}{}^{\{0\}}(\mathbf{q},\boldsymbol{\xi},\boldsymbol{\xi}'). \tag{146}$$

The point is that, after ρ expansion, the sum $I^{\{3/2\}}_{(2)\mathrm{dir}}+I^{\{3/2\}}_{(4)\mathrm{dir}}$ is a partial derivative.

The first term in the ρ expansion is calculated by using Eq. (108) at order ρ^0, $\int[d\mathbf{q}/(2\pi)^3][\tilde\phi(q)]^2=2\pi$, and the formula (92) in order to calculate

$$\int \frac{d\mathbf{q}}{(2\pi)^3}[\tilde{\phi}(\mathbf{q})]^2 \int d\mathbf{x}[e^{i\mathbf{q}\cdot\mathbf{x}}-1]\frac{e^{-3x}}{x^3} = 8\pi^2[\ln 3 - 2\ln 2]. \tag{147}$$

The contribution from Fig. 4 at order $\rho^{3/2}$ may be decomposed in two terms

$$I^{\{3/2\}}_{(4)\mathrm{dir}} = \pi\beta^2 e_\alpha^2 \frac{1}{\kappa_D}\sum_{\gamma,\gamma'} \rho_\gamma e_\gamma \rho_{\gamma'} e_{\gamma'} Q^*_{\gamma\gamma'}(4\kappa_D) \tag{148}$$

and

$$I^{\{3/2\}}_{(4)\mathrm{diff}} = -\pi\beta^2 e_\alpha^2 \kappa_D \sum_\gamma \left[C_0 + \frac{2}{3}\delta C(u_{C\gamma})\right]\rho_\gamma e_\gamma^2 \lambda_\gamma^2. \tag{149}$$

After inspection of diagrams, it turns out that the sum of the direct terms at order $\rho^{3/2}$ is given by Eqs. (139) and (148),

$$\begin{aligned}\int D_{\mathbf{B}_0}(\boldsymbol{\xi}) J^{\{3/2\}}_{\mathrm{dir}}(\boldsymbol{\xi}) &= I^{\{3/2\}}_{(2)\mathrm{dir}} + I^{\{3/2\}}_{(4)\mathrm{dir}} \\ &= \frac{\partial}{\partial\rho_\alpha}\left(\frac{1}{2}\beta\kappa_D \sum_{\gamma,\gamma'} \rho_\gamma e_\gamma \rho_{\gamma'} e_{\gamma'} Q^*_{\gamma\gamma'}(4\kappa_D)\right) \\ &\quad - \frac{\partial}{\partial\rho_\alpha}\left(\frac{\pi}{3}\beta^4\kappa_D\left[\sum_\gamma \rho_\gamma e_\gamma^4\right]^2\right)\end{aligned} \tag{150}$$

while the sum of the diffraction terms at order $\rho^{3/2}$ arises from Eqs. (134), (140), and (149) with the result

$$\begin{aligned}\int D_{\mathbf{B}_0}(\boldsymbol{\xi}) J^{\{3/2\}}_{\mathrm{diff}} &= I^{\{3/2\}}_{(1)\mathrm{diff}} + I^{\{3/2\}}_{(2)\mathrm{diff}} + I^{\{3/2\}}_{(4)\mathrm{diff}} \\ &= -\frac{\beta}{4}\frac{\partial}{\partial\rho_\alpha}\left(\kappa_D^3 \sum_\gamma \left[C_0 + \frac{2}{3}\delta C(u_{C\gamma})\right]\rho_\gamma e_\gamma^2 \lambda_\gamma^2\right).\end{aligned} \tag{151}$$

Both sums prove to be partial derivatives, as they should.

D. Purely classical contributions at order $\rho^{3/2}$

The last figures contain purely classical diagrams which are exactly of order $\rho^{3/2}(\mathcal{L})$ and which do not involve short-ranged effects. We have chosen to collect all diagrams whose sum is a derivative with respect to ρ_α in one given figure. Moreover, we already notice that diagrams in Fig. 5 give a contribution to $\ln(\rho_\alpha/\rho_\alpha^{\mathrm{id}\star,\mathrm{MB}})$ of the form $(\rho e^3)^2\rho e^4$ whereas the terms arising from the diagrams in Figs. 6 and 7 have the same $(\rho e^3)^4$ structure. The detailed calculations are displayed in Appendix B and the results are the following.

The total contribution from diagrams in Fig. 5 at order $\rho^{3/2}$ is

$$I^{\{3/2\}}_{(5)} = -4\pi K_1 \beta^5 \frac{\partial}{\partial\rho_\alpha}\left[\frac{1}{\kappa_D}\left(\sum_\gamma \rho_\gamma e_\gamma^3\right)^2\left(\sum_{\gamma'} \rho_{\gamma'} e_{\gamma'}^4\right)\right], \tag{152}$$

where K_1 is defined as

$$K_n \equiv \int_0^\infty dq \frac{1}{[1+q^2]^n}\left[\arctan\left(\frac{q}{2}\right)\right]^2. \tag{153}$$

The sum of the terms arising from Fig. 6 at order $\rho^{3/2}$ reads

$$I^{\{3/2\}}_{(6)} = 8\pi^2 K_2 \beta^6 \frac{\partial}{\partial\rho_\alpha}\left[\frac{1}{\kappa_D^3}\left(\sum_\gamma \rho_\gamma e_\gamma^3\right)^4\right]. \tag{154}$$

Each contribution from the two diagrams in Fig. 7 may be expressed in terms of a dimensionless integral $\tilde{I}_{\mathrm{bridge}\,6} \equiv (1/\kappa_D^3) I_{\mathrm{bridge}\,6}$ with the result

$$I^{\{3/2\}}_{(7)} = \frac{1}{24}\beta^6 \tilde{I}_{\mathrm{bridge}\,6}\frac{\partial}{\partial\rho_\alpha}\left[\frac{1}{\kappa_D^3}\left(\sum_\gamma \rho_\gamma e_\gamma^3\right)^4\right], \tag{155}$$

where

$$\begin{aligned}\tilde{I}_{\mathrm{bridge}\,6} \equiv \int \frac{d\mathbf{q}}{(2\pi)^3}\int \frac{d\mathbf{q}'}{(2\pi)^3}\int \frac{d\mathbf{q}''}{(2\pi)^3}\tilde{\phi}(\mathbf{q})\tilde{\phi}(\mathbf{q}')\tilde{\phi}(\mathbf{q}'') \\ \times \tilde{\phi}(\mathbf{q}-\mathbf{q}')\tilde{\phi}(\mathbf{q}-\mathbf{q}'')\tilde{\phi}(\mathbf{q}'-\mathbf{q}'').\end{aligned} \tag{156}$$

The integral $\tilde{I}_{\mathrm{bridge}\,6}$ can be reduced to a triple integral by the following transformation. In the same way as in Ref. [14], the integral is written in Fourier space in spherical coordinates. Then, according to the method in Ref. [15], the three functions that depend on relative angles are expanded in terms of Legendre polynomials. The addition and orthogonality theorems for Legendre polynomials lead to

$$\begin{aligned}\tilde{I}_{\mathrm{bridge}\,6} = 384\sum_{b=0}^{+\infty}(2b+1)\int_0^\infty du_1 \frac{1}{1+u_1^2} \\ \times \int_{u_1}^\infty du_2 \frac{Q_b(x_{12})}{1+u_2^2}\int_{u_2}^\infty du_3 \frac{Q_b(x_{13})Q_b(x_{23})}{1+u_3^2},\end{aligned} \tag{157}$$

where $Q_b(x)$ is a Legendre function of the second kind, $Q_b(z) \equiv (1/2)\int_{-1}^1 dt P_b(t)/(z-t)$, and $x_{ij} \equiv (1+u_i^2+u_j^2)/2u_iu_j$.

E. Free-energy expression

By collecting the previous exact results up to order $\rho^{5/2}$, we find that, for sets of densities that satisfy the local neutrality relation $\Sigma_\alpha e_\alpha \rho_\alpha = 0$, we get Eqs. (16)–(18d). The difference, up to order $\rho^{5/2}$, between the exact volume densities f of free energies with or without $\mathbf{B}_0$ reads

$$\beta f(\beta,\{\rho_\alpha\},B_0)-\beta f(\beta,\{\rho_\alpha\},B_0=0)$$
$$=\sum_\alpha \rho_\alpha \ln\left(\frac{\sinh u_{C\alpha}}{u_{C\alpha}}\right)+\sum_\alpha \rho_\alpha \ln\left(\frac{(2S_\alpha+1)\sinh u_{S\alpha}}{\sinh[(2S_\alpha+1)u_{S\alpha}]}\right)-\frac{1}{2}\sum_\alpha \frac{(-1)^{2S_\alpha}}{2S_\alpha+1}[1+\beta\kappa_D e_\alpha^2]\rho_\alpha^2(4\pi\lambda_\alpha^2)^{3/2}$$
$$\times\int d\mathbf{r}\left[\frac{(2S_\alpha+1)\tanh u_{S\alpha}}{\tanh[(2S_\alpha+1)u_{S\alpha}]}\frac{\sinh u_{C\alpha}}{u_{C\alpha}}\langle -\mathbf{r}|e^{-\beta h_{\mathrm{rel},\alpha}(\mathbf{B}_0)}|\mathbf{r}\rangle-\langle -\mathbf{r}|e^{-\beta h_{\mathrm{rel},\alpha}(\mathbf{B}_0=\mathbf{0})}|\mathbf{r}\rangle\right]$$
$$-\frac{1}{2}\sum_{\alpha,\gamma}[1+\beta\kappa_D e_\alpha e_\gamma]\rho_\alpha\rho_\gamma(2\pi\lambda_\alpha\lambda_\gamma)^3\int d\mathbf{r}\left[\frac{\sinh u_{C\alpha}}{u_{C\alpha}}\frac{\sinh u_{C\gamma}}{u_{C\gamma}}\langle\mathbf{0},\mathbf{r}|e^{-\beta H_{\alpha\gamma}(\mathbf{B}_0)}|\mathbf{0},\mathbf{r}\rangle-\langle\mathbf{0},\mathbf{r}|e^{-\beta H_{\alpha\gamma}(\mathbf{B}_0=\mathbf{0})}|\mathbf{0},\mathbf{r}\rangle\right]$$
$$+\frac{1}{6}\frac{\beta\hbar c}{B_0}\kappa_D^3\sum_\alpha \rho_\alpha e_\alpha L^{[3]}(\beta\mu_{B\alpha}B_0)+o(\rho^{5/2}). \tag{158}$$

In the direct terms, the difference between quantities with $B_0\neq 0$ and $B_0=0$ automatically performs the truncations needed for extended states, and the corresponding contribution at order ρ^2 is the same as in the case of short-ranged interactions.

The pressure can be derived from Eqs. (16)–(18d) by using the thermodynamic relation $P=\Sigma_\alpha\rho_\alpha(\partial f/\partial\rho_\alpha)-f$. Terms of the form $F_n^{\{p\}}\equiv\kappa_D^n\Sigma_{\alpha_1,\ldots,\alpha_p}\rho_{\alpha_1}\times\cdots\times\rho_{\alpha_p}B_{\alpha_1,\ldots,\alpha_p}$ in f are just multiplied by $[(n/2)+p-1]$ in the pressure P, whereas terms with the structure $(\ln\kappa_D)F_n^{\{p\}}$ in f lead to $\{(1/2)+[(n/2)+p-1]\ln\kappa_D\}F_n^{\{p\}}$ in P. Eventually,

$$\beta P=\sum_\alpha \rho_\alpha-\frac{1}{24\pi}\kappa_D^3 \tag{159a}$$
$$-\frac{1}{2}\sum_\alpha (-1)^{2S_\alpha}\left(1+\frac{3}{2}\beta\kappa_D e_\alpha^2\right)\frac{\tanh u_{S\alpha}}{\tanh[(2S_\alpha+1)u_{S\alpha}]}\rho_\alpha^2\frac{\sinh u_{C\alpha}}{u_{C\alpha}}(4\pi\lambda_\alpha^2)^{3/2}\int d\mathbf{r}\langle -\mathbf{r}|e^{-\beta h_{rel,\alpha}}|\mathbf{r}\rangle \tag{159b}$$
$$-\frac{1}{2}\sum_{\alpha,\gamma}\left(1+\frac{3}{2}\beta\kappa_D e_\alpha e_\gamma\right)\rho_\alpha\rho_\gamma\lim_{R\to\infty}\left\{\int_{r<R}d\mathbf{r}\left[\frac{\sinh u_{C\alpha}}{u_{C\alpha}}\frac{\sinh u_{C\gamma}}{u_{C\gamma}}(2\pi\lambda_\alpha\lambda_\gamma)^3\langle\mathbf{0},\mathbf{r}|e^{-\beta H_{\alpha\gamma}}|\mathbf{0},\mathbf{r}\rangle-1+\frac{\beta e_\alpha e_\gamma}{r}-\frac{(\beta e_\alpha e_\gamma)^2}{2r^2}\right]\right.$$
$$\left.+\frac{2\pi}{3}(\beta e_\alpha e_\gamma)^3\ln(\kappa_D R)\right\}-\frac{\pi}{3}\left[\frac{1}{2}+C+\ln 3\right]\beta^3\left(\sum_\alpha \rho_\alpha e_\alpha^3\right)^2+\pi\left[\frac{1}{3}-\frac{C}{2}-\ln 2\right]\beta^4\kappa_D\left(\sum_\alpha \rho_\alpha e_\alpha^4\right)^2 \tag{159c}$$
$$+\frac{1}{16}\hbar^2\beta^2\kappa_D^3\sum_\alpha \rho_\alpha\frac{e_\alpha^2}{m_\alpha}+\frac{1}{4}\frac{\beta\hbar c}{B_0}\kappa_D^3\sum_\alpha \rho_\alpha e_\alpha L^{[3]}(u_{C\alpha}) \tag{159d}$$
$$+C_1\beta^5\frac{1}{\kappa_D}\left(\sum_\alpha \rho_\alpha e_\alpha^3\right)^2\left(\sum_\gamma \rho_\gamma e_\gamma^4\right)+C_2\beta^6\frac{1}{\kappa_D^3}\left(\sum_\alpha \rho_\alpha e_\alpha^3\right)^4. \tag{159e}$$

When $\mathbf{B}_0=\mathbf{0}$, we retrieve the result given in [4].

VI. CASE OF THE ONE-COMPONENT PLASMA

A. From the two-component plasma to the OCP

1. Limit procedure

The free energy of the OCP is derived from the formulas valid for a two-component plasma (TCP) by the procedure used in Ref. [4]. First, the mass of one given species goes to infinity so that the positions of the corresponding particles are fixed; then its charge vanishes as its density becomes infinite while their product is kept constant so that global neutrality remains valid. This second step ensures that the fixed particles turn into a uniform neutralizing background. In this section, we will call (ρ_1,m_1,e_1) $[(\rho_2,m_2,e_2)]$ the density, mass, and charge of the light [heavy)] particles. The procedure has already been tested successfully in the absence of any magnetic field. In this case its results have been compared with those of a direct derivation of the low-density free energy for the OCP through Mayer expansions [16]. Moreover, the classical terms for the OCP that are derived through this procedure in Ref. [4] coincide with those of Ref. [14] which are directly calculated for the OCP.

In the limit where m_2 goes to infinity, the heavy particles become classical because m_2 always appears through the ratio $m_2/\hbar$. The matrix elements involving species 2 can be derived in this limit from the generalized Wigner-Kirkwood expansions in the presence of a magnetic field of any intensity (see [8]). Indeed, if $H_{\alpha\gamma}$ is written as $h^{(0)}_{B_0,\gamma}+V_{\mathbf{r}_\alpha}(\mathbf{r}_\gamma)$ where $h^0_{B_0,\gamma}(\mathbf{r}_\gamma,\boldsymbol{\nabla}_{\mathbf{r}_\gamma})$ is the Hamiltonian (43) of a heavy particle of species γ alone in the magnetic field, then in the classical limit for the heavy particle, according to Eq. (5.2) of Ref. [8],

$$\lim_{m_\gamma\to\infty}\frac{\sinh u_{C_\gamma}}{u_{C_\gamma}}(2\pi\lambda_\gamma^2)^{3/2}\times\langle\mathbf{r}_\gamma|\exp\{-\beta[h^{(0)}_{B_0,\gamma}(\mathbf{r}_\gamma)+V_{\mathbf{r}_\alpha}(\mathbf{r}_\gamma)]\}|\mathbf{r}_\gamma\rangle=\exp[-\beta V_{\mathbf{r}_\alpha}(\mathbf{r}_\gamma)][1+O(\hbar^2)]. \tag{160}$$

In the right-hand side of Eq. (160) $\mathbf{r}_\gamma$ is no longer an operator, but $V_{\mathbf{r}_\alpha}(\mathbf{r}_\gamma)$ may involve operators acting on the coordinates $\mathbf{r}_\alpha$ of the light particle.

2. Straightforward limits

When m_2 goes to infinity, λ_2 goes to zero as well as u_{C_2}. In this limit, the purely classical terms (18c) are unchanged, and when e_2 goes to zero the summation over α and γ reduces to the contribution from one species. The diffraction contributions (18d) from the heavy particles, which are proportional to λ_2^2 at u_{C_2} fixed, vanish while those from the light particles remain unaltered. For light particles, the exchange term remains unchanged, whereas it tends to its classical limit in the case of heavy particles.

Since the variables of the center of mass and of the relative particle are decoupled for identical particles, even in the presence of $\mathbf{B}_0$, they can be separated. According to Eq. (21) the exchange (direct) term for two heavy particles of species 2 is proportional to $(\sinh u_{C_2}/u_{C_2})$ times $(4\pi\lambda_2^2)^{3/2}\langle\mathbf{r}'|\exp[-\beta h_{\mathrm{rel},2}]|\mathbf{r}\rangle$ with $\mathbf{r}'=-\mathbf{r}$ $(\mathbf{r}'=\mathbf{r})$. In the classical limit, u_{C_2} goes to zero, the term $\sinh u_{C_2}/u_{C_2}$ tends to 1. The exchange integral $\int d\mathbf{r}\langle-\mathbf{r}|\exp[-\beta h_{\mathrm{rel},2}]|\mathbf{r}\rangle$ indeed vanishes. It becomes exponentially small when $\hbar$ goes to zero; more precisely, this was shown for small B_0 in Ref. [9] and for infinite B_0 and in two dimensions in Ref. [8]. For $\langle\mathbf{r}|\exp[-\beta h_{\mathrm{rel},2}]|\mathbf{r}\rangle$, the classical limit of the contribution from the relative particle is obtained from Eq. (160) where there is no particle of species α and the role of species γ is played by the relative particle with mass $m_2/2$, charge $e_2/2$, and $V(\mathbf{r}_2)=e_2^2/r$. Then the classical Boltzmann factor is retrieved,

$$\lim_{m_2\to\infty}\left(\frac{\sinh u_{C_2}}{u_{C_2}}\right)^2(2\pi\lambda_2^2)^3\langle\mathbf{0},\mathbf{r}|e^{-\beta H_{22}}|\mathbf{0},\mathbf{r}\rangle=e^{-\beta e_2^2/r}. \tag{161}$$

By inserting Eq. (161) in the truncated integral of Eq. (18b) and taking the limit e_2 going to zero, we obtain that the direct term for species 2 vanishes.

3. Direct term with two different species

In the case of the direct terms involving the two species 1 and 2, there is no separation of variables, but when species 2 becomes classical, an effective separation turns out through Eq. (160) where the role of species α (γ) is played by 1 (2) and $V_{\mathbf{r}_\alpha}(\mathbf{r}_\gamma)=h^{(0)}_{B_0,1}+e_1e_2/r$. By inserting Eq. (160) into $Q^*_{12}(n\kappa_D)$ given by Eq. (110) we get

$$\lim_{e_2\to0}\Big|_{e_2\rho_2=-e_1\rho_1}\left[\lim_{m_2\to\infty}\rho_1\rho_2Q^*_{12}(n\kappa_D)\right]=\lim_{e_2\to0}\Big|_{e_2\rho_2=-e_1\rho_1}\rho_1\rho_2A_{12}, \tag{162}$$

with

$$A_{12}=\int_{r<R}d\mathbf{r}\left\{(2\pi\lambda_1^2)^{3/2}\frac{\sinh u_{C_1}}{u_{C_1}}\times\langle\mathbf{r}|e^{-\beta[h^{(0)}_{B_0,1}+e_1e_2/r]}|\mathbf{r}\rangle-1+\frac{\beta e_1e_2}{r}\right\}, \tag{163}$$

with $\lambda_1=\sqrt{\beta\hbar^2/m_1}=\lim_{m_2\to\infty}\lambda_{12}$, where λ_{12} is associated with the relative particle with reduced mass $\sqrt{m_1m_2/(m_1+m_2)}$.

In order to study the limit of A_{12} when e_2 vanishes, we use the expansion of Dyson equation up to order e_2,

$$e^{-\beta h^{(0)}_{B_0,1}}-\beta\int_0^1ds\,e^{-\beta(1-s)h^{(0)}_{B_0,1}}\frac{e_1e_2}{r}e^{-\beta s h^{(0)}_{B_0,1}}+O(e_2^2), \tag{164}$$

where $h^{(0)}_{B_0,1}$ and $1/r$ denote operators. (In fact the small dimensionless parameter is $\Gamma_{12}=\beta e_1e_2/a\ll1$). According to Eq. (42) and the closure relation $\int d\mathbf{r}'|\mathbf{r}'\rangle\langle\mathbf{r}'|=\mathcal{I}$ (where $\mathcal{I}$ is the identity operator), the e_2 expansion of A_{12} reads

$$A_{12}=-\beta e_1e_2\int_0^1ds\int_{r<R}d\mathbf{r}\int d\mathbf{r}'\left[\frac{1}{r'}-\frac{1}{r}\right]\times(2\pi\lambda_1^2)^{3/2}\frac{\sinh u_{C_1}}{u_{C_1}}\langle\mathbf{r}|e^{-\beta(1-s)h^{(0)}_{B_0,1}}|\mathbf{r}'\rangle\times\langle\mathbf{r}'|e^{-\beta s h^{(0)}_{B_0,1}}|\mathbf{r}\rangle+O(e_2^2). \tag{165}$$

We set $\mathbf{r}'=\mathbf{r}+\mathbf{t}$ and make a Taylor expansion of $(1/r')-(1/r)$ around $1/r$. After integration over orientations of $\mathbf{r}$, all derivatives eventually lead to $\Delta^p(1/r)$ terms. Equation (82) can be generalized to an integral over a finite volume with the result $\int_{r<R}d\mathbf{r}\Delta^p(1/r)=0$ for $p\geqslant2$, already used in Eq. (97). Eventually,

$$A_{12}=-\beta\,e_1e_2\int_0^1ds\int_{r<R}d\mathbf{r}\frac{1}{2}\sum_\mu\,g_\mu(s)\partial_{\mu\mu}\left(\frac{1}{r}\right)+O(e_2^2), \tag{166}$$

with, according to Eq. (42),

$$g_\mu(s)\equiv(2\pi\lambda_1^2)^{3/2}\frac{\sinh u_{C_1}}{u_{C_1}}\int d\mathbf{t}[\mathbf{t}_\mu]^2\langle\mathbf{0}|e^{-\beta(1-s)h^{(0)}_{B_0,1}}|\mathbf{t}\rangle\times\langle\mathbf{t}|e^{-\beta s h^{(0)}_{B_0,1}}|\mathbf{0}\rangle=\frac{\langle\mathbf{0}|\exp[-\beta h^{(0)}_{B_0,1}][x_H(s)]^2|\mathbf{0}\rangle}{\langle\mathbf{0}|\exp[-\beta h^{(0)}_{B_0,1}]|\mathbf{0}\rangle}, \tag{167}$$

where $x_H(s)$ is the position operator in Heisenberg representation at ''imaginary time'' s. According to the relation between the path integral of a function and the average value of the corresponding operator in Heisenberg representation (see, for instance, page 174 of Ref. [17]),

$$\langle \mathbf{0}|\exp[-\beta h^{(0)}_{B_0,1}][x_H(s)]^2|\mathbf{0}\rangle = \lambda_1^2 \mathrm{cov}_{\mu\mu}(s,s;u_{C_1})\langle \mathbf{0}|\exp[-\beta h^{(0)}_{B_0,1}|\mathbf{0}\rangle, \tag{168}$$

where $\mathrm{cov}_{\mu\mu}(s,s;u_{C_1})$ is defined in Eq. (45). Moreover, by using the identity $\int d\mathbf{r}\partial_{\mu\mu}(1/r) = -4\pi/3$ (without any implicit summation over the index μ), we find a result similar to the diffraction term (98) in $\int d\mathbf{r}\int D(\boldsymbol{\xi}_1)\int D(\boldsymbol{\xi}_2)F_{RT}(\mathbf{r},\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)$,

$$A_{12} = \beta e_1 e_2 2\pi\lambda_1^2\left[C_0 + \frac{2}{3}\delta C(u_{C_1})\right] + O(e_2^2). \tag{169}$$

As a conclusion, in the limit where e_2 goes to zero with the neutrality constraint $\rho_2 e_2 = -\rho_1 e_1$, $\rho_1\rho_2 Q^*_{12}(n\kappa_D)$ leads to a nonvanishing diffraction term whereas $e_1 e_2 \kappa_D \rho_1\rho_2 Q^*_{12}(n\kappa_D)$ disppears. Henceforth, in the case of the OCP, the diffraction term is nonzero at order ρ^2, whereas this term does not appear in the free energy of a multicomponent plasma because of the neutrality equation for the densities of moving particles. On the contrary, diffraction contributions at order $\rho^{5/2}$ have the same structure in both kinds of systems. We notice that this section is a demonstration of the result

$$\lim_{e_2\to 0}\Big|_{e_2\rho_2 = -e_1\rho_1}\left[\lim_{m_2\to\infty}\rho_1\rho_2 Q^*_{12}(n\kappa_D)\right] = \hbar^2\rho_1^2\frac{\pi}{3}\beta^2\frac{e_1^2}{m_1}\left[\frac{1}{2} + \frac{1}{u_{C_1}}L^{[3]}(u_{C_1})\right]. \tag{170}$$

4. OCP free energy

From now on, we change the notation (ρ_1, m_1, e_1) into (ρ, m, e). In the absence of $\mathbf{B}_0$, the quantities Q^* and E^* introduced in the calculations of the present paper differ from Q and E introduced in Ref. [7] only by a multiplicative factor $1/(4\pi\lambda^3)$ and an additive constant in the case of Q. According to the operator representation of Q^* given in Eq. (110),

$$Q\left(-\frac{\beta e^2}{\lambda}, u_C\right) = \frac{1}{4\pi\lambda^3}\left[Q^*(3\kappa_D) - \frac{2\pi}{3}\beta^3 e^6 \ln(\kappa_D\lambda)\right] \tag{171}$$

while, according to Eqs. (21) and (65), $E(-\beta e^2/\lambda, u_C) = (1/4\pi\lambda^3)E^*$.

In order to point out the difference arising from the presence of $\mathbf{B}_0$ we may write Eq. (26) as

$$\beta f^{\mathrm{OCP}}(\beta,\rho,B_0) - \beta f^{\mathrm{OCP}}(\beta,\rho,B_0=0) = \rho\ln\left(\frac{(2S+1)\sinh u_S}{\sinh[(2S+1)u_S]}\right) + \rho\ln\left(\frac{\sinh u_C}{u_C}\right) + 2\pi\frac{(-1)^{2S+1}}{2S+1}\rho^2\lambda^3[1+\beta\kappa_D e^2]$$
$$\times\left[\frac{(2S+1)\tanh u_S}{\tanh[(2S+1)u_S]}E(-\beta e^2/\lambda, u_C) - E(-\beta e^2/\lambda, u_C=0)\right] \tag{172a}$$
$$-2\pi\rho^2\lambda^3[1+\beta\kappa_D e^2][Q(-\beta e^2/\lambda, u_C) - Q(-\beta e^2/\lambda, u_C=0)] \tag{172b}$$
$$+\frac{2\pi}{3}\frac{\beta^2\hbar^2}{m}\rho^2 e^2\left(1+\frac{1}{2}\beta\kappa_D e^2\right)\frac{1}{u_C}L^{[3]}(u_C) + O(\rho^3\ln\rho). \tag{172c}$$

Up to order $\rho^{5/2}$, the results with or without $\mathbf{B}_0$ (see Ref. [4]) are similar to those for a multicomponent plasma (158), apart from the diffraction term (26e), which does not vanish at order ρ^2 (see the end of the preceding section).

We notice that the origin of the diffraction terms in the method of Ref. [3] is essentially the same, though it turns out in a different technical way. Indeed, the diffraction terms of a multicomponent plasma come both from bonds of the same nature as our bond F^{cm} and from the integration of $f_{T,g}$ over g, where $f_{T,g}$ is the value of f_T when the Coulomb interaction is multiplied by the dimensionless coupling parameter g. The latter integration over g, with $0 \leqslant g \leqslant 1$, involves calculations similar to those that we used to get the limit of the OCP, in particular an expression analogous to Eq. (165) is used.

B. Semiclassical limit for the OCP

In regimes of low degeneracy ($\lambda/a \leqslant 1$) and weak quantum dynamical effects at $u_C \equiv \beta\mu_B B_0$ fixed, the expression of the OCP free energy can be expanded with respect to $\hbar$, because the exchange density-matrix element in position space vanishes exponentially fast when $\hbar$ goes to zero as discussed above, and because the OCP has a well-defined thermodynamic limit even with MB statistics.

1. Semiclassical regime

The system is semiclassical for any value of a given coupling if the length scale λ_{qu} beneath which the quantum effects are important is negligible with respect to the smallest

length l that characterizes the coupling, whether the latter is weak or strong. The corresponding dimensionless parameter reads

$$t_{\mathrm{qu}} \equiv \left(\frac{\lambda_{\mathrm{qu}}}{l} \right)^2 . \tag{173}$$

There is no semiclassical parameter associated with the spin because the latter internal degree of freedom is intrinsically quantum.

Let us first consider a system with only one kind of interactions. Then λ_{qu} may be chosen to be equal to the amplitude λ_α of the quantum position fluctuations of free particles with MB statistics at temperature T. The semiclassical parameter t_{qu} which measures the importance of quantum dynamical effects for particles submitted to only one given interaction at a given temperature may be interpreted as the ratio

$$t_{\mathrm{qu}} = \frac{\varepsilon_{\mathrm{qu}}(l)}{\varepsilon_{\mathrm{th}}}, \tag{174}$$

where $\varepsilon_{\mathrm{qu}}(l)$ is the kinetic energy of the quantum dynamical position fluctuations with an amplitude l which is the smallest length characteristic of the interaction, while $\varepsilon_{\mathrm{th}}$ is the average kinetic energy of free particles at equilibrium at temperature T. $\varepsilon_{\mathrm{qu}}(l)$ is derived from the uncertainty principle and the form of the interaction, while, in a low-degeneracy regime, $\varepsilon_{\mathrm{th}}$ is given by the Maxwell-Boltzmann expression, $\varepsilon_{\mathrm{th}} = 1/\beta$.

In the case of Coulomb interaction, the two-body potential has no intrinsic characteristic length. When collective coulombic effects are taken into account, according to Eq. (7), the smallest length l associated with these effects at temperature $1/\beta$ is either the classical closest distance of approach $b_{\alpha\alpha}$ (when $\Gamma \ll 1$) or the screening length ξ_D (when $\Gamma \gg 1$).

In the case of the magnetic orbital interaction, there is no coupling between particles and l is merely the intrinsic length derived from the one-body interaction with the external field. From the point of view of statistical mechanics, the orbital magnetic interaction is essentially quantum in its fundamental origin, though the one-body problem may be accounted for by classical relativistic dynamics. Therefore the corresponding characteristic length l is chosen to arise from quantum instead of classical dynamics; namely, we use the characteristic quantum length $l_{C\alpha}$ in place of the classical thermal gyromagnetic radius $R_{C\alpha}$ defined after Eq. (8) and

$$t_{\mathrm{qu,\,mag}} \equiv \left(\frac{\lambda_\alpha}{l_{C\alpha}} \right)^2 = 2u_{C\alpha} . \tag{175}$$

[However, $R_{C\alpha}$ is the relevant scale for semiclassical expansions of thermodynamic quantities, because then the reference quantities used in the statistical framework are calculated with classical dynamics. Moreover, $\lambda_\alpha / R_{C\alpha}$ is equal to the square of $\lambda_\alpha / l_{C\alpha}$, as mentioned in Eq. (8), so that both parameters increase with the intensity B_0 of the magnetic field according to Eq. (4).]

We point out that when B_0 (i.e., $u_{C\alpha}$) is increased, the system becomes more and more quantum [i.e., $\lambda_\alpha / l_{C\alpha}$ increases, according to Eq. (9), and $t_{\mathrm{qu,\,mag}}$ gets larger]. However, at the same time the length λ_{qu} beneath which quantum dynamics is crucial changes from λ_α to $l_{C\alpha}$ in the plane perpendicular to B_0 [8], because, when the magnetic field is very strong, the particles are in the lowest Landau level while the radius $l_{C\alpha}$ of the Landau orbits is negligible with respect to λ_α, according to (9). Then the Feynman-Kac-Itô integral is controlled by the phase factor, whose amplitude is proportional to $u_{C\alpha}$ and oscillates very fast, as discussed in Ref. [8]. In the plane perpendicular to the axis of $\mathbf{B}_0$, the Landau orbits behave as heavy point particles with mass m_α times $u_{C\alpha}/\tanh u_{C\alpha}$ that increases with B_0 and only the z component of the quantum fluctuations does survive.

Subsequently, when both Coulomb and magnetic interactions are taken into account, a semiclassical regime shows up for any values of the coupling parameters Γ and $u_{C\alpha}$ when the four following conditions are met. If the magnetic field B_0 is weak ($u_{C\alpha} \ll 1$), quantum fluctuations appear over scales smaller than or of order $\lambda_{\mathrm{qu}} = \lambda_\alpha$. On the other hand, according to Eq. (7), the smallest length l associated with Coulomb interaction is $l = b_{\alpha\alpha}$ when $\Gamma \ll 1$ and $l = \xi_D$ when $\Gamma \gg 1$. Thus the system is semiclassical for any strength of the Coulomb coupling if both $\lambda_\alpha \ll b_{\alpha\alpha}$ and $\lambda_\alpha \ll \xi_D$, namely,

$$\frac{\lambda_\alpha}{a} \ll \Gamma_{\alpha\gamma}, \tag{176a}$$

$$\left(\frac{\lambda_\alpha}{a} \right)^2 \ll \frac{1}{\Gamma_{\alpha\alpha}} . \tag{176b}$$

These two inequalities are indeed satisfied in the semiclassical calculations of Ref. [9] which are performed in a limit of weak magnetic field. On the other hand, when the magnetic field is strong, quantum phenomena in the plane perpendicular to $\mathbf{B}_0$ show up over scales smaller than $\lambda_{\mathrm{qu}} = l_{C\alpha}$. Thus, by changing λ_α into $l_{C\alpha}$ in the previous argument and by using Eq. (8), the semiclassical conditions (176a) and (176b) become

$$\frac{\lambda_\alpha}{a} \ll \Gamma_{\alpha\alpha} \sqrt{u_{C\alpha}}, \tag{177a}$$

$$\left(\frac{\lambda_\alpha}{a} \right)^2 \ll \frac{1}{\Gamma_{\alpha\alpha}} u_{C\alpha} . \tag{177b}$$

2. Semiclassical and low-density limits

According to the preceding section, conditions for low degeneracy, $(\lambda/a)^2 \ll 1$, weak Coulomb coupling, and weakly quantum dynamics at u_C fixed may be fulfilled simultaneously. In the semiclassical expansions of Ref. [8], statistics is that of Maxwell-Boltzmann and dynamics is weakly quantum for any strength of the Coulomb and orbital magnetic couplings: conditions (176a), (176b), (177a), and (177b) are satisfied. On the other hand, in our low-density expansions, which correspond to low degeneracy and weak Coulomb coupling regimes, dynamics is fully quantum and magnetic coupling is arbitrary. Thus our low-density results may be expanded in powers of $\hbar$ at u_C fixed in situations where Eqs. (176a) and (177a) are valid. Thus it is legitimate to compare the double expansions in ρ and $\hbar$ obtained by

making either $\hbar$ or ρ go first to zero when Eqs. (176a) and (177a) are fulfilled. In this section we omit the superscript OCP in f in order to simplify notations.

First, the exchange terms are exponentially small when $\hbar$ goes to zero in the absence [11] as well as in the presence [8,9] of $\mathbf{B}_0$. Moreover, the strict weak-coupling limit of the semiclassical formula for exchange effects coincides with the first-order term in the low-density exchange contribution (26b). First, we consider the strict low-density limit of the result (26b). In a strict weak-coupling limit $1+\beta\kappa_D e^2$ tends to 1 (the collective effects disappear) and the latter exchange contribution tends to

$$\beta f_{\rm exch}=\rho^2(2\sqrt{\pi}\lambda)^3\frac{\tanh u_S}{\tanh[(2S+1)u_S]}\frac{\sinh u_C}{u_C}\frac{1}{2}\times\int d\mathbf{r}\langle-\mathbf{r}|e^{-\beta h_{\rm rel}}|\mathbf{r}\rangle+O(\rho^{5/2}). \tag{178}$$

On the other hand, if we generalize the semiclassical formula (5.10) of Ref. [9] which is derived for $S=1/2$ and small u_C to an expression valid for any S and a finite value of u_C, the contribution from exchange effects to the free energy at finite density when $\hbar$ goes to zero at u_C fixed reads

$$\beta f_{\rm exch}=\rho^2 e^C(2\sqrt{\pi}\lambda)^3\frac{\tanh u_S}{\tanh[(2S+1)u_S]}\frac{1}{2}\frac{\sinh u_C}{u_C}\times\int d\mathbf{r}\langle-\mathbf{r}|e^{-\beta h_{\rm rel}}|\mathbf{r}\rangle[1+O_{u_C}(\hbar^2)]. \tag{179}$$

In Eq. (179) $O_{u_C}(\hbar^2)$ is to be understood as a term of order $\hbar^2$ times a function of u_C which remains finite when u_C goes to zero. C is related to the short-ranged behavior of the classical pair distribution function and takes the many-body effects into account. In the low-density (weak Coulomb coupling) limit C vanishes and we get the announced result.

Now, we turn to terms of other kinds, namely, the part $f^{\rm MB}$ of f that is calculated with MB statistics. We show that up to order $\rho^{5/2}\hbar^2$ the ρ expansion of the semiclassical result up to order $\hbar^2$ coincides with the $\hbar$ expansion of the low-density result up to order $\rho^{5/2}$. First, we consider the double expansion where we expand the free energy with respect to ρ then to $\hbar$. According to Eq. (26) the low-density expansion of $f^{\rm MB}$ takes the form

$$f^{{\rm MB}\{\leqslant 5/2\}}(\beta,\rho,B_0)-f_{\rm para}(\beta,\rho,B_0)-f_{\rm dia}(\beta,\rho,B_0)=f_{\rm cl}^{\{\leqslant 5/2\}}(\beta,\rho)+\Delta_{\rm qu}f_{\rm dir}^{\{\leqslant 5/2\}}(\beta,\rho,B_0)+\Delta_{\rm qu}f_{\rm diff}^{\{\leqslant 5/2\}}(\beta,\rho,B_0), \tag{180}$$

where $f^{\{\leqslant n\}}$ denotes the low-density expansion of f up to order ρ^n included, and the quantum corrections read

$$\beta\Delta_{\rm qu}f_{\rm dir}^{\{\leqslant 5/2\}}(\beta,\rho,B_0)\equiv-\rho^2[1+\beta\kappa_D e^2]\left\{2\pi\lambda^3 Q\left(\frac{-\beta e^2}{\lambda},u_C\right)-\lim_{\hbar\to 0}2\pi\lambda^3 Q\left(\frac{-\beta e^2}{\lambda},u_C=0\right)\right\} \tag{181}$$

and

$$\beta\Delta_{\rm qu}f_{\rm diff}^{\{\leqslant 5/2\}}(\beta,\rho,B_0)\equiv\rho^2\left[1+\frac{1}{2}\beta\kappa_D e^2\right]\frac{\pi}{3}\frac{\beta\hbar^2}{m}e^2\left[1+\frac{2}{u_C}L^{(3)}(u_C)\right]. \tag{182}$$

Let us consider the semiclassical limit of the direct term $\beta\Delta_{\rm qu}f_{\rm dir}^{\{\leqslant 5/2\}}$. The $\hbar$ expansion at u_C fixed of the diagonal matrix element for a particle with mass $m/2$ and charge $e/2$ in a potential e^2/r and submitted to $\mathbf{B}_0$ is given by formula (5.2) of Ref. [8] where u_C is unchanged and λ^2 is multiplied by 2. This formula gives the correction of order $\hbar^2/m$ at Γ and u_C fixed to formula (160) of the present paper. When $\mathbf{B}_0=\mathbf{0}$, it reads

$$(4\pi\lambda^2)^{3/2}\langle\mathbf{r}|e^{-\beta h_{{\rm rel},\mathbf{B}_0=\mathbf{0}}}|\mathbf{r}\rangle=e^{-\beta e^2/r}\left(1+\frac{\hbar^2\beta^2e^2}{12m}\left\{\beta e^2\left[\boldsymbol{\nabla}\left(\frac{1}{r}\right)\right]^2-2\Delta\left(\frac{1}{r}\right)\right\}\right)+O(\hbar^4). \tag{183}$$

When $\mathbf{B}_0\neq\mathbf{0}$, it may be expressed as the sum

$$(4\pi\lambda^2)^{3/2}\frac{\sinh u_C}{u_C}\langle\mathbf{r}|e^{-\beta h_{{\rm rel},B_0}}|\mathbf{r}\rangle=(4\pi\lambda^2)^{3/2}\langle\mathbf{r}|e^{-\beta h_{{\rm rel},\mathbf{B}_0=\mathbf{0}}}|\mathbf{r}\rangle+e^{-\beta e^2/r}\frac{\hbar^2\beta^2e^2}{4m}\times\frac{1}{u}L^{[3]}(u)\left\{\beta e^2\left[\boldsymbol{\nabla}_\perp\left(\frac{1}{r}\right)\right]^2-2\Delta_\perp\left(\frac{1}{r}\right)\right\}+O_{u_C}(\hbar^4). \tag{184}$$

We recall that, by definition, $L^{[3]}(u_C=0)=0$. By using

$$\int d\mathbf{r}\left[\partial_\mu\left(\frac{1}{r}\right)\right]^2e^{-\beta e^2/r}=\frac{1}{3}\int d\mathbf{r}\left[\boldsymbol{\nabla}\left(\frac{1}{r}\right)\right]^2e^{-\beta e^2/r}=\frac{4\pi}{3}\frac{e^2}{\beta} \tag{185}$$

and

$$\int d\mathbf{r}\,\partial_{\mu\mu}\left(\frac{1}{r}\right)e^{-\beta e^2/r}=\frac{1}{3}\int d\mathbf{r}\Delta\left(\frac{1}{r}\right)e^{-\beta e^2/r}=0 \tag{186}$$

together with the definition (24) of Q, the formulas (183) and (184) lead to

$$4\pi\lambda^3 Q\left(-\frac{\beta e^2}{\lambda},u_C=0\right)-\lim_{\hbar\to 0}\left[4\pi\lambda^3 Q\left(-\frac{\beta e^2}{\lambda},u_C=0\right)\right]=\hbar^2\frac{\pi}{3}\frac{\beta^2e^2}{m}+O(\hbar^4) \tag{187}$$

and

$$4\pi\lambda^3\left[Q\left(-\frac{\beta e^2}{\lambda},u_C\right)-Q\left(-\frac{\beta e^2}{\lambda},u_C=0\right)\right]=\hbar^2\frac{2\pi}{3}\frac{\beta^2e^2}{m}\frac{1}{u_C}L^{[3]}(u_C)+O_{u_C}(\hbar^4). \tag{188}$$

As a consequence, we get

$$\Delta_{\rm qu} f_{\rm dir}^{\{\leqslant 5/2\}}(\beta,\rho,B_0) = -\hbar^2\rho^2[1+\beta\kappa_D e^2]\frac{\pi}{6}\frac{\beta^2 e^2}{m}\left[1+\frac{2}{u_C}L^{[3]}(u_C)\right] + O_{u_C}(\rho^2\hbar^4). \quad (189)$$

Equation (189) shows one that in the semiclassical limit, the difference between the direct term with or without $\mathbf{B}_0$ and its classical corresponding contribution up to order $\rho^{5/2}$ generates diffraction terms at order $\hbar^2$. When the latter ones are added to the diffraction contributions already present in $\beta\Delta f_{\rm diff}^{\{\leqslant 5/2\}}$ given by Eq. (182), the diffraction terms at order $\hbar^2\rho^{5/2}$ cancel each other. Eventually, the $\hbar$ expansion of Eq. (180) starts as

$$\beta[f^{\rm MB\{\leqslant 5/2\}} - f_{\rm para} - f_{\rm dia}] = \beta f_{\rm cl}^{\{\leqslant 5/2\}} + \hbar^2\rho^2\frac{\pi}{6}\frac{\beta^2 e^2}{m}\left[1+\frac{2}{u_C}L^{[3]}(u_C)\right] + O_{u_C}(\rho^2\hbar^4). \quad (190)$$

On the other hand, the $\hbar$ expansion of the free energy around its classical value $f_{\rm cl}$ at finite density may be expressed from Eqs. (5.4)–(5.9) of Ref. [8] as

$$\beta[f^{\rm MB} - f_{\rm para} - f_{\rm dia}] = \beta f_{\rm cl} + \frac{\pi}{6}\beta^2\hbar^2\rho^2\frac{e^2}{m}\left[1+\frac{2}{u_C}L^{[3]}(u_C)\right] + O_{u_C}(\hbar^4), \quad (191)$$

where the term of order $\hbar^2$ is exactly of order ρ^2. Comparison of Eqs. (190) and (191) shows one that the ρ expansion up to order $\rho^{5/2}$ of the semiclassical $f^{\rm MB}$ given in Eq. (191) up to order $\hbar^2$ coincides with the $\hbar$ expansion (190) up to order $\hbar^2$ of $f^{\rm MB\{5/2\}}$. As a conclusion, we have checked that the double expansion with respect to ρ and $\hbar$ of the free energy is independent of the order in which the two expansions are performed.

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APPENDIX A

There are three different methods that allow one to obtain the values (47) and (48) of the covariances for independent particles in a magnetic field. The first method provides one only with $\mathrm{cov}_{xx}^{\alpha}(s,s';\boldsymbol{B}_0)$ and may be found in [18] as a part of a more intricate calculation. We summarize it very quickly and we give two other methods which we have devised.

(a) In the first method, the two-dimensional measure

$$D(\xi_x)D(\xi_y)\exp\left(iC\int_0^1[\xi_x d\xi_y(s)-\xi_y d\xi_x(s)]\right), \quad (A1)$$

with $C\equiv e_\alpha\lambda_\alpha^2 B_0/2\hbar c$, is expressed as an effective one-dimensional Gaussian measure

$$D(\xi_x)\exp\left[-\int_0^1 ds\int_0^1 ds'\,\xi_x(s)\hat{A}(s,s')\xi_x(s')\right] \quad (A2)$$

and $\mathrm{cov}_{xx}^{\alpha}(s,s';\mathbf{B}_0)$ is identified as the Green function $K(s,s')$ of the quadratic operator $\hat{A}(s,s')$. The calculations can be made explicitly by using basic properties of the Gaussian measure $D(\boldsymbol{\xi})$. Indeed, a characteristic property of generalized Gaussian measure is the following. If $F[\xi_y]$ is a linear functional of ξ_y and $d\xi_y$,

$$\int D(\xi_y)\exp\{iF[\xi_y]\} = \exp\left\{-\frac{1}{2}\int D(\xi_y)(F[\xi_y])^2\right\}, \quad (A3)$$

where $\int D(\xi_y)(F[\xi_y])^2$ is in fact a function of the covariances $\int D(\xi_y)\xi_y(s)\xi_y(s')$, $\int D(\xi_y)d\xi_y(s)\xi_y(s')$, and $\int D(\xi_y)d\xi_y(s)d\xi_y(s')$. The expressions of the free covariances involving derivatives are derived from Eq. (46). By using the Itô lemma introduced in Sec. III B of Paper I, we get

$$D(\xi_x)\int D(\xi_y)\exp\left\{iC\int_0^1[\xi_x d\xi_y(s)-\xi_y d\xi_x(s)]\right\} = D(\xi_x)\exp\left\{-2C^2\int_0^1 ds\int_0^1 ds' \times[\delta(s-s')-1]\xi_x(s)\xi_x(s')\right\}. \quad (A4)$$

The quadratic form in the exponential of the right-hand side can be written in terms of an operator $\hat{A}(s,s')$ as in Eq. (A2). The corresponding Green function $K(s,s')$, such that $\int ds''\int ds'\hat{A}(s,s'')K(s'',s') = \delta(s-s')$, is the solution of the equation

$$-\frac{d^2}{ds^2}K(s,s') + 2C^2\left[K(s,s') - \int_0^1 ds''K(s'',s')\right] = \delta(s-s'), \quad (A5)$$

with $K(0,s') = K(s,0) = 0$.

(b) In the ''sources'' method, the covariance is derived as the second functional derivative of the generating functional $Z(\mathbf{E})$,

$$\lambda_\alpha^2\mathrm{cov}_{\mu\nu}^{\alpha}(s,s';\mathbf{B}_0) = \frac{1}{\beta^2}\left.\frac{\delta^2(Z[\mathbf{E}])}{\delta E_\mu(s)\,\delta E_\nu(s')}\right|_{\mathbf{E}=\mathbf{0}}, \quad (A6)$$

where $Z(\mathbf{E})$ is the integrated measure in the presence of an external field $\mathbf{E}$ that is linearly coupled to the field $\boldsymbol{\xi}$,

$$Z[\mathbf{E}] \equiv \int D_{\mathbf{B}_0}(\boldsymbol{\xi})\exp\left\{\lambda_\alpha\beta\int_0^1 ds''\,\mathbf{E}(s'')\cdot\boldsymbol{\xi}(s'')\right\}. \quad (A7)$$

This functional can be calculated explicitly when it is expressed as a path integral in the phase space (with positions

and momenta as variables) and when the gauge is chosen to be $\mathbf{A}=(0,B_0x,0)$. Indeed, with the latter choice, the integrals over the variables ξ_y and ξ_z are trivial and the problem is reduced to the calculation of the path integral of a one-dimensional harmonic oscillator in the presence of a uniform external force that depends on time s as on page 131 of Ref. [17]. The result may be expressed as the exponential of a quadratic form

$$Z[\mathbf{E}]\equiv\exp\left\{(\lambda_\alpha^2\beta^2/2)\int_0^1 ds\int_0^1 ds' E_\mu(s)E_\nu(s')K_{\mu\nu}^\alpha(s,s')\right\} \tag{A8}$$

so that $\mathrm{cov}_{\mu\nu}^\alpha(s,s';\boldsymbol{B}_0)=[K_{\mu\nu}^\alpha(s,s')+K_{\nu\mu}^\alpha(s',s)]/2$.

(c) In the third method, the covariance may be expressed in terms of time-displaced correlations for position operators in Heisenberg representation,

$$\lambda_\alpha^2\mathrm{cov}_{\mu\nu}^\alpha(s,s';\mathbf{B}_0)=\theta(s-s')G_{\mu\nu}(s,s')+\theta(s'-s)G_{\nu\mu}(s',s), \tag{A9}$$

where

$$G_{\mu\nu}(s,s')=\frac{\langle\mathbf{0}|e^{-\beta h_{\mathbf{B}_0,\alpha}^{(0)}}[\mathbf{r}_H(s)]_\mu[\mathbf{r}_H(s')]_\nu|\mathbf{0}\rangle}{\langle\mathbf{0}|e^{-\beta h_{\mathbf{B}_0,\alpha}^{(0)}}|\mathbf{0}\rangle}. \tag{A10}$$

The calculations can be made explicitly because the equations of motion for the position operators are those of a uniform rotation around the axis of $\mathbf{B}_0$. This method is the most elementary as regards its application. The details can be summarized in four steps as follows.

First, the calculation of the covariance is changed into the determination of matrix elements of operators in Heisenberg representation. According to Eqs. (42)–(45), and the relation between path integrals of functions and the corresponding operators in Heisenberg representation (see page 174 of Ref. [17]),

$$\begin{aligned}\langle\mathbf{r}_\star|e^{-\beta h_{\mathbf{B}_0,\alpha}^{(0)}}|\mathbf{r}_\star\rangle\{[\mathbf{r}_\star]_\mu[\mathbf{r}_\star]_\nu+\lambda_\alpha^2\mathrm{cov}_{\mu\nu}^\alpha(s,s';\mathbf{B}_0)\}&=\theta(s-s')\langle\mathbf{r}_\star|e^{-\beta h_{\mathbf{B}_0,\alpha}^{(0)}}[\mathbf{r}_H(s)]_\mu[\mathbf{r}_H(s')]_\nu|\mathbf{r}_\star\rangle\\&+\theta(s'-s)\langle\mathbf{r}_\star|e^{-\beta h_{\mathbf{B}_0,\alpha}^{(0)}}[\mathbf{r}_H(s')]_\nu[\mathbf{r}_H(s)]_\mu|\mathbf{r}_\star\rangle,\end{aligned} \tag{A11}$$

where $\langle\mathbf{r}_\star|\exp[-\beta h_{\mathbf{B}_0,\alpha}^{(0)}]|\mathbf{r}_\star\rangle$ is given by Eq. (42) and $\mathbf{r}_H(s)$ is the position operator in Heisenberg representation at the imaginary time $t=-i\beta\hbar s$. The operators without any subscript are in Schrödinger representation.

Then $\mathbf{r}_H(s)$ is determined by the equations of motion in Heisenberg representation in imaginary time. They read

$$\frac{d\mathbf{p}_H(s)}{ds}=\beta[h_H^{(0)}(s),\mathbf{p}_H(s)], \tag{A12a}$$

$$\frac{d\mathbf{r}_H(s)}{ds}=\beta[h_H^{(0)}(s),\mathbf{r}_H(s)]. \tag{A12b}$$

By introducing the velocity operator $\mathbf{v}=(1/m_\alpha)[\mathbf{p}-(e_\alpha/c)\mathbf{A}]$, the Hamiltonian can be rewritten as $h_{\mathbf{B}_0,\alpha}^{(0)}(s)=(m_\alpha/2)\mathbf{v}^2(s)$. For a uniform magnetic field $\mathbf{B}_0$, $[v_x,v_y]=i\hbar\omega_{C\alpha}/m_\alpha$ in any gauge [19] and in Schrödinger as well as in Heisenberg representation. In the symmetric gauge $\mathbf{A}=(1/2)\mathbf{B}_0\wedge\mathbf{r}$,

$$\frac{1}{-i\beta\hbar}\frac{d\mathbf{v}_H(s)}{ds}=-\omega_{C\alpha}\hat{\mathbf{B}}_0\wedge\mathbf{v}_H(s), \tag{A13a}$$

$$\frac{1}{-i\beta\hbar}\frac{d\mathbf{r}_H(s)}{ds}=\mathbf{v}_H(s). \tag{A13b}$$

Thus $\mathbf{r}_H(s)$ and $\mathbf{v}_H(s)$ turn around the axis $\hat{\mathbf{B}}_0$ with an imaginary frequency $i\beta\hbar\omega_{C\alpha}=2iu_{C\alpha}$,

$$\mathbf{x}_H(s)=\mathbf{x}+\frac{1}{\omega_{C\alpha}}\{-i[\mathbf{v}]_x\sinh(2u_{C\alpha}s)+[\mathbf{v}]_y[1-\cosh(2u_{C\alpha}s)]\}, \tag{A14}$$

$$\mathbf{y}_H(s)=\mathbf{y}+\frac{1}{\omega_{C\alpha}}\{-[\mathbf{v}]_x[1-\cosh(2u_{C\alpha}s)]-i[\mathbf{v}]_y\sinh(2u_{C\alpha}s)\}. \tag{A15}$$

Thus the calculation of the covariance reduces to the calculation of matrix elements $\langle\mathbf{r}_\star|\exp[-\beta h_{\mathbf{B}_0,\alpha}^{(0)}]O_1O_2|\mathbf{r}_\star\rangle$, where O_1 and O_2 are the components of either the position or the velocity Schrödinger operators.

The value of the thermal propagator between two noncoincident points may be found in the literature (see, for instance, [17]). For a magnetic field $\mathbf{B}_0=B_0\hat{\mathbf{e}}_z$, it reads

$$\begin{aligned}\langle\mathbf{r}_b|e^{-\beta h_{\mathbf{B}_0,\alpha}^{(0)}}|\mathbf{r}_a\rangle=&\frac{1}{(2\pi\lambda_\alpha^2)^{3/2}}\frac{u_{C\alpha}}{\sinh u_{C\alpha}}\\&\times\exp\left\{-\frac{1}{2\lambda_\alpha^2}\left[(z_b-z_a)^2+\frac{u_{C\alpha}}{\tanh u_{C\alpha}}\right.\right.\\&\times[(x_b-x_a)^2+(y_b-y_a)^2]\\&\left.\left.+i2u_{C\alpha}(x_by_a-x_ay_b)\right]\right\}.\end{aligned} \tag{A16}$$

Since the position operator is diagonal in the basis $|\mathbf{r}\rangle$, the matrix elements with $O_1=[\mathbf{r}]_\mu$ and $O_2=[\mathbf{r}]_\nu$ are just equal to $[\mathbf{r}_\star]_\mu[\mathbf{r}_\star]_\nu$ times $\langle\mathbf{r}_\star|\exp[-\beta h_{\mathbf{B}_0,\alpha}^{(0)}]|\mathbf{r}_\star\rangle$. When $O_1=[\mathbf{r}]_\mu$ and $O_2=[\mathbf{v}]_\nu$, we use the commutation relation $[[\mathbf{r}]_\mu,[\mathbf{v}]_\nu]=i(\hbar/m_\alpha)\delta_{\mu,\nu}$, the Hermiticity of operators and the result

$$\langle\mathbf{r}_\star|\mathbf{v}e^{-\beta h_{\mathbf{B}_0,\alpha}^{(0)}}|\mathbf{r}_\star\rangle=\mathbf{0}. \tag{A17}$$

When O_1 and O_2 are velocity components, we use the Hermiticity of operators again to reduce the calculation to that of the following matrix elements:

$$\langle \mathbf{r}_\star|[\mathbf{v}]_x^2 e^{-\beta h^{(0)}_{\mathbf{B}_0,\alpha}}|\mathbf{r}_\star\rangle = \langle \mathbf{r}_\star|[\mathbf{v}]_y^2 e^{-\beta h^{(0)}_{\mathbf{B}_0,\alpha}}|\mathbf{r}_\star\rangle = \frac{1}{\beta m_\alpha}\frac{u_{C\alpha}}{\tanh u_{C\alpha}}\langle \mathbf{r}_\star|e^{-\beta h^{(0)}_{\mathbf{B}_0,\alpha}}|\mathbf{r}_\star\rangle, \tag{A18}$$

$$\langle \mathbf{r}_\star|[\mathbf{v}]_x[\mathbf{v}]_y e^{-\beta h^{(0)}_{\mathbf{B}_0,\alpha}}|\mathbf{r}_\star\rangle = -\langle \mathbf{r}_\star|[\mathbf{v}]_y[\mathbf{v}]_x e^{-\beta h^{(0)}_{\mathbf{B}_0,\alpha}}|\mathbf{r}_\star\rangle = i\frac{1}{\beta m_\alpha}u_{C\alpha}\langle \mathbf{r}_\star|e^{-\beta h^{(0)}_{\mathbf{B}_0,\alpha}}|\mathbf{r}_\star\rangle. \tag{A19}$$

Finally, we get the results given in Eqs. (47) and (48).

APPENDIX B

The present appendix is devoted to the derivation of the contributions from classical diagrams to $\ln(\rho_\alpha/\rho_\alpha^{\mathrm{id}\star,\mathrm{MB}})$ at order $\rho^{3/2}$.

1. Contributions from diagrams of Fig. 5

Contributions from diagrams of Figs. 5(a), 5(b), and 5(c) are simply expressed in terms of Fourier tranform of convolutions that involve $\tilde{\phi}(\mathbf{q})$ and $\tilde{\phi}^2(\mathbf{q})$. We notice that $e_\alpha^n = \partial(\Sigma_\gamma \rho_\gamma e_\gamma^n)/\partial\rho_\alpha$. By using Eq. (129) and introducing the definition (153) we find

$$I_{5(\mathrm{a})} = -4\pi K_1\beta^5 \frac{\partial}{\partial\rho_\alpha}\left[\left(\sum_\gamma \rho_\gamma e_\gamma^3\right)^2\right]\left(\sum_{\gamma'}\rho_{\gamma'}e_{\gamma'}^4\right)\frac{1}{\kappa_D}, \tag{B1}$$

$$I_{5(\mathrm{b})} = -4\pi K_1\beta^5\left(\sum_\gamma \rho_\gamma e_\gamma^3\right)^2 \frac{\partial}{\partial\rho_\alpha}\left[\sum_{\gamma'}\rho_{\gamma'}e_{\gamma'}^4\right]\frac{1}{\kappa_D}, \tag{B2}$$

$$I_{5(\mathrm{c})} = -8\pi K_2\beta^5\left(\sum_\gamma \rho_\gamma e_\gamma^3\right)^2\left(\sum_{\gamma'}\rho_{\gamma'}e_{\gamma'}^4\right)\frac{\partial}{\partial\rho_\alpha}\left[\frac{1}{\kappa_D}\right]. \tag{B3}$$

The contribution from the diagram 5(d) contains L_1 with the definition

$$L_n \equiv \int \frac{d\mathbf{q}}{(2\pi)^3}[\tilde{\phi}(\mathbf{q})]^2 \int \frac{d\mathbf{q}'}{(2\pi)^3}\tilde{\phi}(\mathbf{q}-\mathbf{q}') \times [\tilde{\phi}(\mathbf{q}')]^n \frac{1}{2}\tilde{\phi}^2(\mathbf{q}'). \tag{B4}$$

If we notice that

$$\int \frac{d\mathbf{q}}{(2\pi)^3}[\tilde{\phi}(\mathbf{q})]^2\tilde{\phi}(\mathbf{q}-\mathbf{q}') = \frac{2\pi^2}{1+(\mathbf{q}'^2/4)} = \frac{\pi}{2}\tilde{\phi}\left(\frac{\mathbf{q}'}{2}\right) \tag{B5}$$

then a mere integration by parts allows one to reexpress L_1 in terms of K_1 and K_2,

$$L_1 = 8\pi^2[K_1 - 2K_2]. \tag{B6}$$

Eventually,

$$I_{5(\mathrm{d})} = -4\pi[K_1-2K_2]\beta^5\left(\sum_\gamma \rho_\gamma e_\gamma^3\right)^2 \times\left(\sum_{\gamma'}\rho_{\gamma'}e_{\gamma'}^4\right)\frac{\partial}{\partial\rho_\alpha}\left[\frac{1}{\kappa_D}\right] \tag{B7}$$

and we get the formula (152).

2. Contributions from diagrams of Fig. 6

Again, the contributions from Figs. 6(a) and 6(b) are readily expressed as Fourier transforms of convolutions and involve functions K_n defined in Eq. (153). The results are

$$I_{6(\mathrm{a})} = 8\pi^2 K_2\beta^6\frac{\partial}{\partial\rho_\alpha}\left[\left(\sum_\gamma \rho_\gamma e_\gamma^3\right)^4\right]\frac{1}{\kappa_D^3}, \tag{B8}$$

$$I_{6(\mathrm{b})} = \frac{32}{3}\pi^2 K_3\beta^6\left(\sum_\gamma \rho_\gamma e_\gamma^3\right)^4\frac{\partial}{\partial\rho_\alpha}\left[\frac{1}{\kappa_D^3}\right]. \tag{B9}$$

The contribution from diagram 6(c) may be written in terms of L_2 with the definition (B4). As in the case of L_1, we use Eq. (B5) and an integration by parts to write L_2 in terms of K_2 and K_3,

$$L_2 = 32\pi^3[3K_2 - 4K_3]. \tag{B10}$$

Finally

$$I_{6(\mathrm{c})} = \frac{8}{3}\pi^2[3K_2-4K_3]\beta^6\left(\sum_\gamma \rho_\gamma e_\gamma^3\right)^4\frac{\partial}{\partial\rho_\alpha}\left[\frac{1}{\kappa_D^3}\right] \tag{B11}$$

and we get formula (154).

3. Contributions from diagrams of Fig. 7

We introduce the ''bridge'' integral with six bonds F^{cc},

$$I_{\text{bridge 6}} \equiv \int\frac{d\mathbf{k}}{(2\pi)^3}\int\frac{d\mathbf{k}'}{(2\pi)^3}\int\frac{d\mathbf{k}''}{(2\pi)^3}\phi(\mathbf{k})\phi(\mathbf{k}')\phi(\mathbf{k}'') \times\phi(\mathbf{k}-\mathbf{k}')\phi(\mathbf{k}-\mathbf{k}'')\phi(\mathbf{k}'-\mathbf{k}''). \tag{B12}$$

The symmetry factor of diagram 7(a) is equal to 3!, because any permutation of the three internal points does not change the integrand. Thus

$$I_{7(\mathrm{a})} = \frac{1}{3!}\beta^6 e_\alpha^3\left(\sum_\gamma \rho_\gamma e_\gamma^3\right)^3 I_{\text{bridge 6}} = \frac{1}{24}\beta^6\frac{\partial}{\partial\rho_\alpha}\left[\left(\sum_\gamma \rho_\gamma e_\gamma^3\right)^4\right]I_{\text{bridge 6}}. \tag{B13}$$

The symmetry factor of diagram 7(b) is 4, because, if 1 and 2 denote the two points that are linked to the root point while 3 and 4 label the two other internal points, the allowed permutations that do not change the integrand are the following:

the identity, the permutation of 1 with 2, that of 3 with 4, and the simultaneous permutations of 1 with 2 and 3 with 4. The contribution from 7(b) is

$$I_{7(b)}=-\frac{1}{4}\beta^7 e_\alpha^2\left(\sum_\gamma \rho_\gamma e_\gamma^3\right)^4 I^{[2]}_{\text{bridge 6}}, \tag{B14}$$

where $I^{[2]}_{\text{bridge 6}}$ denotes an expression similar to Eq. (B12) where $\phi(\mathbf{k})$ is replaced by $[\phi(\mathbf{k})]^2$. Moreover, we notice that $I^{[2]}_{\text{bridge 6}}=-(2\pi/3)dI_{\text{bridge 6}}/d(\kappa_D^2)$ and $e_\alpha^2=(1/4\pi\beta)\partial\kappa_D^2/\partial\rho_\alpha$. Henceforth

$$I_{7(b)}=\frac{1}{24}\beta^6\frac{\partial I_{\text{bridge 6}}}{\partial\rho_\alpha}\left(\sum_\gamma \rho_\gamma e_\gamma^3\right)^4 \tag{B15}$$

and we obtain Eq. (155).

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