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Quantum plasmas with or without a uniform magnetic field. III. Exact low-density algebraic tails of correlations

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For a multicomponent plasma of point charges with Coulomb interactions, the exact analytical low-density expressions for leading algebraic tails of various quantum static correlations at large distances are derived at the first two orders in density in the absence as well as in the presence of a uniform magnetic field $\mathbf{B}_0$. The calculation is nonperturbative in the Planck constant $\hbar$ and in the coupling with the magnetic field. It settles the existence of algebraic quantum screening for position correlations with or without charge summations. In the case of a one-component plasma (OCP) our calculation does coincide with another expression which we derive from an exact sum rule specific to the OCP. A simple physical picture emerges from our results. At low density, each algebraic tail arises merely from one effective quantum interaction: a squared dipolar energy when $\mathbf{B}_0=\mathbf{0}$ and a quadrupolar interaction in the anisotropic case $\mathbf{B}_0\neq\mathbf{0}$. Only Maxwell-Boltzmann statistics and free quantum motion are involved at the first two orders in density. Thus for a given value of the magnetic coupling constant the coefficient is exactly proportional to $\hbar^4$, and when $\mathbf{B}_0=\mathbf{0}$, it coincides with the low-density limit of the semiclassical calculation for a Coulomb potential regularized at the origin. Quantum dynamics and quantum statistics in the presence of Coulomb interactions show up only at the third order in density where a singular dependence in $\hbar$ appears. In the case $\mathbf{B}_0=\mathbf{0}$, the classical the Debye screening proves to be sufficient to enforce a cascade in the exponents of the leading algebraic tails when charges are summed over. When $\mathbf{B}_0\neq\mathbf{0}$ a cancellation between the Debye screening effect and its semiclassical diffraction correction may be interpreted as a consequence of the intrinsic quantum nature of statistical magnetic properties. Subsequently, the charge density induced by an external charge δq is nonlinear in δq at the first order in density when $\mathbf{B}_0\neq\mathbf{0}$. Finally, the crossover distance between classical exponential and quantum algebraic falloffs is estimated: it is a few Debye lengths, because quantum effects are still quantitatively small in the regime where exact analytical results are available. [S1063-651X(98)02810-4]

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I. INTRODUCTION

In the present paper we exhibit the exact analytical low-density limits of leading algebraic tails for various correlations at large distances r in a quantum plasma of point charges with Coulomb interaction at finite temperature $1/\beta$: we address the particle-particle, particle-charge, and charge-charge correlations, $\rho_{\alpha\gamma}^{(2)T}(\mathbf{r})$, $\Sigma_\gamma e_\gamma \rho_{\alpha\gamma}^{(2)T}(\mathbf{r})$, and $\Sigma_{\alpha,\gamma} e_\alpha e_\gamma \rho_{\alpha\gamma}^{(2)T}(\mathbf{r})$, respectively. (In fact correlation means two-body distribution function.) We also produce the tail of the induced charge density $\Sigma_\gamma e_\gamma \rho_\gamma^{\mathrm{ind}}(\mathbf{r};q)$ in the presence of an external point charge q, which may be either finite or infinitesimal. [In the latter case we use the notation $\Sigma_\gamma e_\gamma \rho_\gamma^{\mathrm{ind,L}}(\mathbf{r};\delta q)$, where the extra superscript L (not introduced in previous papers) refers to the linear response theory.] These *exact analytical* results settle the existence of algebraic screening in quantum plasmas whose Hamiltonian is that given in Paper I. Indeed, the argument of Paper I about the algebraic decay of quantum correlations in Coulomb systems at any finite density is only perturbative in the sense that it is derived from expansions with respect to an auxiliary variable, namely, the loop density. The low-density regime corresponds to physical situations of low degeneracy and weak Coulomb coupling, and the present low-density limits are derived by using the same techniques as those introduced in Paper II. Moreover, we point out that an exact sum rule for the model of the one-component plasma (OCP) implies the existence of an algebraic tail for the quantum static correlation in the presence of $\mathbf{B}_0$, as already noticed in the analogous case of the classical time-displaced correlation for the same system [1]. Our result for the OCP seen as some limit of a two-component plasma does coincide with the low-density coefficient derived from the exact sum rule. The various qualitative and quantitative results have been summarized elsewhere in the cases $\mathbf{B}_0=\mathbf{0}$ [2] and $\mathbf{B}_0\neq\mathbf{0}$ [3].

The paper is organized as follows. The main results are displayed in Sec. II. In Sec. II A we give the expressions and the formal structures of low-density tails of $\rho_{\alpha\gamma}^{(2)T}(\mathbf{r})$ in order to exhibit physical mechanisms at stake. At low density, algebraic tails arise merely from squared dipolar interactions when $\mathbf{B}_0=\mathbf{0}$ or from quadrupolar forces in the anisotropic case $\mathbf{B}_0\neq\mathbf{0}$. Their coefficients at order ρ^2 and $\rho^{5/2}$ are exactly proportional to $\hbar^4$ (for a given value of the orbital magnetic coupling). A singular dependence shows up only from order ρ^3 when quantum statistics and quantum dynamics with Coulomb interactions begin playing a role. Similarly, spin and position variables are coupled only by the latter ones so that the spin does not appear at the first two orders in densities. In Sec. II B, we point out how the exact low-density tail for the OCP may be derived from an exact sum rule established in another framework. In Sec. II C the order of magnitude of the crossover distance at which the algebraic quantum tail begins dominating the exponential classical tail is shown to be a few Debye lengths: the effect proves to be quantitatively small in the low-density regime. In Sec. III we recall relations between distribution functions for quantum particles and for loops (Sec. III A). The invariance of the measure for loop shapes under inversion allows one to simplify the discussion of the small-**k** behavior of correlations in Fourier space (see Sec. III B). In Sec. IV we sketch the scheme for low-density expansions. First, we perform loop-density expansions by using the scaling analysis of diagrams introduced in Paper II (Sec. IV A) and then we replace loop densities by their expressions in terms of particle densities (Sec. IV B). Section V is devoted to the case $\mathbf{B}_0=\mathbf{0}$. As at any finite density, there occurs a cascade of power laws when charges are summed over: a simple mechanism involving only classical Debye screening is shown to be efficient enough at the first two orders in density (Sec. V A). Explicit results at these orders are produced (Secs. V B and V C) and diagrams that contribute at next order in density are listed in order to discuss other quantum effects that appear only from third order (Sec. V D). In particular, we discuss the dependence of coefficients upon the Planck constant $\hbar$. The case $\mathbf{B}_0\neq\mathbf{0}$ is investigated in Sec. VI. Arguments of invariance under inversion enforce constraints on the diagrams to be considered (Sec. VI A). A cancellation between classical Debye screening and its semiclassical diffraction correction, which occurs only at the first order in density, is analyzed (Sec. VI B). The explicit values of the $1/r^5$ tails and their limits in weak or strong magnetic field are given (Sec. VI C). In Sec. VII the case of the OCP is handled as a limiting case of a two-component plasma in the absence (Sec. VII A) as well as in the presence (Sec. VII B) of $\mathbf{B}_0$. In the conclusion (Sec. VIII), after discussion of some qualitative results (Sec. VIII A), we consider other approaches to get a deeper insight into physical mechanisms at stake. The compatibility of algebraic screening with exact sum rules which characterize perfect screening is discussed in the low-density regime (Sec. VIII B). When $\mathbf{B}_0=\mathbf{0}$, low-density expansions of exact quantum tails coincide with low-density expansions of semiclassical expressions, because the exact result is exactly proportional to $\hbar^4$ at the first two orders in density (Sec. VIII C). When $\mathbf{B}_0\neq\mathbf{0}$ (Sec. VIII D), the intrinsic quantum nature of the algebraic tails which occur only in the presence of the magnetic field is discussed by comparison with the simple model recalled in Paper I. The nonlinearity of the induced charge at the first order in density is explained. Finally (Sec. VIII E), we recall that algebraic tails with the same exponents as in the quantum case also appear in classical time-displaced correlations. In the latter case the polarization cloud cannot follow instantaneously the motion of the charge, because of inertia effects that are involved as soon as time-displaced averages are considered; then the dynamical classical fluctuations of the instantaneous dipole associated with a charge and its screening cloud play a role similar to that of the static quantum fluctuations.

II. MAIN RESULTS

A. Formal structures of low-density tails

In the present section we resume the description that emerges from our final expressions for the low-density particle-particle correlation at the first two orders ρ^2 and $\rho^{5/2}$. (As in Paper II, ρ is a generic notation for particle densities, while a term which is exactly of order ρ^n is denoted by $f^{\{n\}}$.) A particle of species α is characterized by its mass m_α, its charge e_α, and its spin $\hbar S_\alpha$. In fact, the spin will not be involved in the expressions at order ρ^2 and $\rho^{5/2}$.

When $\mathbf{B}_0=\mathbf{0}$, at the first order in density,

$$\rho^{(2)T}_{\alpha\gamma}(\mathbf{r})\big|_{\mathbf{B}_0=\mathbf{0}} \underset{r\to\infty}{\sim} \frac{\beta^4\hbar^4}{240}\rho_\alpha\rho_\gamma e_\alpha e_\gamma\left[\frac{e_\alpha}{m_\alpha}-\frac{\kappa^2_{e/m}}{\kappa^2_D}\right]\left[\frac{e_\gamma}{m_\gamma}-\frac{\kappa^2_{e/m}}{\kappa^2_D}\right], \tag{1}$$

where $\beta=(k_BT)^{-1}$ is the inverse temperature, κ_D is equal to the inverse Debye screening length, $\kappa_D=\sqrt{4\pi\beta\Sigma_\alpha\rho_\alpha e^2_\alpha}$, and $\kappa^2_{e/m}\equiv 4\pi\beta\Sigma_\alpha(\rho_\alpha e^3_\alpha/m_\alpha)$. The leading algebraic decay $A_{\alpha\gamma}/r^6$ of $\rho^{(2)T}_{\alpha\gamma}(r)|_{\mathbf{B}_0=\mathbf{0}}$ at orders ρ^2 and $\rho^{5/2}$ turns out to arise only from the squared fluctuations of some dipolar interaction combined with classical Debye screening effects. More precisely, $A^{\{n\}}_{\alpha\gamma}/r^6$, with $n=2$ or 5/2, may be interpreted as the tail of the convolution

$$\sum_{\alpha_1,\alpha_2} S^{\rm cl}_{D,\alpha\alpha_1}*[-\beta V^{\rm eff(6)}_{\alpha_1\alpha_2}|^{\{n\}}]*S^{\rm cl}_{D,\alpha_2\gamma}, \tag{2}$$

where $V^{\rm eff(6)}_{\alpha_1\alpha_2}(r)|^{\{n\}}$ is a purely algebraic effective potential proportional to $1/r^6$. $S^{\rm cl}_{D,\alpha\gamma}(r)$, which decays exponentially fast, is the Debye part of the classical structure factor. The structure factor is defined as

$$S_{\alpha\gamma}(\mathbf{r})=\rho^{(2)T}_{\alpha\gamma}(\mathbf{r})+\delta_{\alpha,\gamma}\rho_\alpha\delta(\mathbf{r}) \tag{3}$$

and its classical (linearized) Debye approximation $S^{\rm cl}_{D,\alpha\gamma}$ corresponds to

$$\rho^{(2)T\rm cl}_{D,\alpha\gamma}(\mathbf{r})=\rho_\alpha\rho_\gamma\left[-\beta e_\alpha e_\gamma\frac{e^{-\kappa_D r}}{r}\right]\equiv\rho_\alpha\rho_\gamma F^{cc}_{D,\alpha\gamma}. \tag{4}$$

In terms of dimensionless Brownian bridges $\boldsymbol{\xi}_i$ which describe quantum position fluctuations (see Sec. III B of Paper I), $V^{\rm eff(6)}_{\alpha_1\alpha_2}(r)|^{\{n\}}$ is the average of some squared dipolar interaction over all possible shapes $\lambda_{\alpha_i}\boldsymbol{\xi}_i$ with a normalized Gaussian measure $D(\boldsymbol{\xi}_i)$ and a typical extent equal to the de Broglie thermal wavelength $\lambda_\alpha\equiv\sqrt{\beta\hbar^2/m_\alpha}$. At zero density

$$-\beta V^{\rm eff(6)}_{\alpha_1\alpha_2}(r)|^{\{0\}}=\frac{1}{2}\int D(\boldsymbol{\xi}_1)\int D(\boldsymbol{\xi}_2)\times[W_3(\mathbf{r},\boldsymbol{\xi}_1,\boldsymbol{\xi}_2;\alpha_1,\alpha_2)]^2, \tag{5}$$

where W_3 is a purely quantum dipole-dipole potential,

$$W_3(\mathbf{r},\boldsymbol{\xi}_1,\boldsymbol{\xi}_2;\alpha_1,\alpha_2)\equiv\beta\int_0^1 ds_1\int_0^1 ds_2[\delta(s_1-s_2)-1]\times[\lambda_{\alpha_1}\boldsymbol{\xi}_1(s_1)\cdot\boldsymbol{\nabla}][\lambda_{\alpha_2}\boldsymbol{\xi}_2(s_2)\cdot\boldsymbol{\nabla}]\times\left(\frac{e_{\alpha_1}e_{\alpha_2}}{r}\right), \tag{6}$$

and $W_3(\mathbf{r},\boldsymbol{\xi}_1,\boldsymbol{\xi}_2;\alpha_1,\alpha_2)$ is the $1/r^3$ tail of

$$W\equiv-\beta e_{\alpha_1}e_{\alpha_2}\int_0^1 ds_1\int_0^1 ds_2[\delta(s_1-s_2)-1]\times v_C(\mathbf{r}+\lambda_{\alpha_2}\boldsymbol{\xi}_2(s_2)-\lambda_{\alpha_1}\boldsymbol{\xi}_1(s_1)),\tag{7}$$

where $v_C(r)=1/r$. The mean value $\int D(\boldsymbol{\xi}_1)\int D(\boldsymbol{\xi}_2)W_3(\mathbf{r},\boldsymbol{\xi}_1,\boldsymbol{\xi}_2;\alpha_1,\alpha_2)$ vanishes by arguments of invariance under $\boldsymbol{\xi}_i\to-\boldsymbol{\xi}_i$. All $1/r^n$ terms W_n arising from the large-r Taylor expansion of W and which are not canceled by inversion invariance of the measure $D(\boldsymbol{\xi})$ are in fact short ranged, because, after the rotational invariance of $D(\boldsymbol{\xi})$ has been taken into account, the latter terms are reduced to powers of the Laplacian of $1/r$ and $\Delta(1/r)=-4\pi\delta(\mathbf{r})$. The explicit values of $V^{\mathrm{eff}(6)}_{\alpha_1\alpha_2}(r)|^{\{n\}}$ with $n=0,1/2$ are given in Sec. V C. We notice that in the case of a two-component plasma $A_{\alpha\gamma}/\rho_\alpha\rho_\gamma$ is independent from species α or γ in the low-density limit and the force corresponding to $A^{\{2\}}_{\alpha\gamma}+A^{\{5/2\}}_{\alpha\gamma}$ is attractive.

When $\mathbf{B}_0\neq\mathbf{0}$, rotational invariance is broken in one space direction and nonsquared multipole-multipole interactions partially survive after statistical average. At the first order in density,

$$\rho^{(2)T}_{\alpha\gamma}(\mathbf{r})|_{\mathbf{B}_0}\underset{r\to\infty}{\sim}-\rho_\alpha\rho_\gamma\beta^3\hbar^4\frac{e_\alpha e_\gamma}{m_\alpha m_\gamma}A(u_{C\alpha},u_{C\gamma})\frac{P_4(\cos\theta)}{r^5},\tag{8}$$

where $P_4(x)$ is a Legendre polynomial and θ is the angle between $\mathbf{B}_0$ and $\mathbf{r}$, as already noticed in Paper I, while $P_4(x)=[35\cos4\theta+20\cos2\theta+9]/64$. In Eq. (8)

$$A(u_{C\alpha},u_{C\gamma})=\frac{3}{2}\Bigg\{-\frac{1}{u^2_{C\alpha}-u^2_{C\gamma}}\left[\frac{u^2_{C\gamma}}{u^3_{C\alpha}}\coth u_{C\alpha}-\frac{u^2_{C\alpha}}{u^3_{C\gamma}}\coth u_{C\gamma}\right]+\frac{1}{45}-\frac{1}{3u^2_{C\alpha}}-\frac{1}{3u^2_{C\gamma}}-\frac{1}{u^4_{C\alpha}}-\frac{1}{u^4_{C\gamma}}-\frac{1}{u^2_{C\alpha}u^2_{C\gamma}}\Bigg\},\tag{9}$$

where $u_{C\alpha}$ is the dimensionless coupling constant for orbital magnetic interaction $u_{C\alpha}\equiv\beta\mu_{B\alpha}B_0$. The interpretation is the following. The leading $D^{\{2\}}_{\alpha\gamma}/r^5$ tail of $\rho^{(2)T}_{\alpha\gamma}(\mathbf{r})|_{\mathbf{B}_0}$ at the first order ρ^2 reads

$$\frac{D^{\{2\}}_{\alpha\gamma}}{r^5}=\rho_\alpha\rho_\gamma[-\beta V^{\mathrm{eff}(5)}_{\alpha\gamma}(\mathbf{r})|^{\{0\}}],\tag{10}$$

where $V^{\mathrm{eff}(5)}_{\alpha\gamma}$ is a purely quantum quadrupole-quadrupole interaction

$$-\beta V^{\mathrm{eff}(5)}_{\alpha_1\alpha_2}(\mathbf{r})|^{\{0\}}=\int D_{\mathbf{B}_0}(\boldsymbol{\xi}_1)\int D_{\mathbf{B}_0}(\boldsymbol{\xi}_2)W_5(\mathbf{r},\boldsymbol{\xi}_1,\boldsymbol{\xi}_2;\alpha_1,\alpha_2),\tag{11}$$

with

$$W_5(\mathbf{r},\boldsymbol{\xi}_1,\boldsymbol{\xi}_2;\alpha_1,\alpha_2)=-\beta\frac{1}{4}\int_0^1 ds_1\int_0^1 ds_2[\delta(s_1-s_2)-1]\times[\lambda_{\alpha_1}\boldsymbol{\xi}_1(s_1)\cdot\boldsymbol{\nabla}]^2[\lambda_{\alpha_2}\boldsymbol{\xi}_2(s_2)\cdot\boldsymbol{\nabla}]^2\left(\frac{e_{\alpha_1}e_{\alpha_2}}{r}\right).\tag{12}$$

W_5 is not short ranged in spite of the harmonicity of the Coulomb potential, because the measure $D_{\mathbf{B}_0}(\boldsymbol{\xi})$ is anisotropic in the presence of $\mathbf{B}_0$. By using the covariance of a Brownian motion driven by a uniform magnetic field (given in Sec. III C of Paper II), we get Eq. (8). $D^{\{2\}}_{\alpha\gamma}/r^5$ contains only one direct quadrupolar contribution without any Debye screening [contrary to what happens in Eq. (2) for $A^{\{2\}}_{\alpha\gamma}$] and there is no dipole-dipole or dipole-quadrupole quantum interaction screened by monopole-dipole Debye interaction either. Indeed, there occurs a cancellation mechanism between classical exponential Debye screening and semiclassical diffraction corrections to it, so that the medium has no net contribution to Eq. (10). At next order, the same cancellation mechanism is involved but it does not suppress all screening contributions.

The cancellation of Debye screening contributions may be related to the intrinsically quantum dynamical nature of statistical effects due to the magnetic field. It does not appear in the case of the $1/r^6$ subleading tail in the presence of $\mathbf{B}_0$. Indeed, according to Sec. VIII A of Paper I, a $1/r^6$ decay arises from the same diagrammatic structure whether $\mathbf{B}_0$ is switched on or not, and $A^{\{2\}}_{\alpha\gamma}(\hat{\mathbf{r}};\mathbf{B}_0)/r^6$ (with $\hat{\mathbf{r}}\equiv\mathbf{r}/|\mathbf{r}|$) is analogous to the tail of the convolution (2) with the measure $D_{\mathbf{B}_0}(\boldsymbol{\xi})$ in place of $D(\boldsymbol{\xi})$ in Eq. (5). The coefficient $A^{\{2\}}_{\alpha\gamma}(\hat{\mathbf{r}};\mathbf{B}_0)$ is anisotropic, because the effective potential $V^{\mathrm{eff}(6)}_{\alpha_1\alpha_2}|^{\{0\}}$ is now calculated with the measure $D_{\mathbf{B}_0}(\boldsymbol{\xi})$.

The coefficients of all considered tails at orders lower than ρ^3 are entirely determined by quantum dynamics of independent particles (in the absence or in the presence of $\mathbf{B}_0$), Maxwell-Botzmann (MB) statistics, and classical Debye screening. However, from order ρ^3 on, mechanisms are more intricate and contributions from quantum dynamics and quantum statistics for interacting charges appear in the structures of algebraic tails.

When charges are summed over, internal screening is involved in the relation between exponents of correlation decays at any finite density (as already discussed in Fourier space in Paper I). At the first two orders in density, when $\mathbf{B}_0=\mathbf{0}$, classical Debye screening by itself is sufficient to lead to the cascade of power laws $A_{\alpha\gamma}/r^6$, B_α/r^8, and C/r^{10} for $\rho^{(2)T}_{\alpha\gamma}(r)|_{\mathbf{B}_0=\mathbf{0}}$, $\Sigma_\alpha e_\gamma\rho^{(2)T}_{\alpha\gamma}(r)|_{\mathbf{B}_0=\mathbf{0}}$, and $\Sigma_{\alpha,\gamma}e_\alpha e_\gamma\rho^{(2)T}_{\alpha\gamma}(r)|_{\mathbf{B}_0=\mathbf{0}}$, respectively. Indeed, at the first two orders in density, B_α/r^8 and C/r^{10} may also be interpreted as the tails of the convolution (2) with corresponding charge summations with the results

$$\sum_\gamma e_\gamma\rho^{(2)T}_{\alpha\gamma}(r)\bigg|_{\mathbf{B}_0=\mathbf{0}}\underset{r\to\infty}{\sim}\frac{1}{r^8}\frac{1}{32\pi}\beta^3\hbar^4\rho_\alpha e_\alpha\frac{\kappa^2_{e/m}}{\kappa^2_D}\left[\frac{\kappa^2_{e/m}}{\kappa^2_D}-\frac{e_\alpha}{m_\alpha}\right],\tag{13}$$

$$\sum_{\alpha,\gamma} e_\alpha e_\gamma \rho^{(2)T}_{\alpha\gamma}(r)\Big|_{\mathbf{B}_0=\mathbf{0}} \underset{r\to\infty}{\sim} \frac{1}{r^{10}} \frac{7}{16\pi^2}\beta^2\hbar^4 \frac{\kappa^4_{e/m}}{\kappa^4_D}. \quad (14)$$

When $\mathbf{B}_0 \neq \mathbf{0}$, all correlations (with or without summation over charges) decay with the same $1/r^5$ inverse power law. At first orders ρ^2 and $\rho^{5/2}$ this may be accounted for by the fact that internal screening mechanisms, involving both the classical monopole-monopole Debye potential and its semiclassical ''diffraction'' correction (more precisely a dipole-monopole Debye interaction) partially cancel one another at first orders in density.

Screening of an external charge is also investigated. At low density, when $\mathbf{B}_0 = \mathbf{0}$, classical Debye screening of an external charge enforces a cascade in the density orders at which the coefficients $A_{\alpha\gamma}$, B_α, and C start. The leading $B^\star/r^8$ tail of the induced charge density reads

$$\sum_{\gamma} e_\gamma \rho^{\text{ind,L}}_{\gamma}(r;\delta q)\Big|_{\mathbf{B}_0=\mathbf{0}} \underset{r\to\infty}{\sim} \frac{1}{r^{8}} \frac{1}{32\pi}\beta^3\hbar^4 \frac{\kappa^4_{e/m}}{\kappa^4_D}\delta q. \quad (15)$$

It starts at order ρ^0; henceforth the $1/r^8$ tail of the induced charge density does not vanish in the strict zero-density limit. The reason is that such a tail is valid for distances which are far larger than the Debye screening length so that the existence of the medium is always taken into account implicitly. When $\mathbf{B}_0 \neq \mathbf{0}$, the tail $D_{\alpha\gamma}/r^5$ of $\rho^{(2)T}_{\alpha\gamma}(r)|_{\mathbf{B}_0}$ at the lowest order ρ^2 (but not at order $\rho^{5/2}$) results only from nonlinear effects in each charge e_α. Indeed, at order ρ^2, it does not involve any net contribution from Debye screening (which is linear in e_α) while the bare Coulomb interaction is proportional to e_α and the quadrupolar moment associated with the quantum motion of each charge e_α in Eq. (12) is quadratic in small e_α when $\mathbf{B}_0$ is switched on. [More precisely it varies as e^2_α/m^3_α; this can be checked from the explicit formula (8).] As a consequence, the tail of the induced charge density $\Sigma_\gamma e_\gamma \rho^{\text{ind}}_\gamma(\mathbf{r};q)|_{\mathbf{B}_0}$, which can be derived from $\rho^{(2)T}_{\alpha\gamma}(\mathbf{r})|_{\mathbf{B}_0}$ (see Sec. VIII C of Paper I), is nonlinear in q at the corresponding order ρ: it becomes cubic with respect to the ratio q/m_q for the external charge when the latter goes to zero—and henceforth vanishes for a fixed infinitesimal external point charge which cannot be sensitive to the magnetic field (since its mass must be seen as infinite). Indeed, according to the loop formalism, the tail of the induced charge density $\Sigma_\gamma e_\gamma \rho^{\text{ind,L}}_\gamma|_{\mathbf{B}_0}(\mathbf{r};\delta q)$ given by the linear response theory vanishes at order ρ and shows up only at order $\rho^{3/2}$ (see Sec. VI B 4).

B. The OCP as a test bench

An interesting test bench for our calculations is the model of the quantum OCP. The OCP is special in the sense that there exists an exact sum rule [4] that determines the value of the term of order $|\mathbf{k}|^2$ in the Fourier transform of the structure factor $S_{\text{OCP}}(\mathbf{r}) = \rho\delta(\mathbf{r}) + \rho^{(2)T}_{\text{OCP}}(\mathbf{r})$. (This sum rule arises from the fact that the center of mass decouples from relative coordinates and is only submitted to the harmonic force of the background.) When $\mathbf{B}_0 = \mathbf{0}$, this term is analytic, which is in agreement with the $1/r^{10}$ leading algebraic decay of the charge-charge correlation $e^2 S_{\text{OCP}}(\mathbf{r})$. When $\mathbf{B}_0 \neq \mathbf{0}$, the exact $|\mathbf{k}|^2$ term in the small-$\mathbf{k}$ expansion of $S_{\text{OCP}}(\mathbf{k})$ oscillates in its dependence upon the angle $\theta_{\mathbf{k}}$ between $\mathbf{B}_0$ and $\mathbf{k}$. A part of this oscillating term is nonanalytic in the components of $\mathbf{k}$ and it gives the coefficient of the D_{OCP}/r^5 tail of $S_{\text{OCP}}(\mathbf{r})$ at any density implicitly.

More precisely, according to Eq. (5.63) of Ref. [4], the $|\mathbf{k}|^2$ term is given by

$$4\pi\beta \lim_{|\mathbf{k}|\to 0}\Big|_{\theta_{\mathbf{k}}\text{fixed}} \frac{e^2 S_{\text{OCP}}(\mathbf{k})|_{\mathbf{B}_0}}{|\mathbf{k}|^2} = \frac{1}{\omega_+^2 - \omega_-^2}\left[(\omega_+^2 - \omega_c^2)\frac{\beta\hbar\omega_+}{2}\coth\left(\frac{\beta\hbar\omega_+}{2}\right) - (\omega_-^2 - \omega_c^2)\frac{\beta\hbar\omega_-}{2}\coth\left(\frac{\beta\hbar\omega_-}{2}\right)\right]. \quad (16)$$

The term in the right-hand side of Eq. (16) corresponds to the double limit $\lim_{L\to\infty}\lim_{R\to\infty}$ of $\int_{\mathcal{C}} d\mathbf{r}' \int d\mathbf{r} v_C(\mathbf{r} - \mathbf{r}')e^2 S_{\text{OCP}}(\mathbf{r}')$ where $\mathbf{r}'$ is integrated over a cylinder $\mathcal{C}$ with radius R and length $2L$ and whose axis is parallel with $\mathbf{B}_0$. The $|\mathbf{k}|^2$ term in $e^2 S_{\text{OCP}}|_{\mathbf{B}_0}(\mathbf{k})$ is not analytic in the components of $\mathbf{k}$, because frequencies ω_+ and ω_- oscillate with the angle $\theta_{\mathbf{k}}$,

$$\omega^2_\pm = \frac{1}{2}\{\omega_p^2 + \omega_c^2 \pm [(\omega_p^2+\omega_c^2)^2 - 4(\omega_p\omega_c\cos\theta_{\mathbf{k}})^2]^{1/2}\}, \quad (17)$$

where $\omega_c \equiv (e/mc)B_0$ is the cyclotronic frequency and $\omega_p = \sqrt{4\pi e^2\rho/m}$ is the plasmon frequency. When $\mathbf{B}_0 \neq \mathbf{0}$ such an oscillatory behavior in the angle $\theta_{\mathbf{k}}$ between $\mathbf{B}_0$ and $\mathbf{k}$ also appears in density waves with large wavelength in the random phase approximation [5]. We notice that if $\mathbf{B}_0 = \mathbf{0}$, then $\omega_+^2 = \omega_p^2$ and $\omega_-^2 = 0$, and Eq. (16) reduces to the sum rule derived in [6,7],

$$4\pi\beta \lim_{|\mathbf{k}|\to 0} \frac{e^2 S_{\text{OCP}}(|\mathbf{k}|)|_{\mathbf{B}_0=\mathbf{0}}}{|\mathbf{k}|^2} = \frac{\beta\hbar\omega_p}{2}\coth\left(\frac{\beta\hbar\omega_p}{2}\right). \quad (18)$$

The dependence on $\cos\theta_{\mathbf{k}}$ of the exact term (16) is intricate. However, explicit analytical results may be obtained in the low-density regime at $u_C = \beta\hbar\omega_c/2$ fixed. At order ρ, the $|\mathbf{k}|^2$ term in $S_{\text{OCP}}(\mathbf{k})$ involves a term proportional to $\mathbf{k}^2$ plus a contribution with $\mathbf{k}^2(\cos\theta_{\mathbf{k}})^2 = [\mathbf{k}]^2_z$, which is anisotropic but analytic. At order ρ^2 there appears an extra $\mathbf{k}^2(\cos\theta_{\mathbf{k}})^4 = [\mathbf{k}]^4_z/\mathbf{k}^2$ term, which is nonanalytic. We have checked that the low-density coefficient $D^{\{2\}}_{\text{OCP}}$ at order ρ^2 which is derived from our low-density results for a two-component plasma does coincide with the inverse Fourier transform of the nonanalytic part in the low-density limit of the exact $|\mathbf{k}|^2$ term in $S_{\text{OCP}}(\mathbf{k})$.

C. Orders of magnitudes

The crossover distance r_* between the Debye exponential decay and the algebraic tail may be calculated in some physical situations, where conditions for low-density expansions are met. In regimes of weak Coulomb coupling and low degeneracy, the numerical value of the algebraic tail of the particle-particle correlation becomes more important than that of the Debye exponential decay only at distances of about ten Debye lengths, in the absence [2] as well as in the presence [3] of $\mathbf{B}_0$.

The latter estimations are obtained by considering tails of particle-particle correlations for a symmetric two-component plasma in order to get very simple analytical expressions in terms of dimensionless parameters of the problem. For a symmetric two-component plasma in which $e_+ = -e_- \equiv e$, the local neutrality relation implies that $\rho_+ = \rho_- \equiv \rho$. The Debye expression, which is exact for a classical plasma in the limit of weak Coulomb coupling [8], reads

$$\rho^{(2)T\mathrm{cl}}_{\alpha\gamma,D}(\mathbf{r}) \underset{r\to\infty}{\sim} -\mathrm{sgn}(e_\alpha)\mathrm{sgn}(e_\gamma)\rho^2\sqrt{3}\Gamma^{3/2}\frac{e^{-x}}{x}, \quad (19)$$

where $\mathrm{sgn}(e_\alpha)$ is the sign of the charge e_α, $x \equiv r/\xi_D$, and ξ_D is the Debye length, $\xi_D \equiv \kappa_D^{-1}$. The mean interparticle distance a is calculated from the relation $(4\pi/3)a^3(2\rho) = 1$, so that the coupling constant is $\Gamma \equiv \beta e^2/a = (1/3)(a/\xi_D)^2$. When $\mathbf{B}_0 = \mathbf{0}$, the first-order term (1) in the $1/r^6$ tail of $\rho^{(2)T}_{\alpha\gamma}(\mathbf{r})|_{\mathbf{B}_0=\mathbf{0}}$ is reduced to

$$\rho^{(2)T}_{\alpha\gamma}(\mathbf{r})|_{\mathbf{B}_0=\mathbf{0}} \underset{r\to\infty}{\sim} \rho^2\frac{9}{320}\Gamma^5\left(\frac{\lambda_-}{a}\right)^4\left[1+\frac{m_-}{m_+}\right]^2\frac{1}{x^6}, \quad (20)$$

where λ_- is the de Broglie thermal wavelength of negative charges, $\lambda_- \equiv \sqrt{\beta\hbar^2/m_-}$. For instance, the core of the Sun may be seen as a hydrogen plasma almost fully ionized by pressure and temperature, with a mass density $\rho_m \sim 160$ g/cm^3 at temperature $T \sim 1.5\times 10^7$ K. Thus $a \sim 0.1$ Å, the system is rather weakly degenerated, $\lambda_-/a \sim 0.7$, and weakly coupled, $\Gamma \sim 0.1$. The contribution from the algebraic tail $A_{\alpha\gamma}/r^6$ becomes as large as the classical Debye-Hückel approximation at a crossover distance $r_* \sim 31\xi_D$. In the presence of $\mathbf{B}_0$, we use the simple form (8) valid in the limit where $B_0 = |\mathbf{B}_0|$ becomes infinite,

$$\rho^{(2)T}_{\alpha\gamma}(\mathbf{r})|_{\mathbf{B}_0} \underset{r\to\infty}{\sim} -\mathrm{sgn}(e_\alpha)\mathrm{sgn}(e_\gamma)\rho^2 \times\frac{3\sqrt{3}}{10}\Gamma^{7/2}\left(\frac{\lambda_\alpha}{a}\right)^2\left(\frac{\lambda_\gamma}{a}\right)^2\frac{P_4(\cos\theta)}{x^5}. \quad (21)$$

In the opposite regime of very weak B_0, the limiting law is just equal to Eq. (21) multiplied by (1/105) times $(\beta\mu_{B\alpha}B_0)^2(\beta\mu_{B\gamma}B_0)^2$, where $\mu_{B\alpha}$ is the Bohr magneton, $\mu_{B\alpha} = e_\alpha\hbar/2m_\alpha c$.

An example where we can compare results with or without magnetic field is that of an intrinsic semiconductor where the charge-carrier gas (made of electrons and holes) is indeed at finite temperature and weakly degenerated as well as weakly coupled. For instance, in germanium, holes have the same mass as electrons, $a \sim 1530$ Å and $T \sim 300$ K, so that $\lambda_-/a \sim 0.01$, $\Gamma \sim 0.4$, and $r_* \sim 43\xi_D$. In laboratory experiments, magnetic fields are not much stronger than a few teslas. For $B_0 \sim 1$ T, $\beta\mu_B B_0 \sim 2.2\times 10^{-3}$ T and we are in conditions of weak coupling with the external magnetic field. Then we find that $r_* \sim 66\xi_D$. We notice that if it were possible to generate magnetic fields so intense that the coefficient of the $1/r^5$ tail would no longer depend on B_0 at leading order, then $r_* \sim 36\xi_D$.

It is not surprising that r_* is about ten ξ_D. Indeed, our analytical expressions do correspond to fully quantum dynamics but quantum effects are not quantitatively important at low density. However, there may exist some system with stronger quantum features where r_* might be of the same order as ξ_D.

III. BASIC FORMULAS AT ANY DENSITY

A. Particle versus loop correlations

The part of the correlation that arises from configurations where particles at positions $\mathbf{0}$ and $\mathbf{r}$ are not exchanged under a cyclic permutation of quantum statistics may be expressed as [9]

$$\rho^{(2)T}_{\alpha_a\alpha_b}|_{\mathrm{nonexch}}(\mathbf{r}) = \sum_{p_a,p_b} p_a p_b \int D(\mathbf{X}_a)\int D(\mathbf{X}_b)\rho(\chi_a)\rho(\chi_b)h(\mathbf{r},\chi_a,\chi_b), \quad (22)$$

with notations of Paper I. In the part of the paper devoted to the derivation of the coefficients of algebraic tails, we use the notation α_a and α_b in order to keep track of loops that are root points in the diagrammatic representation of the Ursell function h. We recall that a "point" in a diagram denotes the loop variables $(\mathbf{R},\chi)$. The analogous nonexchange part of the induced charge density reads [10]

$$\frac{\sum_\gamma e_\gamma \rho^{\mathrm{ind}}_\gamma\Big|_{\mathrm{nonexch}}}{\delta q(\mathbf{k})} = -\frac{4\pi\beta}{\mathbf{k}^2}\int d\chi_a\rho(\chi_a)e_{\alpha_a}p_a G_h(\mathbf{k},\chi_a), \quad (23)$$

where G_f is defined as

$$G_f(\mathbf{k},\chi) \equiv \int d\chi_b\rho(\chi_b)e_{\alpha_b}\int_0^{p_b} d\tau e^{-i\mathbf{k}\cdot\mathbf{X}_b(\tau)}f(\mathbf{k},\chi;\chi_b). \quad (24)$$

The structures of leading algebraic tails of various correlations have been given in terms of diagrams with weight $\rho(\mathcal{L})$ before any expansion in Paper I. They involve diagrams $\tilde{\Pi}$ with bonds F^{cc}, F^{cm}, $F_R - W$, and W. For all kinds of correlations, they are expressed in terms of one and the same function h^{nn}, with proper "dressings" which describe screening of monopole-monopole and monopole-multipole quantum interactions. h^{nn} is the sum of all diagrams in h in which root points are not Coulomb root points, i.e., in which each root point $\mathcal{L}_r$ is neither the end point of a single bond $F^{cc}(\mathcal{L}_r,\mathcal{L}_i)$ or $F^{cm}(\mathcal{L}_r,\mathcal{L}_i)$. For instance, the $1/r^6$ tail of $\rho^{(2)T}_{\alpha_a\alpha_b}$ comes from

$$\Sigma_D * h^{nn} * \Sigma_D, \tag{25}$$

where Σ_D describes a loop surrounded by a polarization cloud in the mean-field ''Debye'' approximation (see Sec. VIII A of Paper I). We recall that * denotes a convolution in position space and an integration over the internal degrees of freedom of the intermediate loop.

B. Structure of Fourier transforms of correlation tails

Since diagrams of interest are convolutions, we work in Fourier space. Indeed, the large-distance behavior of a convolution $f*g$ is merely given by the nonanalytical singularities in the product of small-**k** expansions of Fourier transforms of f and g (see Sec. VI B of Paper I).

According to the general study performed in Paper I, the orders in $|\mathbf{k}|$ of dressings and of h^{nn} are determined by two kinds of symmetry arguments: on one hand, parity arguments which occur when loop shapes are integrated over and, on the other hand, arguments of rotational invariance around any axis (or only around some given direction) which make (or do not make) some terms analytical in the components of **k**. In the very simple example which we mentioned in Eq. (25) when $\mathbf{B}_0=\mathbf{0}$, the terms of order $|\mathbf{k}|^0$, $|\mathbf{k}|$, and $|\mathbf{k}|^2$ in the singular part $S_{h^{nn}}(\mathbf{k},\chi_1,\chi_2)$ of the Fourier transform of h^{nn} are canceled by symmetry arguments at any finite density and singularities of $h^{nn}(\mathbf{k})$ appear only from order $|\mathbf{k}|^3$. We notice that, if the first term in the Fourier transform of the dressing is of order $|\mathbf{k}|^p$ with $p>0$, then the order of the singular term to be looked for in $S_{h^{nn}}$ is lowered for a given inverse power law in the decay of the considered correlation.

Moreover, the part of $S_{h^{nn}}(\mathbf{k},\mathbf{X}_1,\mathbf{X}_2)$ which is relevant has a given parity in $\mathbf{X}_i$, because each term of a given order in $|\mathbf{k}|$ in the small-**k** expansion of the dressing has a given parity in the shapes of its arguments. As in Paper I, we use the notation $f^{[q,q']}(\chi,\chi')$ where $q=0$ $(q'=0)$ if f is even under inversion of $\mathbf{X}$ $(\mathbf{X}')$ and $q=1$ $(q'=1)$, if it is odd. According to Sec. VIII A of Paper I

$$\Sigma_D(\mathbf{k},\chi_1;\chi_a)=\delta_{\chi_1,\chi_a}-e_{\alpha_a}p_a\frac{4\pi\beta\rho(\chi_1)e_{\alpha_1}p_1}{\kappa^2}\left[1-\frac{\mathbf{k}^2}{\kappa^2}\right]+O_{\text{anal}}(|\mathbf{k}|^4), \tag{26}$$

where $O_{\text{anal}}(|\mathbf{k}|^4)$ denotes a term of order $|\mathbf{k}|^4$ which is analytical in the components of **k**, and at any order in $|\mathbf{k}|$

$$\Sigma_D=\Sigma_D^{[0,0]}. \tag{27}$$

With the convention that the root point is always a Coulomb point in dressings by Σ_D and F^{cm}, so that $F^{cm}(\mathbf{k},\chi_1;\chi_a)\equiv F^{cm}(\mathbf{k},\chi_a,\chi_1)$,

$$\rho(\chi_1)F^{cm}(\mathbf{k},\chi_1;\chi_a)=e_{\alpha_a}p_a\frac{4\pi\beta e_{\alpha_1}p_1\rho(\chi_1)}{\kappa^2}\left[\int_0^{p_1}\frac{d\tau}{p_1}i\mathbf{k}\cdot\mathbf{X}_1(\tau)+\int_0^{p_1}\frac{d\tau}{p_1}\frac{1}{2}(\mathbf{k}\cdot\mathbf{X}_1(\tau))^2\right]+O(|\mathbf{k}|^3). \tag{28}$$

Thus the term $F^{cm(p)}(\mathbf{k},\chi,\chi')$ of order $|\mathbf{k}|^p$ in $F^{cm}(\mathbf{k},\chi,\chi')$ is such that

$$F^{cm(p)}=F^{cm(p)[0,\theta(p)]}, \tag{29}$$

where $\theta(p)=0$ if p is even and $\theta(p)=1$ if p is odd. In fact, F^{cm} is independent of $\mathbf{X}_a$. Moreover, according to Sec. V A of Paper II,

$$\int d\mathbf{r}F^{cm}(\mathbf{r},\chi_1;\chi_a)=0. \tag{30}$$

Other properties arising from parity arguments are useful to simplify formulas. First,

$$\int d\mathbf{r}[F_R-W](\mathbf{r},\chi,\chi')=\int d\mathbf{r}[F_R-W_3](\mathbf{r},\chi,\chi'), \tag{31}$$

where W_3 is the $1/r^3$ part in the purely algebraic term

$$W(\mathbf{r},\chi,\chi')=-\beta e_{\alpha}e_{\alpha'}\int_0^p d\tau\int_0^{p'}d\tau'\{\delta([\tau-P(\tau)]-[\tau'-P(\tau')])-1\}\frac{1}{|\mathbf{r}+\mathbf{X}'(\tau')-\mathbf{X}(\tau)|}. \tag{32}$$

Only W_3 contributes to $\int d\mathbf{r}W(\mathbf{r})$ in Eq. (31), because every derivative of greater order than 2 that is not canceled by parity arguments involves an even power of the Laplacian of $1/r$, the integral of which vanishes. (This property has already been used in Sec. IV B of Paper II.) Since $\Delta(1/r)=-4\pi\delta(\mathbf{r})$,

$$\int d\mathbf{r}W_3(\mathbf{r},\chi,\chi')=-\frac{4\pi}{3}\beta e_{\alpha}e_{\alpha'}\int_0^p d\tau\int_0^{p'}d\tau'\{\delta([\tau-P(\tau)]-[\tau'-P(\tau')])-1\}\mathbf{X}(\tau)\cdot\mathbf{X}'(\tau'). \tag{33}$$

Moreover,

$$W_3^{[0,0]}=0 \tag{34}$$

and subsequently, according to Eq. (31),

$$\int d\mathbf{r}[F_R-W]^{[0,0]}(\mathbf{r},\chi,\chi')=\int d\mathbf{r}F_R^{[0,0]}(\mathbf{r},\chi,\chi'). \tag{35}$$

In the present simple example (25), only the part $S_{h^{nn}}^{(3)[0,0]}(\mathbf{k},\chi_1,\chi_2)$ of $S_{h^{nn}}^{(3)}(\mathbf{k},\chi_1,\chi_2)$ is to be considered because $\Sigma_D(\mathbf{k}=\mathbf{0},\chi;\chi')$ is even under inversion of both its arguments. As a conclusion, the part from h that does contribute to the $1/r^6$ tail of $\rho_{\alpha_a\alpha_b}^{(2)T}(r)|_{\mathbf{B}_0=\mathbf{0}}$ arises only from $\Sigma_D*h^{nn}*\Sigma_D$ and has a very simple structure in Fourier space,

$$S_h^{(3)}|_{\mathbf{B}_0=\mathbf{0}}(\mathbf{k},\chi_a,\chi_b)=\int d\chi_1\int d\chi_2 \times\Sigma_D^{(0)}(\mathbf{k},\chi_1;\chi_a)S_{h^{nn}}^{(3)[0,0]}(\mathbf{k},\chi_1,\chi_2) \times\Sigma_D^{(0)}(\mathbf{k},\chi_2;\chi_b). \quad (36)$$

IV. SCHEME FOR LOW-DENSITY EXPANSIONS

Once structures of correlation tails have been analyzed in Fourier space, the low-density expansions may be achieved in two steps: a first expansion in terms of the loop density, and a second one in terms of the particle density. As in Paper II, the order in density (of loops or of particles) is denoted by braces in order to avoid confusion with the order in $\mathbf{k}$ which is referred to in parentheses. We use the notation $\rho_{\rm loop}^n[\rho^n]$ for the order in loop [particle] density.

A. Loop-density expansions

1. Dressings at low loop density

Dressings are entirely scaled by $\kappa=\sqrt{4\pi\beta\int d\chi p^2e^2\rho(\chi)}$ according to the definitions of F^{cc} and F^{cm} given in Sec. IV A of Paper I. Simple results are the following. $\rho(\chi')F^{cc}(\mathbf{k}=\mathbf{0},\chi,\chi')$ is of order $\rho_{\rm loop}^0$ exactly, and, according to Eq. (26),

$$\Sigma_D(\mathbf{k}=\mathbf{0},\chi,\chi')=O(\rho_{\rm loop}^0). \quad (37)$$

Moreover, the term of order $|\mathbf{k}|^{2q}$ is proportional to $\rho_{\rm loop}^{-q}$, so that the order in ρ decreases when the order in $|\mathbf{k}|$ increases. On the contrary, F^{cm} has a more complex structure. Nevertheless, we only need a property derived from Eq. (28), namely,

$$\rho(\chi')F^{cm(1)}(\mathbf{k},\chi;\chi')=O(\rho_{\rm loop}^0). \quad (38)$$

2. Loop-density expansions for tails of h^{nn}

According to Ref. [10] and Sec. VII A of Paper I, a tail $S_{h^{nn}}^{(\gamma)}(\mathbf{r},\chi,\chi')$ is expressed as a $1/r^\gamma$ term independent of any loop density times integrals of functions $\mathcal{G}$ that involve graphs $\tilde{\Pi}$ with weights $\rho(\mathcal{L})$ and bonds F^{cc}, F^{cm}, F_R-W, and W. These $\mathcal{G}$'s are expanded in powers of loop density and then in powers of particle density by decomposing diagrams $\tilde{\Pi}$ in terms of diagrams $\tilde{\Pi}_T$ with bonds F^{cc}, F^{cm}, $[F^{cc}]^2/2$, $F_{RT}-W$, and W, with $F_{RT}\equiv F_R-[F^{cc}]^2/2$. Indeed, the $\rho_{\rm loop}$ then ρ expansions rely on a scale analysis in $\rho_{\rm loop}$ introduced in Sec. IV A of Paper II for the derivation of the low-density free energy, with the only difference that the bond F_{RT} is split into $F_{RT}-W$ and W, because the latter decomposition is the proper one for selecting algebraic tails.

The orders in $\rho_{\rm loop}$ for F^{cc} and F^{cm} have already been given. $F_R(\mathbf{k})$ and $W(\mathbf{k})$ are known from Sec. IV B of Paper II, and

$$\rho(\chi')[F_{RT}-W](\mathbf{k}=\mathbf{0},\chi,\chi')=O(\rho_{\rm loop}). \quad (39)$$

We notice that properties (31) and (35) also hold when F_R is replaced by F_{RT}. As shown below, the expansions of algebraic tails of $S_{h^{nn}}$ start at order zero in $\rho_{\rm loop}$.

B. Particle-density expansions

Expansions in terms of quantum particle densities are performed by using the ρ expansions of $\rho(\mathcal{L})$, κ, and of bonds. We recall that integration over loop shapes cannot diminish the order in density, but, on the contrary, it can only increase it.

The fundamental properties for the ρ-expansion of the loop density have been derived in Sec. III D of Paper II: $\rho_{\alpha,p}(\mathbf{X}_p)$ starts at order ρ^p,

$$D(\boldsymbol{\xi})\rho_{\alpha,1}(\boldsymbol{\xi})=\rho_\alpha D_{\mathbf{B}_0}(\boldsymbol{\xi})+O(\rho^2) \quad (40)$$

and $\kappa=\kappa_D+O(\rho^{3/2})$. Equation (40) is valid in the presence as well in the absence of a magnetic field (because it comes from a diagrammatic structure).

As shown below, only loops with $p=1$ will have to be taken into account in the study of the coefficients of leading tails at the first two orders in density. For instance, the coefficient $A_{\alpha_a\alpha_b}$ of the $1/r^6$ tail of the particle-particle correlation $\rho_{\alpha_a\alpha_b}^{(2)T}|_{\mathbf{B}_0=\mathbf{0}}(r)$ will be easily calculated up to order $\rho^{5/2}$, because, according to Eqs. (22), (36), and (37), it involves only $S_{h^{nn}}^{(6)[0,0]\{0\}}(\mathbf{r},\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)$ and $S_{h^{nn}}^{(6)[0,0]\{1/2\}}(\mathbf{r},\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)$ which are of order ρ^0 and $\rho^{1/2}$, respectively.

The tail of the particle-particle correlation will not be calculated up to order ρ^3, because sources of mistakes are too numerous. Indeed, $\rho(\mathcal{L})$ should be expanded up to order ρ^2, dressings in Fourier space should be taken into account further than up to the first simple orders in ρ and $\mathbf{k}$, and diagrams with more numerous internal points would be involved. For the same reasons, we will not give the subleading tails of correlations.

However, in view of the qualitative discussion of higher-order terms in ρ, we give more precise formulas for $\rho_{\alpha,1}(\boldsymbol{\xi})$ and $\rho_{\alpha,2}(\mathbf{X}_2)$ at orders ρ^2 and $\rho^{5/2}$. These expressions are derived from results in Sec. III D of Paper II. In the difference $J^{\{1\}}(\boldsymbol{\xi})-\int D_{\mathbf{B}_0}(\boldsymbol{\xi}')J^{\{1\}}(\boldsymbol{\xi}')$, only the part of $J^{\{1\}}(\boldsymbol{\xi})$ that explicitly depends on $\boldsymbol{\xi}$ does contribute and, according to a formula in Sec. III E and calculations in Sec. IV B of Paper II,

$$D(\boldsymbol{\xi})\rho_{\alpha,1}^{\{2\}}(\boldsymbol{\xi})=\sum_\gamma \rho_\gamma\lim_{R\to\infty}\int_{r<R}d\mathbf{r}\int D_{\mathbf{B}_0}(\boldsymbol{\xi}')\left[e^{-\beta e_\alpha e_\gamma v(\mathbf{r},\boldsymbol{\xi},\boldsymbol{\xi}')} -\int D_{\mathbf{B}_0}(\boldsymbol{\xi}'')e^{-\beta e_\alpha e_\gamma v(\mathbf{r},\boldsymbol{\xi}'',\boldsymbol{\xi}')}\right]-\rho_\alpha E_\alpha^*. \quad (41)$$

In Eq. (41) we have used notations of Sec. III E of Paper II where $E_\alpha^*\equiv\int D(\mathbf{X}_2)E_{\text{exch},\alpha}^*(\mathbf{X}_2)$. The term of order $\rho^{3/2}$ has the structure

$$J_\alpha^{\{3/2\}}(\boldsymbol{\xi})-\int D_{\mathbf{B}_0}(\boldsymbol{\xi}')J_\alpha^{\{3/2\}}(\boldsymbol{\xi}')-\beta e_\alpha^2\rho_\alpha\kappa_D E_\alpha^*. \quad (42)$$

The loop density with $p=2$ is

$$\rho_{\alpha,2}^{\{2\}}(\mathbf{X}_2)=\rho_\alpha^2\frac{1}{2}E_{\text{exch},\alpha}^*(\mathbf{X}_2). \quad (43)$$

The correction of order $\rho^{5/2}$ is equal to the term of order ρ^2 given in Eq. (43) times $\beta e_\alpha^2\kappa_D$. Equations (41)–(43) will not be written more precisely, because the only important point is that, after integration over $\boldsymbol{\xi}$ or $\mathbf{X}_2$, these terms will give rise to diagonal or exchange matrix elements of two-body quantum Gibbs factors.

V. CASE $\mathbf{B}_0=\mathbf{0}$

The diagrammatic structures of leading tails of various correlations have been given in Secs. VIII B and VIII C of Paper I. In the present section we immediately select the part of the Fourier transform of the dressing that gives the contribution to the first orders in $\rho_{\rm loop}$. This selection is done from the start in order to avoid writing the more complex formulas that hold before $\rho_{\rm loop}$ expansions.

A. Structure in Fourier space and dressing at low loop density

The structure of the Fourier transform of the $1/r^6$ tail of the particle-particle correlation has been derived as an example in Sec. II A. We recall briefly the main lines of the arguments leading to Eq. (36). According to Paper I, the $A_{\alpha_a\alpha_b}/r^6$ tail of $\rho^{(2)T}_{\alpha_a\alpha_b}$ comes from the $1/r^6$ tail of $\Sigma_D * h^{nn} * \Sigma_D$, and more precisely only from the values of the Σ_D's for $\mathbf{k}=\mathbf{0}$ and from the $1/r^6$ tail $S^{(6)}_{h^{nn}}(\mathbf{r},\mathbf{X}_a,\mathbf{X}_b)$ of h^{nn} before integration over the shapes $\mathbf{X}_a$ and $\mathbf{X}_b$. Moreover, parity arguments show that only the part $S^{(6)[0,0]}_{h^{nn}}(\mathbf{r},\chi_1,\chi_2)$ of the $1/r^6$ tail of h^{nn} that is even under inversion of each argument is to be considered.

Since $\Sigma_D(\mathbf{k}=\mathbf{0},\chi_1;\chi_a)$ is exactly of order $\rho^0_{\rm loop}$ and since $S^{(6)[0,0]}_{h^{nn}}(\mathbf{r},\chi_1,\chi_2)$ starts at order $\rho^0_{\rm loop}$ (according to Appendix A), the $A_{\alpha_a\alpha_b}/r^6$ tail of the particle-particle correlation at the first two orders $\rho^2_{\rm loop}$ and $\rho^{5/2}_{\rm loop}$ is given by inserting Eq. (36), where $S^{(3)[0,0]}_{h^{nn}}(\mathbf{k},\chi_1,\chi_2)$ is reduced to its parts of orders $\rho^0_{\rm loop}$ and $\rho^{1/2}_{\rm loop}$, into Eq. (22). The result is

$$\frac{A_{\alpha_a\alpha_b}}{r^6}=\int d\chi_1 p_1\rho(\chi_1)\left[\delta_{e_{\alpha_1},e_{\alpha_a}}-4\pi\beta e_{\alpha_1}\times\frac{e_{\alpha_a}\sum_{p_a}p_a^2\int D(\mathbf{X}_a)\rho(\chi_a)}{\kappa^2}\right]\int d\chi_2 p_2\rho(\chi_2)\times\left[\delta_{e_{\alpha_2},e_{\alpha_b}}-4\pi\beta e_{\alpha_2}\frac{e_{\alpha_b}\sum_{p_b}p_b^2\int D(\mathbf{X}_b)\rho(\chi_b)}{\kappa^2}\right]\times\{S^{(6)[0,0]}_{h^{nn}}|^{\{0\}}_{\rm loop}(\mathbf{r},\chi_1,\chi_2)+S^{(6)[0,0]}_{h^{nn}}|^{\{1/2\}}_{\rm loop}(\mathbf{r},\chi_1,\chi_2)\}+O(\rho^3_{\rm loop}). \tag{44}$$

The B_{α_a}/r^8 tail of the particle-charge correlation $\Sigma_{\alpha_b}e_{\alpha_b}\rho^{(2)T}_{\alpha_a\alpha_b}|_{\mathbf{B}_0=\mathbf{0}}(r)$ comes from $\Sigma_D * h^{nn} * \Sigma_D^{**}$ with $\Sigma_D^{**}=\Sigma_D+\rho F^{mc}+\rho F^{cc}*\rho F^{mc}$, according to Sec. VIII B of Paper I. The analysis of the structure in Fourier space is the following. After summation over charges, the Fourier transform of Σ_D^{**} gives a contribution that begins at order $|\mathbf{k}|$. Indeed,

$$\int d\chi_b\rho(\chi_b)e_{\alpha_b}p_b\Sigma_D^{**}(\mathbf{k},\chi_2;\chi_b)=\rho(\chi_2)p_2e_{\alpha_2}\left[\frac{\mathbf{k}^2}{\kappa^2}+A^{(2)}(\mathbf{k})-\int_0^{p_2}\frac{d\tau}{p_2}(e^{i\mathbf{k}\cdot\mathbf{X}_2(\tau)}-1)+O(|\mathbf{k}|^3)\right] \tag{45}$$

and *a priori* both $1/r^5$ and $1/r^6$ tails of h^{nn} should contribute to the $1/r^8$ tail of interest. However, only the term $\mathbf{k}^2[\rho(\chi_2)/\kappa^2]$ does contribute at the first orders $\rho^2_{\rm loop}$ and $\rho^{5/2}_{\rm loop}$ in B_α/r^8. In other words, terms of interest come only from $\Sigma_D * h^{nn} * \Sigma_D$, as in the case of the particle-particle correlation, though the general diagrammatic structures of $\rho^{(2)T}_{\alpha_a\alpha_b}|_{\mathbf{B}_0=\mathbf{0}}(r)$ and $\Sigma_{\alpha_b}e_{\alpha_b}\rho^{(2)T}_{\alpha_a\alpha_b}|_{\mathbf{B}_0=\mathbf{0}}(r)$ are different at finite density. Therefore these terms involve only $S^{(6)[0,0]}_{h^{nn}}(\mathbf{r},\chi_1,\chi_2)$ [and not $S^{(5)[0,0]}_{h^{nn}}(\mathbf{r},\chi_1,\chi_2)$]. They read

$$-\frac{1}{\kappa^2}\int d\chi_1 p_1\rho(\chi_1)\left[\delta_{e_{\alpha_1},e_{\alpha_a}}-4\pi\beta e_{\alpha_1}\times\frac{e_{\alpha_a}\sum_{p_a}p_a^2\int D(\mathbf{X}_a)\rho(\chi_a)}{\kappa^2}\right]\times\int d\chi_2 p_2e_{\alpha_2}\rho(\chi_2)\{\Delta S^{(6)[0,0]}_{h^{nn}}|^{\{0\}}_{\rm loop}(\mathbf{r},\chi_1,\chi_2)+\Delta S^{(6)[0,0]}_{h^{nn}}|^{\{1/2\}}_{\rm loop}(\mathbf{r},\chi_1,\chi_2)\}+O(\rho^2_{\rm loop}). \tag{46}$$

The C/r^{10} tail of the charge-charge correlation

$$\sum_{\alpha_a,\alpha_b}e_{\alpha_a}e_{\alpha_b}\rho^{(2)T}_{\alpha_a\alpha_b}|_{\mathbf{B}_0=\mathbf{0}}(r)$$

comes from $\Sigma_D^{**} * h^{nn} * \Sigma_D^{**}$. An analysis similar to that performed for the particle-charge correlation shows that, at the first two orders in $\rho_{\rm loop}$, the C/r^{10} tail arises only from $\Sigma_D * h^{nn} * \Sigma_D$. It is equal to

$$\frac{1}{(\kappa^2)^2}\int d\chi_1 p_1e_{\alpha_1}\rho(\chi_1)\int d\chi_2 p_2e_{\alpha_2}\rho(\chi_2)\times\{\Delta\Delta S^{(6)[0,0]}_{h^{nn}}|^{\{0\}}_{\rm loop}(\mathbf{r},\chi_1,\chi_2)+\Delta\Delta S^{(6)[0,0]}_{h^{nn}}|^{\{1/2\}}_{\rm loop}\times(\mathbf{r},\chi_1,\chi_2)\}+O(\rho_{\rm loop}). \tag{47}$$

The nonexchange part of the induced charge density $\Sigma_\gamma e_\gamma\rho^{\rm ind}_\gamma(r;\delta q)$ in the presence of an infinitesimal external charge δq located at the origin is given by the linear response expression (23). It decays as $1/r^8$ [10], as the particle-charge correlation. According to Sec. VIII C of Paper I, the B/r^8 tail of the induced charge density originates from $\Sigma_D^* * h^{nn} * \Sigma_D^{**}$, with

$$\Sigma_D^*(\mathbf{k},\chi_1;\chi_a)\equiv\Sigma_D(\mathbf{k},\chi_1;\chi_a)+\rho(\chi_1)F^{cm}(\mathbf{r},\chi_a,\chi_1). \tag{48}$$

According to Eqs. (26) and (28)

$$\int d\chi_a \rho(\chi_a) e_{\alpha_a} p_a \Sigma_D^*(\mathbf{k},\chi_1;\chi_a) = \rho(\chi_1) e_{\alpha_a} p_1 \left[\frac{\mathbf{k}^2}{\kappa^2} - \int_0^{p_1} \frac{d\tau}{p_1} (e^{-i\mathbf{k}\cdot\mathbf{X}_1(\tau)} - 1) + O(|\mathbf{k}|^3) \right]. \tag{49}$$

This expression is similar to Eq. (45) without the term $A^{(2)}(\mathbf{k})$ and with the change of $\mathbf{X}_2$ into $-\mathbf{X}_1$. Another screening equation reads

$$G_{\Sigma_D^{**}}(\mathbf{k},\chi_2) = \frac{\rho(\chi_2) p_2 e_{\alpha_2}}{\kappa^2} \mathbf{k}^2 + O(|\mathbf{k}|^3), \tag{50}$$

where G_f is defined in Eq. (24). Equations (49) and (50) imply that the $1/r^8$ tail of the induced charge density at the lowest order in ρ_{loop} is entirely determined by $\Sigma_D * h^{nn} * \Sigma_D$. It is reduced to

$$\frac{4\pi\beta}{(\kappa^2)^2} \int d\chi_1 p_1 e_{\alpha_1} \rho(\chi_1) \int d\chi_2 p_2 e_{\alpha_2} \rho(\chi_2) \times \{ \Delta S_{h^{nn}}^{(6)[0,0]}|_{\text{loop}}^{\{0\}}(\mathbf{r},\chi_1,\chi_2) + \Delta S_{h^{nn}}^{(6)[0,0]}|_{\text{loop}}^{\{1/2\}}(\mathbf{r},\chi_1,\chi_2) \} + O(\rho_{\text{loop}}). \tag{51}$$

B. Loop-density expansions of the $1/r^6$ tail of h^{nn}

According to Appendixes A and B, the $1/r^6$ tails of h^{nn} at orders ρ_{loop}^0 and $\rho_{\text{loop}}^{1/2}$ arise from purely $1/r^6$ structures with two intermediate points. $A_{\alpha_a\alpha_b}/r^6$ at order ρ_{loop}^2 is derived from Eq. (44) with

$$S_{h^{nn}}^{(6)[0,0]}|_{\text{loop}}^{\{0\}} = \frac{1}{2}[W_3]^2, \tag{52}$$

where W_3 is the $1/r^3$ tail of W defined in Eq. (32) and the corresponding part of h comes from the diagram $\tilde{\Pi}$ equal to $\Sigma_D * [F_R - W] * \Sigma_D$. At order $\rho_{\text{loop}}^{5/2}$, $A_{\alpha_a\alpha_b}/r^6$ originates from Eq. (44) with

$$S_{h^{nn}}^{(6)[0,0]}|_{\text{loop}}^{\{1/2\}}(\mathbf{r},\chi_1,\chi_2) = \int d\chi_1' \left(\int d\mathbf{x} \left[\frac{1}{2}[F^{cc}]^2 \rho * \Sigma_D \right](\mathbf{x},\chi_1,\chi_1') \right) \times \frac{1}{2}[W_3(\mathbf{r},\chi_1',\chi_2)]^2 + \int d\chi_2' \frac{1}{2}[W_3(\mathbf{r},\chi_1,\chi_2')]^2 \times \left(\int d\mathbf{y} \left[\Sigma_D * \rho \frac{1}{2}[F^{cc}]^2 \right](\mathbf{y},\chi_2',\chi_2) \right). \tag{53}$$

The corresponding part of h arises from the sum of the diagram $\tilde{\Pi}$,

$$\Sigma_D * \frac{1}{2}[F^{cc}]^2 \rho * \Sigma_D * [F_R - W] * \Sigma_D, \tag{54}$$

and from the symmetric diagram

$$\Sigma_D * [F_R - W] * \Sigma_D * \rho \frac{1}{2}[F^{cc}]^2 * \Sigma_D. \tag{55}$$

The leading large-distance behaviors of these diagrams are obtained by replacing $F_R - W$ by $(1/2)[W_3]^2$ and other functions by their Fourier transforms at $\mathbf{k}=\mathbf{0}$.

$A_{\alpha_a\alpha_b}/r^6$ at order ρ_{loop}^3 comes from an $S_{h^{nn}}^{(6)}$ with an algebraic structure involving either two or three intermediate points. In a simplified notation of the objects introduced in Appendix A, terms with a purely $1/r^6$ tail involving two intermediate points at order ρ_{loop}^3 read

$$\rho_{\text{loop}}^{\{2\}}(1) \mathcal{G}_2^{\{0\}}(1,1') \frac{1}{2}[W_3(1',2')]^2 \mathcal{G}_2^{\{0\}}(2',2) \rho_{\text{loop}}^{\{1\}}(2) + \text{sym. term} \tag{56}$$

or

$$\rho_{\text{loop}}^{\{1\}}(1) \mathcal{G}_2^{\{1/2\}}(1,1') \frac{1}{2}[W_3(1',2')]^2 \mathcal{G}_2^{\{1/2\}}(2',2) \rho_{\text{loop}}^{\{1\}}(2) \tag{57}$$

or

$$\rho_{\text{loop}}^{\{1\}}(1) \mathcal{G}_2^{\{1\}}(1,1') \frac{1}{2}[W_3(1',2')]^2 \mathcal{G}_2^{\{0\}}(2',2) \rho_{\text{loop}}^{\{1\}}(2), \tag{58}$$

where $\rho_{\text{loop}}^{\{1\}}$ denotes a contribution of the loop density $\rho(\chi)$ with $p=1$ while $\rho_{\text{loop}}^{\{2\}}$ corresponds either to the exchange term in the expansion of $\rho(\boldsymbol{\xi})$ given in Eq. (41) or it refers to the leading term in $\rho_{\alpha,2}(\mathbf{X}_2)$ which is equal to Eq. (43). Terms with a purely $1/r^6$ tail involving three intermediate points at order ρ_{loop}^3 are of the form

$$\rho_{\text{loop}}^{\{1\}}(1) \mathcal{G}_3^{\{1\}}(1,1',1'') W_3(1',2') W_3(1'',2') \times \mathcal{G}_2^{\{0\}}(2',2) \rho_{\text{loop}}^{\{1\}}(2), \tag{59}$$

where $\mathcal{G}_2^{\{0\}}$, $\mathcal{G}_2^{\{1/2\}}$, $\mathcal{G}_2^{\{1\}}$, and $\mathcal{G}_3^{\{1\}}$ are given in Appendix B. (We notice that in the present case the index $\{n\}$ refers to the order in density of the function *after* its integration over position variables.)

C. Results at the first two orders in particle density

An important conclusion of Sec. V B is that, at the first two orders in ρ_{loop}, the coefficients of all tails $A_{\alpha\gamma}/r^6$, B_α/r^8, C/r^{10}, and $B^\star/r^{10}$ prove to arise only from $\Sigma_D * h^{nn} * \Sigma_D$. >From now on, we replace α_a by α and α_b by γ.

Now we turn to ρ expansions by using the properties $\rho_{\alpha,p}(\mathbf{X}_p) = O(\rho^p)$ and $\rho_{\alpha,1}(\boldsymbol{\xi}) = \rho_\alpha + O(\rho^2)$ for $\mathbf{B}_0 = \mathbf{0}$ [see Eq. (40)]. At the first two orders in density the coefficients of all tails of interest are determined by diagrams where (root or internal) loops with only $p=1$ are to be considered. Thus

$\int d\chi$ is replaced by $\Sigma_\alpha \int D(\boldsymbol{\xi})$, and $\rho_\alpha(\boldsymbol{\xi})$ by ρ_α. Then the dressing is reduced to a classical Debye contribution

$$\sum_{p_a} p_a \int D(\mathbf{X}_a)\rho(\chi_a)\Sigma_D(\mathbf{k}=\mathbf{0},\chi_1;\chi_a)\Bigg|^{\{0\}}$$

$$=\rho_\alpha\left[\delta_{\alpha_1,\alpha}-e_\alpha\frac{4\pi\beta e_{\alpha_1}\rho_{\alpha_1}}{\kappa_D^2}\right]=\int d\mathbf{x}S^{\rm cl}_{D,\alpha\alpha_1}(\mathbf{x}), \qquad (60)$$

where $S^{\rm cl}_{D,\alpha\alpha_1}$ is given by Eq. (4). Besides, the $1/r^3$ tail W_3 of W in Eq. (32) takes the value (6) for $p_1=p_2=1$.

1. Particle correlation

According to the preceding section, the first two terms in the $\rho_{\rm loop}$ expansion of $A_{\alpha\gamma}/r^6$ are of orders $\rho^2_{\rm loop}$ and $\rho^{5/2}_{\rm loop}$ and are given by Eqs. (44), (52), and (53). When dressings by Σ_D are replaced by Eq. (60), the $1/r^6$ tail of the particle-particle correlation $\rho^{(2)T}_{\alpha\gamma}(r)|_{\mathbf{B}_0=\mathbf{0}}$ at orders ρ^n with $n=2,5/2$ reads

$$\frac{A^{\{n\}}_{\alpha\gamma}}{r^6}=\sum_{\alpha_1,\alpha_2}\int d\mathbf{x}S^{\rm cl}_{D,\alpha\alpha_1}(\mathbf{x})\int d\mathbf{y}S^{\rm cl}_{D,\alpha_2\gamma}(\mathbf{y})$$
$$\times[-\beta V^{\rm eff(6)}_{\alpha_1\alpha_2}(r)|^{\{n\}}], \qquad (61)$$

where $V^{\rm eff(6)}_{\alpha_1\alpha_2}(r)|^{\{0\}}$ is defined in Eq. (5) and

$$V^{\rm eff(6)}_{\alpha_1\alpha_2}(r)|^{\{1/2\}}=\sum_{\alpha'} V^{\rm eff(6)}_{\alpha_1\alpha'}(r)\Bigg|^{\{0\}}$$
$$\times\sum_{\gamma'}\int d\mathbf{x}S^{\rm cl}_{D,\alpha'\gamma'}(\mathbf{x})\int d\mathbf{y}\frac{1}{2}[F^{cc}_{D,\gamma'\alpha_2}(\mathbf{y})]^2$$
$$+\text{sym. term}. \qquad (62)$$

In Eq. (62) ‘‘ sym. term’’ denotes a symmetric term obtained by exchanging the roles of α_1 and α_2. Equation (61) is indeed equal to the tail of the convolution (2).

In terms of the covariance introduced in Sec. III C of Paper II, the squared mean dipolar potential reads

$$-\beta V^{\rm eff(6)}_{\alpha_1\alpha_2}(r)|^{\{0\}}=\frac{(\beta e_{\alpha_1}e_{\alpha_2}\lambda_{\alpha_1}\lambda_{\alpha_2})^2}{2}$$
$$\times\int_0^1 ds_1\int_0^1 ds_2\int_0^1 ds_1'\int_0^1 ds_2'$$
$$\times[\delta(s_1-s_2)-1][\delta(s_1'-s_2')-1]$$
$$\times{\rm cov}_{xx}(s_1,s_1')\,{\rm cov}_{xx}(s_2,s_2')$$
$$\times\sum_{\mu,\nu}\left[\partial_{\mu\nu}\left(\frac{1}{r}\right)\right]^2. \qquad (63)$$

According to the explicit value of the covariance for independent particles recalled in Sec. III C of Paper II, the four integrations over the s variables give a factor 1/720. Since $\Sigma_{\mu,\nu}[\partial_{\mu\nu}(1/r)]^2=6/r^6$, we get

$$-\beta V^{\rm eff(6)}_{\alpha_1\alpha_2}(r)|^{\{0\}}=\frac{1}{240}\beta^4\hbar^4\frac{e^2_{\alpha_1}}{m_{\alpha_1}}\frac{e^2_{\alpha_2}}{m_{\alpha_2}}\frac{1}{r^6}. \qquad (64)$$

The expression of $-\beta V^{\rm eff(6)}_{\alpha_1\alpha_2}(r)|^{\{1/2\}}$ is derived from Eqs. (62), (64), and

$$\sum_{\gamma'}\int d\mathbf{x}S^{\rm cl}_{D,\alpha'\gamma'}(\mathbf{x})\int d\mathbf{y}\frac{1}{2}[F^{cc}_{D,\gamma'\alpha_2}(\mathbf{y})]^2$$
$$=e^2_{\alpha_2}\pi\frac{\beta^2}{\kappa_D}\rho_{\alpha'}e_{\alpha'}\left[e_{\alpha'}-\frac{4\pi\beta\sum_{\gamma'}\rho_{\gamma'}e^3_{\gamma'}}{\kappa_D^2}\right]. \qquad (65)$$

By using Eq. (60) the coefficients $A^{\{n\}}_{\alpha\gamma}$ ($n=2,5/2$) of the $1/r^6$ tail given by Eqs. (61), (62), (64), and (65) may be written in the following concise form:

$$A^{\{n\}}_{\alpha\gamma}=\frac{\beta^4\hbar^4}{240}\rho_\alpha\rho_\gamma\sum_{\alpha_1,\alpha_2}\left[\delta_{\alpha,\alpha_1}-e_\alpha\frac{4\pi\beta e_{\alpha_1}\rho_{\alpha_1}}{\kappa_D^2}\right]$$
$$\times\left[\delta_{\gamma,\alpha_2}-e_\gamma\frac{4\pi\beta e_{\alpha_2}\rho_{\alpha_2}}{\kappa_D^2}\right]\mathcal{A}^{\{n\}}_{\alpha_1\alpha_2}, \qquad (66)$$

with

$$\mathcal{A}^{\{0\}}_{\alpha_1\alpha_2}=\frac{e^2_{\alpha_1}}{m_{\alpha_1}}\frac{e^2_{\alpha_2}}{m_{\alpha_2}}, \qquad (67)$$

$$\mathcal{A}^{\{1/2\}}_{\alpha_1\alpha_2}=e^2_{\alpha_1}e^2_{\alpha_2}\left(\frac{1}{m_{\alpha_1}}+\frac{1}{m_{\alpha_2}}\right)\frac{\pi\beta^2}{\kappa_D}$$
$$\times\left[\sum_{\alpha'}\rho_{\alpha'}\frac{e^4_{\alpha'}}{m_{\alpha'}}-\frac{\kappa^2_{e/m}}{\kappa_D^2}\sum_\gamma\rho_\gamma e^3_\gamma\right], \qquad (68)$$

where $\kappa^2_{e/m}=4\pi\beta\Sigma_\alpha\rho_\alpha e^3_\alpha/m_\alpha$. For instance, summations over α_1 and α_2 are performed with the result (1).

For a two-component plasma of charges e_+ and e_-, with masses m_+ and m_-, the previous general formulas at orders ρ^2 and $\rho^{5/2}$ may be specified by using the neutrality relation $e_-\rho_-=-e_+\rho_+$. We find

$$\rho^{T(2)}_{\alpha\gamma}(r)|_{\mathbf{B}_0=\mathbf{0}}\underset{r\to\infty}{\sim}\frac{\rho_\alpha\rho_\gamma}{r^6}\frac{\beta^4\hbar^4}{240}\left(\frac{e_+e_-}{e_++|e_-|}\right)^2$$
$$\times\left[\frac{e_+}{m_+}+\frac{|e_-|}{m_-}\right]^2\left[1+\frac{1}{2}\beta\kappa_D e_+|e_-|\right]. \qquad (69)$$

The ratio $A^{\{5/2\}}_{\alpha\gamma}/A^{\{2\}}_{\alpha\gamma}$ is of order $(a/\xi_D)^3$, namely, of order $\Gamma^{3/2}$ where Γ is the coupling constant defined after Eq. (19). The correction is indeed negligible in the weak-coupling limit. We notice that, in the case of a two-component plasma, the neutrality relation implies that the coefficients $A^{\{n\}}_{\alpha\gamma}$ of the $1/r^6$ tail of particle-particle correlations at orders ρ^n, with $n=2,5/2$, are positive. The corresponding effective interaction is attractive, whatever the signs of the charges. Moreover these coefficients satisfy the relation

$$\frac{A_{++}^{\{n\}}}{\rho_+^2}=\frac{A_{--}^{\{n\}}}{\rho_-^2}=\frac{A_{+-}^{\{n\}}}{\rho_+\rho_-}, \quad n=2,\frac{5}{2}. \tag{70}$$

We stress that, from a technical point of view, this equality arises only from the neutrality relation and from the structure (66) where $\mathcal{A}_{\alpha_1\alpha_2}^{\{n\}}$ has no special property. From a more physical point of view, the peculiar identity (70) is due to a classical contribution in the screening of every quantum charge by the surrounding plasma, as discussed in Sec. VIII D.

2. Other correlations

The B_α/r^8, C/r^{10}, and $B^\star/r^8$ tails of $\Sigma_\gamma e_\gamma \rho_{\alpha\gamma}^{(2)T}(r)|_{\mathbf{B}_0=\mathbf{0}}$, $\Sigma_{\alpha\gamma} e_\alpha e_\gamma \rho_{\alpha\gamma}^{(2)T}(r)|_{\mathbf{B}_0=\mathbf{0}}$, and $\Sigma_\gamma e_\gamma \rho_\gamma^{\mathrm{ind},L}(r;\delta q)|_{\mathbf{B}_0=\mathbf{0}}$ are given by Eqs. (46), (47), and (51) at the first two orders in ρ_{loop}. According to Eqs. (52) (53), (60), and (65), together with $\Delta(1/r^6)=30/r^8$ and $\Delta(1/r^8)=56/r^{10}$ we get for $n=2,5/2$

$$B_\alpha^{\{n-1\}}=-\frac{\beta^4\hbar^4}{8}\frac{\rho_\alpha}{\kappa_D^2}\sum_{\alpha_1,\alpha_2}\left[\delta_{\alpha,\alpha_1}-e_\alpha\frac{4\pi\beta e_{\alpha_1}\rho_{\alpha_1}}{\kappa_D^2}\right]\times e_{\alpha_2}\rho_{\alpha_2}\mathcal{A}_{\alpha_1\alpha_2}^{\{n\}}, \tag{71}$$

$$C^{\{n-2\}}=7\beta^4\hbar^4\frac{1}{\kappa_D^4}\sum_{\alpha_1,\alpha_2}e_{\alpha_1}\rho_{\alpha_1}e_{\alpha_2}\rho_{\alpha_2}\mathcal{A}_{\alpha_1\alpha_2}^{\{n\}}, \tag{72}$$

$$B^{\star\{n-2\}}=\frac{\pi}{2}\beta^5\hbar^4\frac{1}{\kappa_D^4}\sum_{\alpha_1,\alpha_2}e_{\alpha_1}\rho_{\alpha_1}e_{\alpha_2}\rho_{\alpha_2}\mathcal{A}_{\alpha_1\alpha_2}^{\{n\}}. \tag{73}$$

As already mentioned, at the first two orders in density, these tails arise only from $\Sigma_D * h^{nn} * \Sigma_D$ with adequate summations over charges. On the other hand, according to Eq. (60), the small-$\mathbf{k}$ behavior of $S_{\alpha\gamma,D}^{\mathrm{cl}}(\mathbf{k})$ is similar to the term $\rho(\chi_a)\Sigma_D^{(2)}(\mathbf{k},\chi_1;\chi_a)$ of order $|\mathbf{k}|^2$ given in Eq. (26) and which is of order $1/\rho$,

$$\sum_\gamma e_\gamma S_{\alpha\gamma,D}^{\mathrm{cl}}(\mathbf{k}) \underset{|\mathbf{k}|\to 0}{\sim} \frac{e_\alpha\rho_\alpha}{\kappa_D^2}\mathbf{k}^2 \tag{74}$$

and we retrieve Eqs. (71) and (72) from Eq. (2). Thus, at the first two orders in density, the tails coincide with those of the convolution (2) with adequate charge summation.

At the first order in density we get Eqs. (13), (14), and (15). Comparison of Eq. (15) with the linear term with respect to the given charge e_α in Eq. (13) shows that the algebraic tails of the linearly induced charge density and of the particle-charge correlation satisfy the more general relation

$$\sum_\gamma e_\gamma \rho_\gamma^{\mathrm{ind}}(r;e_\alpha)\bigg|_{\rho_\alpha=0}=\lim_{\rho_\alpha\to 0}\frac{\sum_\gamma e_\gamma \rho_{\alpha\gamma}^{(2)T}(r)}{\rho_\alpha}, \tag{75}$$

which is valid at any distance. Equation (75) also holds for any finite charge e_α and we derive from Eq. (13) that the $B^{\star\{0\}}/r^8$ tail of the charge density induced by a finite external charge q reads

$$\sum_\gamma e_\gamma \rho_\gamma^{\mathrm{ind}}(r;q)\bigg|_{\mathbf{B}_0=\mathbf{0}} \underset{r\to\infty}{\sim} \frac{1}{r^8}\frac{1}{32\pi}\beta^3\hbar^4\frac{\kappa_{e/m}^4}{\kappa_D^4}\left[q-\frac{\kappa_D^2}{\kappa_{e/m}^2}\frac{q^2}{m_q}\right], \tag{76}$$

where m_q is the mass of the particle with charge q. At the first order ρ^0 the response to an external charge q involves both a linear contribution in q and a quadratic one. However, the latter term exists only if the external charge is quantum—according to its origin from the formula (66)—because a classical external charge corresponds to the limit m_q going to infinity.

D. Diagrams that contribute at order ρ^3

First, we recall that in order to perform density expansions we have introduced bonds $F_{RT}=F_R-[F^{cc}]^2/2$ and $[F^{cc}]^2/2$ to calculate the integrated functions that are involved in the $1/r^6$ tail. $\int d\mathbf{r}[F^{cc}]^2(\mathbf{r})$ is exactly of order $1/\rho_{\mathrm{loop}}^{1/2}$ whereas the low-density expansion of $\int d\mathbf{r}F_{RT}(\mathbf{r})$ starts at order ρ_{loop}^0. At order ρ_{loop}^3 the purely algebraic term in the $1/r^6$ tail of the particle-particle correlation may have a structure with either two or three intermediate points. We first consider structures (56)–(58), where the quantum interaction involves only two intermediate points, then the structure (59) where three intermediate points appear in the quantum term.

At order ρ^3 a tail of the form (56) arises from the most simple diagram, $\rho[F_R-W]\rho$, either with $p_a=1$ and $\rho(\boldsymbol{\xi}_a)$ expanded up to order ρ^2 or with $p_a=2$ and $\rho_{\alpha,2}(\mathbf{X}_2)$ taken at order ρ^2. According to Eq. (41), when $p_a=1$ the ρ^3 tail is proportional to $\hbar^4$ times diagonal matrix elements of $\exp[-\beta H_{\alpha\gamma'}]$ and exchange matrix elements of $\exp[-\beta H_{\alpha\alpha}]$. When $p_a=2$, according to Eq. (43), there appears only a term proportional to $\hbar^4$ times an exchange contribution.

At order ρ^3, all density weights in Eqs. (57)–(59) are $\rho_{\mathrm{loop}}^{\{1\}}$, and only loops with $p=1$ do contribute at order ρ^3. A tail of the form (57) appears among the various $1/r^6$ tails of $\rho[F_R*\Sigma_D*\rho F_R*\Sigma_D*F_R]\rho$. At order ρ^3 one of these tails (namely, the tail where every Σ_D is replaced by its part δ) gives a falloff

$$\rho\left[\frac{1}{2}[F^{cc}]^2\rho*\frac{1}{2}[W_3]^2*\rho\frac{1}{2}[F^{cc}]^2\right]\rho. \tag{77}$$

The coefficient of this tail is purely proportional to $\hbar^4$.

The $1/r^6$ tails of the diagrams $\rho[F_R*\rho F_R]\rho$ and $\rho[F_R*\rho F^{cc}*\rho F_R]\rho$ at order ρ^3 are of the form (58). For instance, at order ρ^3 the $1/r^6$ tail of the former diagram decays as

$$\rho\left[\frac{1}{2}[W_3]^2*\rho[F_{RT}-W]+[F_{RT}-W]\rho*\frac{1}{2}[W_3]^2\right]\rho. \tag{78}$$

In fact W does not contribute to $\int d\mathbf{r}[F_{RT}-W]$ according to Eq. (35). A new dependence upon $\hbar$ appears in this tail: a purely $\hbar^6$ term arises from an $\hbar^4$ in $[W_3]^2$ and from an $\hbar^2$ contained in the "diffraction" contribution $\int d\mathbf{r} v^{cm}(\mathbf{r},\chi_2,\chi_b)$ in the decomposition of $\int d\mathbf{r} F_{RT}(\mathbf{r},\chi_2,\chi_b)$ (see Sec. IV B of Paper II).

A structure where the quantum interaction links three intermediate points, with every $p=1$ at order ρ^3, arises from the diagram $\rho F_R \rho[F_R * \rho F_R]$. This tail is of the form (59) because it is the sum of three contributions,

$$\rho W_3 \rho[(F_{RT}-W)\rho * W_3]+\rho W_3 \rho[W_3 * \rho(F_{RT}-W)] \tag{79a}$$

$$+\rho W_3 \rho[W_3 \rho * W_3]. \tag{79b}$$

Equation (79a) involves matrix elements of $\exp[-\beta H_{\alpha\alpha_1}]$ and $\exp[-\beta H_{\alpha_1\gamma}]$ whereas Eq. (79b) is purely proportional to $\hbar^6$.

Eventually, terms of order ρ^3 in $A_{\alpha\gamma}$ are equal to

$$[a_{\alpha\gamma}^{\{3\}}+f_{\alpha\gamma}(\hbar)]\hbar^4+b_{\alpha\gamma}^{\{3\}}\hbar^6. \tag{80}$$

In Eq. (80) $f_{\alpha\gamma}(\hbar)$ is a function of $\hbar$ which is fully quantum because it involves either a diagonal or a nondiagonal matrix element of $\exp[-\beta H_{\alpha'\gamma'}]$.

VI. CASE $\mathbf{B}_0 \neq 0$

In the following, we will omit the indexes $\mathbf{B}_0$. As in the case $\mathbf{B}_0=\mathbf{0}$, the scheme of the discussion is that presented in Sec. III. In particular, we use the results of Paper I about the structures of diagrams that contribute to the various $1/r^5$ tails. Since summations over charges do not change the exponents of the leading algebraic tails—their consequence in the diagrammatic language is only to suppress contributions from some diagrams—the $D_{\alpha\gamma}^{\{2\}}(\hat{\mathbf{r}})/r^5$, $D_{\alpha}^{\{2\}}(\hat{\mathbf{r}})/r^5$, and $D^{\{2\}}(\hat{\mathbf{r}})/r^5$ tails of $\rho_{\alpha\gamma}^{(2)T}(\mathbf{r})$, $\Sigma_\gamma e_\gamma \rho_{\alpha\gamma}^{(2)T}(\mathbf{r})$, and $\Sigma_{\alpha,\gamma} e_\alpha e_\gamma \rho_{\alpha\gamma}^{(2)T}(\mathbf{r})$ (where $\hat{\mathbf{r}} \equiv \mathbf{r}/|\mathbf{r}|$), respectively, satisfy

$$D_\alpha^{\{2\}}=\sum_\gamma e_\gamma D_{\alpha\gamma}^{\{2\}} \tag{81}$$

and

$$D^{\{2\}}=\sum_\gamma e_\alpha e_\gamma D_{\alpha\gamma}^{\{2\}}. \tag{82}$$

These identities are explicitly checked at the first two orders in density in Appendix C.

A. Structure in Fourier space

According to the analysis in Sec. VIII B of Paper I, the $D_{\alpha\gamma}/r^5$ tail of $\rho_{\alpha\gamma}^{(2)\,T}$ is given by Eq. (22) where h is replaced by the $1/r^5$ tail $S_{pp}^{(5)[0,0]}(r,\chi_a,\chi_b)$ of $\Sigma_D^* * h^{nn} * \Sigma_D^*$, with $\Sigma_D^*(\mathbf{r},\chi_1;\chi_a)$ defined in Eq. (48) while $\Sigma_D^*(\mathbf{k},\chi_2;\chi_b) \equiv \Sigma_D(\mathbf{k},\chi_2;\chi_b)+\rho(\chi_2)F^{mc}(\mathbf{r},\chi_2,\chi_b)$. The $1/r^5$ tail of $\rho_{\alpha\gamma}^{(2)T}$ may be written as

$$\frac{D_{\alpha\gamma}}{r^5}=\int d\chi_a \rho(\chi_a)\delta_{e_{\alpha_a},\alpha} \times \int d\chi_b \rho(\chi_b)\delta_{e_{\alpha_b},\gamma} S_{pp}^{(5)[0,0]}(\mathbf{r},\chi_a,\chi_b). \tag{83}$$

Since $h^{nn}(\mathbf{r},\chi_1,\chi_2)$ decays at least as $1/r^3$ before integration over loop shapes $\mathbf{X}_a$ and $\mathbf{X}_b$, $h^{nn}(\mathbf{k},\chi_1,\chi_2)$ contains singularities from order $|\mathbf{k}|^0$ on; thus the Fourier transform $S_{pp}^{(2)}(\mathbf{k},\chi_a,\chi_b)$ involves terms of orders $|\mathbf{k}|^0$, $|\mathbf{k}|$, and $|\mathbf{k}|^2$ in the small-$\mathbf{k}$ expansion of $\Sigma_D^*(\mathbf{k},\chi_i;\chi_r)$ where χ_r is a root point. According to Eqs. (26) and (28), Σ_D^* is even under inversion of loop shapes of root points, while the term $\Sigma_D^{*(q)}$ of order $|\mathbf{k}|^q$ in Σ_D^* is of parity $(-1)^q$ under inversion of each internal loop shape $\mathbf{X}_i$; the latter property constrains the parities of the tails of h^{nn} that may contribute after integration over loop shapes. In order to simplify notations, we will not indicate the parities of Σ_D^* but only those of $S_{h^{nn}}^{(p)[q,q']}$. A first contribution given by the dimensional analysis of the order in $|\mathbf{k}|$ is

$$\int d\chi_1 \int d\chi_2 \Sigma_D^{*(2)}(\mathbf{k},\chi_1;\chi_a) \times S_{h^{nn}}^{(0)[0,0]}(\mathbf{k},\chi_1,\chi_2)\Sigma_D^{*(0)}(\mathbf{k},\chi_2;\chi_b)=0. \tag{84}$$

Equation (84) vanishes, because $S_{h^{nn}}^{(0)[0,0]}=0$ according to the general study of the structure of tails before integration over loop shapes (see Appendix A of Paper I). $S_{pp}^{(2)}(\mathbf{k},\chi_a,\chi_b)$ may be written as the sum of three contributions,

$$\int d\chi_1 \int d\chi_2 \rho(\chi_1)F^{cm(1)}(\mathbf{k},\chi_a,\chi_1) \times S_{h^{nn}}^{(0)[1,1]}(\mathbf{k},\chi_1,\chi_2)\rho(\chi_2)F^{mc(1)}(\mathbf{k},\chi_2,\chi_b), \tag{85a}$$

$$\int d\chi_1 \int d\chi_2 \rho(\chi_1)F^{cm(1)}(\mathbf{k},\chi_a,\chi_1)S_{h^{nn}}^{(1)[1,0]}(\mathbf{k},\chi_1,\chi_2) \times \Sigma_D^{(0)}(\mathbf{k},\chi_2;\chi_b)+\text{sym. term}, \tag{85b}$$

$$\int d\chi_1 \int d\chi_2 \Sigma_D^{(0)}(\mathbf{k},\chi_1;\chi_a)S_{h^{nn}}^{(2)[0,0]}(\mathbf{k},\chi_1,\chi_2) \times \Sigma_D^{(0)}(\mathbf{k},\chi_2;\chi_b), \tag{85c}$$

where we have used the identities $\Sigma_D^{*(0)}=\Sigma_D^{(0)}$ and $\Sigma_D^{*(1)}(\mathbf{k},\chi_i;\chi_r)=\rho(\chi_1)F^{cm(1)}(\mathbf{k},\chi_i;\chi_r)$. The symmetric term in Eq. (85b) is obtained by exchanging the roles of χ_a and χ_b.

B. Density expansions

1. Loop-density expansions

All dressings in the nonvanishing contributions (85), namely, $\Sigma_D^{*(0)}(\mathbf{k},\chi_1;\chi_a)$ and $\Sigma_D^{*(1)}(\mathbf{k},\chi_1;\chi_a)$, are exactly of order zero in density. We notice that, on the contrary, $\Sigma_D^{*(2q)}$ and $\Sigma_D^{*(2q+1)}$ with $q \geq 1$ starts at order $O(\rho_{\text{loop}}^{-q})$, according to Eqs. (26) and (28) and the expansion of $[1+(\mathbf{k}^2/\kappa^2)]^{-1}$. For instance, $\Sigma_D^{*(2)}(\mathbf{k},\chi_1;\chi_a)$ is the sum of two terms of orders ρ_{loop}^{-1} and ρ_{loop}^{0}, respectively, but $\Sigma_D^{*(2)}$ is involved in faster falloffs than those considered here, because of parity arguments [see Eq. (84)].

As a consequence, at the lowest order in ρ_{loop}, $S_{pp}^{(2)}(\mathbf{k},\chi_a,\chi_b)$ is merely given by Eq. (85) where h^{nn} is replaced by its value at first order in ρ_{loop}. The first term in the low-density expansions of the $1/r^3$, $1/r^4$, and $1/r^5$ tails of h^{nn} are of order ρ_{loop}^{0}. They are reduced to the bond W possibly convoluted with bonds ρF^{cc} or ρF^{mc}, because the first term in the $\mathbf{k}$ expansion of ρF^{cc} or ρF^{mc} is of order ρ_{loop}^{0}. In order to take advantage of parity arguments, it is convenient to rewrite W given in Eq. (32) as $W=\Sigma_{n=3}^{\infty}W_n$, with W_n decaying as $1/r^n$, and to push the decomposition further by writing

$$W_n(\mathcal{L}_i,\mathcal{L}_j)=\sum_{l=1}^{n-2}\frac{1}{l!(n-1-l)!}w_n^{[l,n-1-l]}(\mathbf{k},\mathbf{X}_i,\mathbf{X}_j), \tag{86}$$

where

$$w_n^{[l_i,l_j]}(\mathbf{k},\mathbf{X}_i,\mathbf{X}_j)=-\beta e_{\alpha_i}e_{\alpha_j}\int_0^{p_i}d\tau\int_0^{p_j}d\tau'\{\delta([\tau-P(\tau)]-[\tau'-P(\tau')])-1\}[i\mathbf{k}\cdot\mathbf{X}_i(\tau)]^{l_i}\times[-i\mathbf{k}\cdot\mathbf{X}_j(\tau')]^{l_j}\frac{4\pi}{\mathbf{k}^2} \tag{87}$$

is of parity $(-1)^{l_i}$ $[(-1)^{l_j}]$ with respect to $\mathbf{X}_i$ ($\mathbf{X}_j$). With these definitions, we get in Fourier space

$$S_{h^{nn}}^{(0)[1,1]}|_{\text{loop}}^{\{0\}}=w_3^{[1,1]}=w_3, \tag{88}$$

$$S_{h^{nn}}^{(1)[1,0]}|_{\text{loop}}^{\{0\}}=\frac{1}{2}w_4^{[1,2]}, \tag{89}$$

$$S_{h^{nn}}^{(2)[0,0]}|_{\text{loop}}^{\{0\}}=\frac{1}{4}w_5^{[2,2]}. \tag{90}$$

At next order in loop density

$$S_{h^{nn}}^{(0)[1,1]}|_{\text{loop}}^{\{1/2\}}=0, \tag{91}$$

$$S_{h^{nn}}^{(1)[1,0]}|_{\text{loop}}^{\{1/2\}}=\left\{\frac{1}{2}w_4^{[1,2]}+\left[\frac{1}{2}w_4^{[1,2]}\rho F^{cc(0)}+w_3^{[1,1]}\rho F^{mc(1)}\right]\right\}\times\rho\frac{1}{2}[F^{cc}]^2(\mathbf{k}=\mathbf{0}), \tag{92}$$

$$S_{h^{nn}}^{(2)[0,0]}|_{\text{loop}}^{\{1/2\}}=\left\{\frac{1}{4}w_5^{[2,2]}+\left[\frac{1}{4}w_5^{[2,2]}\rho F^{cc(0)}+\frac{1}{2}w_4^{[2,1]}\rho F^{mc(1)}\right]\right\}\rho\frac{1}{2}[F^{cc}]^2(\mathbf{k}=\mathbf{0})+\text{sym. term}. \tag{93}$$

In Eqs. (92) and (93) charge indexes and summation over species of intermediate points are implicit as in the notation of convolutions in position space introduced in Eq. (25).

2. Cancellations at the first two orders in particle density

At the first two orders in density, many contributions in $S_{pp}^{(5)}(\mathbf{r})$ cancel each other by virtue of the identity

$$\int D_{\mathbf{B}_0}(\boldsymbol{\xi})\left\{\frac{1}{2}[\mathbf{k}\cdot\boldsymbol{\xi}(s)]^2-[\mathbf{k}\cdot\boldsymbol{\xi}(s)]\int_0^1 ds'[\mathbf{k}\cdot\boldsymbol{\xi}(s')]\right\}=0. \tag{94}$$

Indeed, the left-hand side of (94) is equal to

$$[\mathbf{k}]_\mu[\mathbf{k}]_\nu\left[\frac{1}{2}\text{cov}_{\mu\nu}^{\alpha}(s,s';\mathbf{B}_0)-\int_0^1 ds'\,\text{cov}_{\mu\nu}^{\alpha}(s,s';\mathbf{B}_0)\right], \tag{95}$$

with an implicit summation over the space indices μ and ν ($\mu,\nu=1,2,3$). The property $\text{cov}_{xy}^{\alpha}(s,s';\mathbf{B}_0)=-\text{cov}_{yx}^{\alpha}(s,s';\mathbf{B}_0)$ derived in Sec. V C in Paper I implies that

$$[\mathbf{k}]_\mu[\mathbf{k}]_\nu\text{cov}_{\mu\nu}^{\alpha}(s,s';\mathbf{B}_0)=[\mathbf{k}]_\mu^2\text{cov}_{\mu\mu}^{\alpha}(s,s';\mathbf{B}_0). \tag{96}$$

Then we use the following property, which can be derived from the explicit value of the covariance given in Sec. III C of Paper II:

$$\int_0^1 ds'\,\text{cov}_{xx}^{\alpha}(s,s';\mathbf{B}_0)=\frac{1}{2}\text{cov}_{xx}^{\alpha}(s,s;\mathbf{B}_0). \tag{97}$$

Of course, this identity is also valid for $\text{cov}_{zz}(s,s')=\lim_{B_0\to 0}\text{cov}_{xx}^{\alpha}(s,s';\mathbf{B}_0)$. Then, Eq. (94) is proved.

The identity (94) allows one to show that at first order in density,

$$S_{pp}^{(5)[0,0]}(\mathbf{r},\boldsymbol{\xi}_a,\boldsymbol{\xi}_b)|^{\{0\}}=\frac{1}{4}w_5^{[2,2]}(\mathbf{r},\boldsymbol{\xi}_a,\boldsymbol{\xi}_b). \tag{98}$$

The reasons for this simple expression are the following. $S_{pp}^{(5)[0,0]}(\mathbf{r},\chi_a,\chi_b)$ is given by Eq. (85) where $\Sigma_D^{(0)}$ and $\rho F^{mc(1)}$ are exactly of order ρ_{loop}^{0}. In order to prove Eq. (98), we reorganize the effective contributions coming from $\Sigma_D^{*}*W*\Sigma_D^{*}$ in order to exhibit the combination (94). In Fourier space we get the sum of $(1/4)w_5^{[2,2]}$ plus

$$F^{cm(1)}\rho\left[w_3^{[1,1]}\rho F^{mc(1)}+\frac{1}{2}w_4^{[1,2]}\rho F^{cc(0)}\right] \tag{99a}$$

$$+\frac{1}{2}F^{cc(0)}\rho\left[w_4^{[2,1]}\rho F^{mc(1)}+\frac{1}{2}w_5^{[2,2]}\rho F^{cc(0)}\right] \tag{99b}$$

$$+\frac{1}{2}\left[w_4^{[2,1]}\rho F^{mc(1)}+\frac{1}{2}w_5^{[2,2]}\rho F^{cc(0)}\right]+(\text{sym. term}), \tag{99c}$$

where (sym. term) denotes the symmetric term $(1/2)[F^{cm(1)}\rho w_4^{[1,2]}+(1/2)F^{cc(0)}\rho w_5^{[2,2]}]$. At first order in density, only loops with $p=1$ contribute. Since $F^{cc(0)}(\mathbf{k},\alpha,\alpha')$ and $F^{cm(1)}(\mathbf{k},\boldsymbol{\xi}_r,\boldsymbol{\xi})$ [or $F^{mc(1)}(\mathbf{k},\boldsymbol{\xi},\boldsymbol{\xi}_r)$] only differ by an extra factor $\mp i\int_0^1 ds\lambda_\alpha\boldsymbol{\xi}(s)\cdot\mathbf{k}$, the definition (87) and the property (94) imply that only the term (98) does contribute to the $1/r^5$ tail $S_{pp}^{(5)[0,0]}$ at the first order in density ρ.

At order $\rho^{5/2}$ the same cancellation mechanism implies that the $1/r^5$ tail $S_{pp}^{(5)[0,0]}$ comes only from

$$\frac{1}{4}w_5^{[2,2]}\rho\frac{1}{2}\int d\mathbf{r}[F^{cc}(\mathbf{r})]^2\Sigma_D^{(0)}+\text{sym. term.} \quad (100)$$

This cancellation mechanism seems to be specific to the first order in density. It still partially operates at order $\rho^{5/2}$, but from order ρ^3 property (94) cannot be applied when the expansion of $\rho(\boldsymbol{\xi})$ is used at orders higher than ρ or when $\rho_{\alpha,2}(\mathbf{X}_2)$ occurs.

3. Particle-particle correlation at the first order in density

In the case of the particle-particle correlation, the $D_{\alpha\gamma}/r^5$ tail is given by Eq. (83) where, at the first order in ρ, $S_{pp}^{(5)}(\mathbf{r},\chi_a,\chi_b)$ is reduced to $(1/4)w_5^{[2,2]}$, according to Eq. (98), with $p_a=p_b=1$. The Fourier transform of the $D_{\alpha\gamma}^{\{2\}}/r^5$ tail at the first order ρ^2 reads

$$\int_0^1 ds_1\int_0^1 ds_2[\delta(s_1-s_2)-1]F_{\alpha\gamma}^{pp}(\mathbf{k},s_1,s_2), \quad (101)$$

where

$$F_{\alpha\gamma}^{pp}(\mathbf{k},s_1,s_2)\equiv-\rho_\alpha\rho_\gamma\beta e_\alpha e_\gamma\lambda_\alpha^2\lambda_\gamma^2 \times\left\{\frac{1}{4}\int D_{\alpha,B_0}(\boldsymbol{\xi}_1)\int D_{\gamma,B_0}(\boldsymbol{\xi}_2) \times[\mathbf{k}\cdot\boldsymbol{\xi}_1(s_1)]^2[\mathbf{k}\cdot\boldsymbol{\xi}_2(s_2)]^2\frac{4\pi}{\mathbf{k}^2}\right\}. \quad (102)$$

In Eq. (102) we have added the index α in the notation of the measure $D_{\mathbf{B}_0}(\boldsymbol{\xi})$ in order to recall the dependence of the measure upon the species when $\mathbf{B}_0\neq\mathbf{0}$. Integrations over $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ lead to the appearance of covariances of Brownian bridges in the presence of $\mathbf{B}_0$. According to Eq. (96) and $\text{cov}_{xx}^\alpha=\text{cov}_{yy}^\alpha$,

$$[\mathbf{k}]_\mu[\mathbf{k}]_\nu\text{cov}_{\mu\nu}^\alpha(s,s';\mathbf{B}_0)=\mathbf{k}^2\text{cov}_{xx}^\alpha(s,s';\mathbf{B}_0=\mathbf{0}) -[\mathbf{k}]_z^2\,\delta C_\alpha(s,s';\mathbf{B}_0), \quad (103)$$

where

$$\delta C_\alpha(s,s';\mathbf{B}_0)\equiv\text{cov}_{xx}^\alpha(s,s';\mathbf{B}_0)-\text{cov}_{zz}^\alpha(s,s';\mathbf{B}_0) =\text{cov}_{xx}^\alpha(s,s';\mathbf{B}_0)-\text{cov}_{xx}^\alpha(s,s';\mathbf{B}_0=\mathbf{0}). \quad (104)$$

So the nonanalytic term in $\{\cdots\}$ in Eq. (102) is equal to

$$4\pi\frac{[\mathbf{k}]_z^4}{\mathbf{k}^2}\frac{1}{4}\delta C_\alpha(s_1,s_1)\delta C_\gamma(s_2,s_2). \quad (105)$$

We notice that $4\pi[\mathbf{k}]_z^4/\mathbf{k}^2$ is the Fourier transform of

$$\partial_{zzzz}\left(\frac{1}{r}\right)=24\frac{P_4(\cos\theta)}{r^5} \quad (106)$$

and, according to the value of the de Broglie thermal wavelength $\lambda_\alpha=\sqrt{\beta\hbar^2/m_\alpha}$, we get Eq. (8) where

$$A(u_{C\alpha},u_{C\gamma})\equiv 6\int_0^1 ds_1\int_0^1 ds_2[\delta(s_1-s_2)-1] \times\delta C_\alpha(s_1,s_1)\delta C_\gamma(s_2,s_2). \quad (107)$$

According to Eqs. (83), (100), and (60) $D_{\alpha\gamma}^{\{5/2\}}/r^5$ may be seen as arising from the convolution

$$\rho_\alpha\sum_{\alpha_2}[-\beta V_{\alpha\alpha_2}^{\text{eff}(5)}|^{\{1/2\}}]*S_{D,\alpha_2\gamma}^{\text{cl}} +\rho_\gamma\sum_{\alpha_1}S_{D,\alpha\alpha_1}^{\text{cl}}*[-\beta V_{\alpha_1\gamma}^{\text{eff}(5)}|^{\{1/2\}}], \quad (108)$$

with

$$V_{\alpha_1\alpha_2}^{\text{eff}(5)}(\mathbf{r})|^{\{1/2\}}=\sum_{\alpha'}V_{\alpha_1\alpha'}^{\text{eff}(5)}(\mathbf{r})\bigg|^{\{0\}}\rho_{\alpha'}\int d\mathbf{x}\frac{1}{2}[F_{D,\alpha'\alpha_2}^{cc}(\mathbf{x})]^2. \quad (109)$$

Now, we turn to the explicit values of the $D_{\alpha\gamma}/r^5$ tail at the first two orders in density. $D_{\alpha\gamma}^{\{n\}}$ with $n=2,5/2$ may be written in the concise forms

$$D_{\alpha\gamma}^{\{n\}}=-\beta^3\hbar^4P_4(\cos\theta)\rho_\alpha\rho_\gamma e_\alpha e_\gamma\mathcal{D}_{\alpha\gamma}^{\{n-2\}}, \quad (110)$$

with

$$\mathcal{D}_{\alpha\gamma}^{\{0\}}=\frac{1}{m_\alpha m_\gamma}A(u_{C\alpha},u_{C\gamma}) \quad (111)$$

and

$$\mathcal{D}_{\alpha\gamma}^{\{1/2\}}=\frac{\pi\beta^2}{\kappa_D}\frac{1}{m_\alpha}\left[e_\gamma-\frac{4\pi\beta\sum_{\gamma'}\rho_{\gamma'}e_{\gamma'}^3}{\kappa_D^2}\right] \times\sum_{\alpha_2}\rho_{\alpha_2}\frac{e_{\alpha_2}^3}{m_{\alpha_2}}A(u_{C\alpha_2},u_{C\alpha})+\text{sym. term.} \quad (112)$$

In Eq. (112) the symmetric term is obtained by exchanging the roles of α and γ. Comparison of Eqs. (111) and (112) shows that, according to Eq. (110),

$$D_{\alpha\gamma}^{\{5/2\}}=-\frac{\pi\beta^2}{\kappa_D}\left\{\rho_\gamma e_\gamma\left[\frac{4\pi\beta\sum_{\gamma'}\rho_{\gamma'}e_{\gamma'}^3}{\kappa_D^2}-e_\gamma\right]\sum_{\alpha_2}e_{\alpha_2}^2D_{\alpha\alpha_2}^{\{2\}} +\rho_\alpha e_\alpha\left[\frac{4\pi\beta\sum_{\gamma'}\rho_{\gamma'}e_{\gamma'}^3}{\kappa_D^2}-e_\alpha\right]\sum_{\alpha_2}e_{\alpha_2}^2D_{\gamma\alpha_2}^{\{2\}}\right\}. \quad (113)$$

The analytical expression of $A(u_{C\alpha},u_{C\gamma})$ is determined by dynamics of independent charges in a magnetic field. The covariance that characterizes this dynamics has been explicitly derived in Appendix A of Paper II. The values of interest are

$$\mathrm{cov}_{xx}^{\alpha}(s,s;\mathbf{B}_0)=\frac{1}{u_{C\alpha}\sinh u_{C\alpha}}\sinh(su_{C\alpha})\sinh([1-s]u_{C\alpha}) \tag{114}$$

and

$$\int_0^1 ds\ \mathrm{cov}_{xx}^{\alpha}(s,s)=\frac{1}{2u_{C\alpha}}L(u_{C\alpha}), \tag{115}$$

where $L(u_{C\alpha})=\coth u_{C\alpha}-(1/u_{C\alpha})$ is the Langevin function. By using the definition (104) and Eq. (115) we get

$$\delta C_{\alpha}\equiv\int_0^1 ds\ \delta C_{\alpha}(s,s)=\frac{1}{2u_{C\alpha}}L^{[3]}(u_{C\alpha}), \tag{116}$$

with $L^{[3]}(u_{C\alpha})\equiv L(u_{C\alpha})-u_{C\alpha}/3$ defined in Sec. II B of Paper II. The value of $A(u_{C\alpha},u_{C\gamma})$ is given in Eq. (9). In the case of particles with the same $u_{C\alpha}$'s, a straightforward limit of Eq. (9) leads to

$$A(u_{C\alpha},u_{C\alpha})=\frac{3}{2}\left\{\frac{1}{2u_{C\alpha}^2}[\coth u_{C\alpha}]^2+\frac{5}{2u_{C\alpha}^3}\coth u_{C\alpha}+\frac{1}{45}-\frac{7}{6u_{C\alpha}^2}-\frac{3}{u_{C\alpha}^4}\right\}. \tag{117}$$

4. Induced charge density

The $D^{\star}/r^5$ tail of the ratio $\Sigma_{\gamma}e_{\gamma}\rho_{\gamma}^{\mathrm{ind,L}}(\mathbf{r})/\delta q$ for the linearly induced charge arises from diagrams which are different from those involved in the D_{α}/r^5 tail of the particle-charge correlation for finite charges of the plasma. $D^{\star}/r^5$ is given by Eq. (23) where $G_h(\mathbf{k},\chi_a)$ is replaced by the Fourier transform $S_{G_h}^{(4)}(\mathbf{k},\chi_a)$ of its $1/r^7$ tail $S_{G_h}^{(7)}(\mathbf{r},\chi_a)$,

$$\frac{D^{\star}}{r^5}=\mathcal{F}^{-1}\left[-\frac{4\pi\beta}{\mathbf{k}^2}\int d\chi_a\rho(\chi_a)p_a e_{\alpha_a}S_{G_h}^{(4)}(\mathbf{k},\chi_a)\right](\mathbf{r}). \tag{118}$$

In fact, according to Sec. VIII C of Paper I, $S_{G_h}^{(7)}(\mathbf{r},\chi_a)$ arises only from the part of h equal to $F^{cm}\rho * h^{nn}*[\Sigma_D^{**}+\rho F^{mc}*\rho F^{mc}]$. In other words, $S_{G_h}^{(4)}(\mathbf{k},\chi_a)$ is the nonanalytic term of order $|\mathbf{k}|^4$ in

$$\int d\chi_1\rho(\chi_1)\int d\chi_2 F^{cm}(\mathbf{k},\chi_a,\chi_1)\times h^{nn}(\mathbf{k},\chi_1,\chi_2)G_{\{\Sigma_D^{**}+\rho F^{mc}*\rho F^{mc}\}}(\mathbf{k},\chi_2). \tag{119}$$

A straightforward calculation leads to

$$G_{\{\Sigma_D^{**}+\rho F^{mc}*\rho F^{mc}\}}(\mathbf{k},\chi_2)=\frac{\rho(\chi_2)}{\kappa^2}e_{\alpha_2}p_2\mathbf{k}^2\left[1+\int_0^{p_2}\frac{d\tau}{p_2}i\mathbf{k}\cdot\mathbf{X}_2(\tau)\right]+O(|\mathbf{k}|^4). \tag{120}$$

Henceforth, since $F^{cm}(\mathbf{k},\chi_a,\chi_1)=O(|\mathbf{k}|)$ and according to parity arguments,

$$S_{G_h}^{(4)}(\mathbf{k},\chi_a)=\mathbf{k}^2\int d\chi_1\rho(\chi_1)\int d\chi_2\frac{\rho(\chi_2)}{\kappa^2}e_{\alpha_2}p_2\times F^{cm(1)}(\mathbf{k},\chi_a,\chi_1)\left\{S_{h^{nn}}^{(1)[1,0]}(\mathbf{k},\chi_1,\chi_2)+S_{h^{nn}}^{(0)[1,1]}(\mathbf{k},\chi_1,\chi_2)\int_0^{p_2}\frac{d\tau_2}{p_2}i\mathbf{k}\cdot\mathbf{X}_2(\tau_2)\right\}. \tag{121}$$

The Fourier transform of $D^{\star}/r^5$ tail is given by Eqs. (118) and (121).

At the first two orders ρ_{loop} and $\rho_{\mathrm{loop}}^{3/2}$, we insert Eqs. (28) in (121). We use Eqs. (88) and (89) for contributions of order ρ_{loop} and Eqs. (91) and (92) for terms of order $\rho_{\mathrm{loop}}^{3/2}$. At order ρ_{loop}, $D^{\star}/r^5$ is the inverse Fourier transform of

$$-\frac{4\pi\beta}{\kappa^2}\int d\chi_1\rho(\chi_1)e_{\alpha_1}p_1\times\int d\chi_2\rho(\chi_2)e_{\alpha_2}p_2\int_0^{p_1}\frac{d\tau_1}{p_1}[i\mathbf{k}\cdot\mathbf{X}_1(\tau_1)]\times\left\{\frac{1}{2}w_4^{[1,2]}(\mathbf{k},\chi_1,\chi_2)+w_3(\mathbf{k},\chi_1,\chi_2)\int_0^{p_2}\frac{d\tau_2}{p_2}i\mathbf{k}\cdot\mathbf{X}_2(\tau_2)\right\}. \tag{122}$$

At next order $\rho_{\mathrm{loop}}^{3/2}$, it is given by Eq. (122) where $\{\cdots\}$ is replaced only by the value (92) of $S_{h^{nn}}^{(1)[1,0]}(\mathbf{k},\chi_1,\chi_2)|_{\mathrm{loop}}^{\{1/2\}}$. As a consequence, only loops with $p=1$ are involved at orders ρ and $\rho^{3/2}$.

According to the cancellation mechanism (94), the tail (122) of the induced charge density, which is derived from the linear response theory in the loop formalism [9], vanishes at order ρ,

$$D^{\star\{2\}}=0, \tag{123}$$

and appears only at higher orders in density. This result can be retrieved from Eq. (75): though the part $\{\rho[F^{cc}+F^{cm}]\rho * h^{nn}*\rho F^{mc}\}(\mathbf{r},\chi_a,\chi_b)$ in $\Sigma_D^{*}*h^{nn}*\rho F^{mc}$ is linear in e_{α}, this linear term does not contribute at order ρ^2 to $\Sigma_{\gamma}e_{\gamma}\rho_{\alpha\gamma}^{(2)T}(r)$ because of the cancellation mechanism (94) at low density.

The latter result is in agreement with the value $D_{\alpha}^{\{2\}}=\Sigma_{\gamma}e_{\gamma}D_{\alpha\gamma}^{\{2\}}$ for the coefficient of the $1/r^5$ tail of the particle-charge correlation at the first order ρ^2. Indeed, the infinitesimal induced charge is generically given by the linear term in e_{α} in the particle-charge correlation $\Sigma_{\gamma}e_{\gamma}\rho_{\alpha\gamma}^{(2)T}(r)/\rho_{\alpha}$, as recalled in Eq. (75). Thus

$$D^{\star}=\lim_{e_{\alpha}\to 0}\frac{1}{e_{\alpha}}\lim_{\rho_{\alpha}\to 0}\frac{\sum_{\gamma}e_{\gamma}D_{\alpha\gamma}}{\rho_{\alpha}}. \tag{124}$$

Indeed, the coefficient of the $1/r^5$ tail of $\Sigma_\gamma e_\gamma \rho_{\alpha\gamma}^{(2)T}(r)$ starts at order ρ^2 and involves the charge e_α through $A(u_{C\alpha},u_{C\gamma})$ at this order. When $u_{C\alpha}=(\beta\hbar B_0/2m_\alpha c)e_\alpha$ tends to zero at $u_{C\gamma}$ fixed,

$$A(u_{C\alpha},u_{C\gamma}) \underset{u_{C\alpha}\to 0}{\sim} u_{C\alpha}^2 \frac{1}{4u_{C\gamma}^5}\left[-\coth u_{C\gamma}+\frac{1}{u_{C\gamma}}+\frac{u_{C\gamma}}{3} -\frac{u_{C\gamma}^3}{45}+\frac{2u_{C\gamma}^5}{945}\right]. \tag{125}$$

Thus $D_\alpha^{\{2\}}$ is nonlinear in e_α when e_α tends to zero, and $D^\star$ vanishes at order ρ. According to the relations (75) and $D_\gamma^{\{2\}}=\Sigma_\gamma e_\gamma D_{\alpha\gamma}^{\{2\}}$, together with Eqs. (110) and (125), the nonlinearly induced charge at order ρ is

$$\sum_\gamma e_\gamma \rho_\gamma^{\rm ind}(\mathbf{r};q)|_{\mathbf{B}_0}^{\{1\}}=-\beta^3\hbar^4\frac{q}{m_q}\left[\sum_\gamma \rho_\gamma \frac{e_\gamma^2}{m_\gamma}A(u_{Cq},u_{C\gamma})\right] \times\frac{P_4(\cos\theta)}{r^5}. \tag{126}$$

It is cubic in q/m_q when q/m_q vanishes.

We now turn to the tail of the linearly induced charge density at next order $\rho^{3/2}$. Since only loops with $p=1$ contribute at this order, the cancellation mechanism (94) at first order in density operates and only the part $(1/2)w_4^{[1,2]}*\rho[F^{cc}]^2/2$ appears in $S_{h^{nn}}^{(1)[1,0]}(\mathbf{k},\chi_1,\chi_2)|^{\{1/2\}}$ given in Eq. (92). According to Eqs. (121) and (87), the Fourier transform of $D^{\star\{3/2\}}/r^5$ comes from

$$-\frac{\pi\beta^2}{\kappa_D}\frac{4\pi\beta\sum_{\gamma'}\rho_{\gamma'}e_{\gamma'}^3}{\kappa_D^2}\sum_\gamma e_\gamma\sum_{\alpha_2} e_{\alpha_2}^2\int_0^1 ds_1\int_0^1 ds_2 \times[\delta(s_1-s_2)-1]F_{\gamma\alpha_2}^{\rm ind,L}(\mathbf{k},s_1,s_2), \tag{127}$$

where $F_{\gamma\alpha_2}^{\rm ind,L}$ is defined as $F_{\gamma\alpha_2}^{pp}$ given in Eq. (102) with the $\{\cdots\}$ replaced by

$$\left\{\frac{1}{2}\int_0^1 ds\int D_{\gamma,B_0}(\boldsymbol{\xi}_1)\int D_{\alpha_2,B_0}(\boldsymbol{\xi}_2)[\mathbf{k}\cdot\boldsymbol{\xi}_1(s)][\mathbf{k}\cdot\boldsymbol{\xi}_1(s_1)] \times[\mathbf{k}\cdot\boldsymbol{\xi}_2(s_2)]^2\frac{4\pi}{\mathbf{k}^2}\right\}. \tag{128}$$

The nonanalytic term in Eq. (128) is similar to the nonanalytic expression (105) with $2\int_0^1 ds\,\delta C_\gamma(s,s_1)$ in place of $\delta C_\alpha(s_1,s_1)$. After integration over s the factor 2 is compensated by the factor 1/2 arising from Eq. (97) and comparison of Eq. (127) with Eq. (113) shows that the identity (124) is indeed satisfied at order $\rho^{3/2}$. The explicit value of $D^{\star\{3/2\}}$ is then derived from Eq. (113) with the result

$$D^{\star\{3/2\}}=-4\pi^2\beta^3\frac{\sum_{\gamma'}\rho_{\gamma'}e_{\gamma'}^3}{\kappa_D^4}\sum_{\alpha'\gamma} e_\gamma e_{\alpha'}^2 D_{\gamma\alpha'}^{\{2\}}. \tag{129}$$

C. Weak or strong magnetic field

In the limit of a weak field $\mathbf{B}_0$, $A(u_{C\alpha},u_{C\gamma})$ behaves as $(1/3150)u_{C\alpha}^2u_{C\gamma}^2$ and $D_{\alpha\gamma}$ is proportional to $\hbar^8B_0^4$. More precisely,

$$\rho_{\alpha\gamma}^{(2)T}(\mathbf{r})|_{\mathbf{B}_0}\underset{r\to\infty}{\sim}-\rho_\alpha\rho_\gamma\frac{1}{50\,400}\beta^7\hbar^8 \times\left(\frac{e_\alpha}{m_\alpha}\right)^3\left(\frac{e_\gamma}{m_\gamma}\right)^3\left(\frac{B_0}{c}\right)^4\frac{P_4(\cos\theta)}{r^5}. \tag{130}$$

In terms of dimensionless parameters

$$\frac{D_{\alpha\gamma}}{\rho_\alpha\rho_\gamma}\propto a^5\Gamma\left(\frac{\lambda_\alpha}{a}\right)^2\left(\frac{\lambda_\gamma}{a}\right)^2(\beta\mu_{B\alpha}B_0)^2(\beta\mu_{B\gamma}B_0)^2. \tag{131}$$

In the strong field limit, $A(u_{C\alpha},u_{C\gamma})$ tends to 1/30, and $D_{\alpha\gamma}$ becomes independent from B_0,

$$\rho_{\alpha\gamma}^{(2)T}(\mathbf{r})|_{\mathbf{B}_0}\underset{r\to\infty}{\sim}-\rho_\alpha\rho_\gamma\frac{1}{30}\beta^3\hbar^4\frac{e_\alpha}{m_\alpha}\frac{e_\gamma}{m_\gamma} \times\left[1-60\frac{c^2}{\beta^2\hbar^2B_0^2}\left(\frac{m_\alpha^2}{e_\alpha^2}+\frac{m_\gamma^2}{e_\gamma^2}\right)+O\left(\frac{1}{B_0^4}\right)\right] \times\frac{P_4(\cos\theta)}{r^5}. \tag{132}$$

We notice that the limits where e_α tends to zero or where B_0 goes to infinity do not commute, since u_C goes to zero in the first case and to infinity in the second case.

VII. ONE-COMPONENT PLASMA

In order to get correlations for the OCP with moving charges e_- from expressions calculated for a two-component plasma (TCP), we first insert the local neutrality relation in order to replace every product $e_+\rho_+$ by $-e_-\rho_-$. In a second step, we use the same procedure as in Sec. VI A of Paper II: m_+ goes to infinity, then e_+ vanishes while ρ_+ becomes infinite. Moreover, we rather consider the following objects which remain finite even if one density goes to infinity: the Ursell function $h_{\alpha\gamma}=\rho_{\alpha\gamma}^{(2)T}(r)/\rho_\alpha\rho_\gamma$,

$$\sum_\gamma e_\gamma\rho_\gamma h_{\alpha\gamma}=-e_-\rho_-\sum_\gamma \operatorname{sgn}(e_\gamma)h_{\alpha\gamma} \tag{133}$$

and

$$\sum_{\alpha,\gamma} e_\alpha e_\gamma\rho_\alpha\rho_\gamma h_{\alpha\gamma}=(e_-\rho_-)^2\sum_{\alpha\gamma} \operatorname{sgn}(e_\alpha)\operatorname{sgn}(e_\gamma)h_{\alpha\gamma}, \tag{134}$$

where $\operatorname{sgn}(e_\alpha)$ denotes the sign of e_α. In the limit of the OCP, only h_{--} is expected to survive and the left-hand side of Eqs. (133) and (134) should be reduced to $e_-\rho_-h_{--}$ and $(e_-\rho_-)^2h_{--}$, respectively.

A. In the absence of $\mathbf{B}_0$

When $\mathbf{B}_0=\mathbf{0}$, for a two-component plasma, according to Eqs. (69), (13), and (14),

$$h_{\alpha\gamma}(r)|_{\mathbf{B}_0=\mathbf{0}} \underset{r\to\infty}{\sim} \frac{\beta^4\hbar^4}{240}\left(\frac{e_+e_-}{e_++|e_-|}\right)^2\left[\frac{e_+}{m_+}+\frac{|e_-|}{m_-}\right]^2\frac{1}{r^6}, \tag{135}$$

$$\sum_\gamma e_\gamma\rho_\gamma h_{\alpha\gamma}(r)|_{\mathbf{B}_0=\mathbf{0}} \underset{r\to\infty}{\sim} -\frac{\beta^3\hbar^4}{32\pi}\frac{e_+|e_-|}{(e_++|e_-|)^2}\times\left[\frac{e_+}{m_+}-\frac{|e_-|}{m_-}\right]\left[\frac{e_+}{m_+}+\frac{|e_-|}{m_-}\right]^2\frac{1}{r^8}, \tag{136}$$

$$\sum_{\alpha\gamma} e_\alpha e_\gamma\rho^{(2)T}_{\alpha\gamma}(r)|_{\mathbf{B}_0=\mathbf{0}} \underset{r\to\infty}{\sim} \frac{7\beta^2\hbar^4}{16\pi^2}\frac{1}{r^{10}}\left(\frac{1}{e_++|e_-|}\right)^2\times\left[\frac{e_+^2}{m_+}-\frac{e_-^2}{m_-}\right]^2. \tag{137}$$

In the limit of the OCP taken as recalled above, the $1/r^6$ tail of $\rho^{(2)T}_{\alpha\gamma}/\rho_\alpha\rho_\gamma$ and the $1/r^8$ tail of $\Sigma_\gamma e_\gamma\rho^{(2)T}_{\alpha\gamma}/\rho_\alpha$, derived from Eqs. (135) and (136), respectively, vanish whatever the values of α or γ, as it should. Indeed, only h_{--} is expected to survive and h_{--}, which is in fact a charge-charge correlation, decays as fast as $1/r^{10}$, according to $\hbar$ expansions [11,12] or perturbative results in the coupling constant [13]. Besides, $\Sigma_{\alpha,\gamma}e_\alpha e_\gamma\rho^{(2)T}_{\alpha\gamma}$ tends to $e_-^2\rho^{(2)T}_{\text{OCP}}$ and, according to Eq. (137),

$$e_-^2\rho^{(2)T}_{\text{OCP}}(r)|_{\mathbf{B}_0=\mathbf{0}} \underset{r\to\infty}{\sim} \frac{7}{16\pi^2}\beta^2\hbar^4\frac{e_-^4}{m_-^2}\frac{1}{r^{10}}. \tag{138}$$

B. In the presence of $\mathbf{B}_0$

In this section, we compare our low-density result with the algebraic tail derived from the low-density expansion of the exact sum rule (16). The singular term of order $|\mathbf{k}|^2$ in the Fourier transform of $S_{\text{OCP}}(\mathbf{r})|_{\mathbf{B}_0}=\rho_-\delta(\mathbf{r})+\rho^{(2)T}_{\text{OCP}}(\mathbf{r})$ is denoted by $S^{(2)}_{\text{OCPsing}}(\mathbf{k})|_{\mathbf{B}_0}$.

First, we calculate the low-density limit of $S^{(2)}_{\text{OCPsing}}(\mathbf{k})|_{\mathbf{B}_0}$, which is derived from our low-density path integral formalism. The ρ^2 term, $S^{(2)}_{\text{OCPsing}}(\mathbf{k})|^{\{2\}}_{\mathbf{B}_0}$, in the low-density expansion of the nonanalytic term of order $|\mathbf{k}|^2$, $S^{(2)}_{\text{OCPsing}}(\mathbf{k})|_{\mathbf{B}_0}$, is given by the Fourier transform of the $D^{\{2\}}_{\text{OCP}}/r^5$ tail calculated at the first order in density. The low-density $D^{\{2\}}_{\text{OCP}}/r^5$ tail of the OCP is derived from the result for a two-component plasma, as in the absence of $\mathbf{B}_0$. According to Eq. (8), the $[D^{\{2\}}_{\alpha\gamma}/\rho_\alpha\rho_\gamma]/r^5$ tail of $h_{\alpha\gamma}$ for a TCP is proportional to $[e_\alpha e_\gamma/m_\alpha m_\gamma]A(u_{C\alpha},u_{C\gamma})$. In the limiting process, m_+ goes to infinity then e_+ vanishes, so that u_{C+} tends to zero; thus, according to Eq. (125), the tails of h_{++} and h_{+-} vanish while $h_{\text{OCP}}=h_{--}$. According to Eq. (106) and $[\mathbf{k}]_z^4/\mathbf{k}^2=|\mathbf{k}|^2(\cos\theta_{\mathbf{k}})^4$, the Fourier transform of the $D^{\{2\}}_{\text{OCP}}/r^5$ tail (8) reads

$$4\pi\beta\frac{e_-^2S^{(2)}_{\text{OCPsing}}(\mathbf{k})|^{\{2\}}_{\mathbf{B}_0}}{\mathbf{k}^2}=-\frac{1}{4!}A_{\text{OCP}}(u_C)(\beta\hbar\omega_p)^4(\cos\theta_{\mathbf{k}})^4, \tag{139}$$

with $u_C=\beta\hbar\omega_c/2=\beta\hbar e_-B_0/2m_-c$ and the plasma frequency $\omega_p=\sqrt{4\pi e_-^2\rho_-m_-}$,

$$A_{\text{OCP}}(u_C)=\frac{1}{30}+\frac{3}{4u_C^2}(\coth u_C)^2+\frac{15}{4u_C^3}\coth u_C-\frac{7}{4u_C^2}-\frac{9}{2u_C^4}. \tag{140}$$

Second, we exhibit the nonanalytic contribution in the ρ^2 term in $S^{(2)}_{\text{OCP}}(\mathbf{k})|_{\mathbf{B}_0}$ which is given by the low-density limit of the exact sum rule (16). The latter expression is expanded up to order ρ^2 at u_C fixed. In other words, the expansion is performed with respect to the dimensionless parameter $\omega_p^2/\omega_c^2=4\pi\rho_-m_-c^2/B_0^2$ which appears in Eq. (17). We obtain the structure

$$\begin{aligned}4\pi\beta\frac{e_-^2S^{(2)}_{\text{OCP}}(\mathbf{k})}{\mathbf{k}^2}&=1+\rho[a^{\{1\}}(u_C)+b^{\{1\}}(u_C)(\cos\theta_{\mathbf{k}})^2]\\&\quad+\rho^2[a^{\{2\}}(u_C)+b^{\{2\}}(u_C)(\cos\theta_{\mathbf{k}})^2\\&\quad+c^{\{2\}}(u_C)(\cos\theta_{\mathbf{k}})^4]+o(\rho^2),\end{aligned} \tag{141}$$

where $a^{\{n\}}$, $b^{\{n\}}$, and $c^{\{n\}}$ are functions of the single variable u_C and $o(\rho^2)$ denotes a term of order greater than ρ^2. The term of order ρ in $4\pi\beta e^2S^{(2)}_{\text{OCP}}(\mathbf{k})$, though anisotropic, is analytical, as it should be, while the term of order ρ^2 contains a nonanalytic term, $\rho^2c^{\{2\}}(u_C)(\cos\theta_{\mathbf{k}})^4|\mathbf{k}|^2$. The latter one does coincide with our low-density result (139).

VIII. CONCLUSION

A. Compared qualitative results

1. Sign of the interaction

When $\mathbf{B}_0=\mathbf{0}$, in the case of a two-component plasma of charges e_+ and e_-, with masses m_+ and m_-, the effective interaction associated with the $1/r^6$ tail of the particle-particle correlation is attractive at the first two orders in density, whatever the signs of charges. [Indeed, the neutrality relation $e_-\rho_-=-e_+\rho_+$ implies that the coefficients $A^{\{n\}}_{\alpha\gamma}$ at order ρ^n, with $n=2,5/2$, are positive according to Eqs. (69) and (70).]

The peculiar identity (70) between all $A^{\{n\}}_{\alpha\gamma}/\rho_\alpha\rho_\gamma$ with $n=2,5/2$ is due to a classical contribution in the screening of every quantum charge by the surrounding plasma. It is no longer satisfied at higher orders in density ρ^n with $n\geqslant 3$, because then quantum dynamical and statistical effects are involved and destroy the symmetry between various species of particles, as shown in Sec. V D.

In the presence of $\mathbf{B}_0$, the sign of $P_4(\cos\theta)$ varies when θ ranges from 0 to π, so that the effective force is either attractive or repulsive according to the relative orientation θ of $\mathbf{r}$ and $\mathbf{B}_0$ as well as according to the relative signs of charges.

For instance, when $\theta=0$ $P_4(\cos\theta)=1$, the force is attractive (repulsive) between charges with opposite (same) signs; when $\theta=\pi/3$, $P_4(\cos\theta)<0$ and previous results are reversed.

2. Dependence upon thermodynamic parameters

First, we consider the variation of the coefficients of algebraic tails with respect to the temperature T and the density ρ. Up to a numerical multiplicative factor that involves masses, the dimensionless coefficient $|A_{\alpha\gamma}|$ is proportional to

$$|A_{\alpha\gamma}|\propto\Gamma^2\left(\frac{\lambda}{a}\right)^4, \tag{142}$$

with notations of Sec. II C. (λ is a generic notation for the de Broglie thermal wave length.) $|D_{\alpha\gamma}|$, which has the dimension of an inverse length, has a similar dependence,

$$|D_{\alpha\gamma}|\propto\Gamma\left(\frac{\lambda}{a}\right)^4\frac{1}{a}. \tag{143}$$

Henceforth, $|A_{\alpha\gamma}|$ and $|D_{\alpha\gamma}|$ have the same sense of variation with ρ and T. They both decrease as the density is lowered or as the temperature becomes higher. The relative corrections may be expressed in terms of dimensionless parameters which are indeed small in the regime $\Gamma\ll 1$ and $(\lambda/a)\ll 1$.

Second, we address the dependence upon the intensity of $\mathbf{B}_0$. When $\mathbf{B}_0$ is weak, $D_{\alpha\gamma}$ is proportional to $\hbar^8B_0^4$, whereas, in the strong field limit, it becomes independent from B_0 and is only of order $\hbar^4$. The coefficient $A_{\alpha\gamma}$ of the $1/r^6$ tail when $\mathbf{B}_0=\mathbf{0}$ [2] has the same dependence upon $\hbar$ as $D_{\alpha\gamma}$ in the strong field limit. An interpretation is that, in some sense, the $1/r^5$ tail that appears only in the presence of $\mathbf{B}_0$ is more quantum than the $1/r^6$ tail, because statistical effects of $\mathbf{B}_0$ are purely quantum (see also Sec. VIII D). However, when the intensity of $\mathbf{B}_0$ increases sufficiently, a new effect appears: a strong field $\mathbf{B}_0$ enforces a localization that drives the system into a semiclassical regime (see Sec. VI B of Paper II), and the $1/r^5$ tail becomes ''less quantum'' as regards its order in $\hbar$.

B. Algebraic screening

Algebraic screening at large distances is compatible with the sum rules enforced by both internal and perfect external screening which must be satisfied in any classical as well as in any quantum regime. As recalled in Sec. II B of Paper I internal screening means that

$$\int d\mathbf{r}\sum_\alpha e_\alpha S_{\alpha\gamma}(\mathbf{r})=0 \tag{144}$$

and

$$\int d\mathbf{r}\mathbf{r}\sum_\alpha e_\alpha S_{\alpha\gamma}(\mathbf{r})=\mathbf{0}. \tag{145}$$

On the other hand, perfect external screening refers to the fact that the total charge induced in the plasma in the vicinity of an infinitesimal external charge δq is exactly equal to $-\delta q$.

1. Consequence of classical internal screening

When $\mathbf{B}_0=\mathbf{0}$, internal screening of monopole-monopole and monopole-multipole interactions plays a role in the cascade of power laws as mentioned in the Introduction. (See also Sec. IV B of Ref. [10].) From a technical point of view, at finite density, the screening of the total charge of a loop surrounded by its Debye polarization cloud (corresponding to the bond F^{cc}) combines with diffraction effects described by a monopole-dipole Debye interaction (contained in the bond F^{cm}) and this interplay leads to cancellations of some tails when particle-charge or charge-charge correlations are considered.

At the first two orders in density, it happens that only the Debye approximation $S^{\rm cl}_{\alpha\gamma,D}$ of the classical structure factor is involved in the mechanism responsible for this cascade. Indeed, at the first two orders in density, the exact expression for $A^{\{n\}}_{\alpha\gamma}/r^6$, with $n=2,5/2$, is equal to the tail of the convolution (2). The Debye-Hückel structure factor by itself satisfies both the charge and dipole sum rules (144) and (145), because the $\mathbf{k}$ expansion of $\Sigma_\gamma e_\gamma S^{\rm cl}_{D,\alpha\gamma}(\mathbf{k})$ starts at order $|\mathbf{k}|^2$ when $|\mathbf{k}|$ goes to zero. Henceforth, Eq. (2) leads to the cascade of power laws in the low-density limit—$\Sigma_\gamma e_\gamma A_{\alpha\gamma}=0$ and $\Sigma_\alpha e_\alpha B_\alpha$, as it should—while $B^{\{n-1\}}_\alpha/r^8$ and $C^{\{n-2\}}/r^{10}$, with $n=2,5/2$, coincide with the tail of the convolution (2) with adequate summation over charges at corresponding orders in density.

We notice that the tail of the convolution (2) may also be interpreted as the tail of

$$\sum_{\alpha_1,\alpha_2}S^{\rm cl,reg}_{\alpha\alpha_1}*[-\beta V^{\rm eff(6)}_{\alpha_1\alpha_2}|^{\{0\}}]*S^{\rm cl,reg}_{\alpha_2\gamma}, \tag{146}$$

where $S^{\rm cl,reg}_{\alpha\gamma}$ is the structure factor of the corresponding classical multicomponent plasma with proper short-distance regularization. Indeed, a quantum multicomponent plasma is stable against *macroscopic* collapse [14,15] only if all its negative or/and positive charges obey Fermi statistics. On the contrary, a multicomponent plasma with Maxwell-Boltzmann statistics (and classical or quantum dynamics) has a well-behaved thermodynamic limit only if the Coulomb interaction is regularized at short distances by addition of a short-ranged repulsive potential $v_{\rm SR}(r)$. (Quantum dynamics alone only prevents the collapse of a *finite* number of point charges with opposite signs.)

The link between Eqs. (2) and (146) is the following. The classical structure factor $S^{\rm cl,reg}_{\alpha\gamma}$ may be represented by Mayer diagrams and

$$S^{\rm cl,reg}_{\alpha\gamma}=S^{\rm cl}_{D,\alpha\gamma}+\sum_{\alpha_1,\alpha_2}S^{\rm cl}_{D,\alpha\alpha_1}*h^{nn\rm cl,reg}_{\alpha_1\alpha_2}*S^{\rm cl}_{D,\alpha_2\gamma}, \tag{147}$$

where $h^{nn\rm cl,reg}_{\alpha_1\alpha_2}$ is defined in Sec. VIII C of Paper I. By virtue of rotational invariance of all interactions $\int d\mathbf{r}\mathbf{r}S^{\rm cl,reg}_{\alpha\gamma}(r)=\mathbf{0}$ at any density. Besides, a remarkable fact is that, at the first two orders in density (but not at higher orders) $\int d\mathbf{r}S^{\rm cl,reg}_{\alpha\gamma}(\mathbf{r})|^{\{n\}}$ and $\int d\mathbf{r}r^2S^{\rm cl,reg}_{\alpha\gamma}(\mathbf{r})|^{\{n-1\}}$, with $n=1,3/2$, are determined only by the Debye potential and are independent from $v_{\rm SR}(r)$. Indeed, according to a scaling analysis similar to that devised in Sec. IV A of Paper II, Eq. (147) implies that the zero and second moments of $S^{\rm cl,reg}_{\alpha\gamma}$ involve only $S^{\rm cl}_{D,\alpha\gamma}$ for $n=1$ and

$$\sum_{\alpha_1,\alpha_2} S^{\rm cl}_{D,\alpha\alpha_1} * \frac{1}{2}[F^{cc}_{D,\alpha_1\alpha_2}]^2 * S^{\rm cl}_{D,\alpha_2\gamma} \tag{148}$$

for $n=3/2$. Thus the tails of Eq. (146) with or without charge summations at the first two orders do coincide with those given in Eqs. (2), (5), and (62).

We notice that $S^{\rm cl,reg}_{\alpha\gamma}$ satisfies both the charge and dipole sum rules (144) and (145) as well as the quantum structure factor $S_{\alpha\gamma}$, independently of the choice of the short-ranged potential $v_{\rm SR}(r)$. Henceforth, the interpretation of the $A^{\{n\}}_{\alpha\gamma}/r^6$ tail, with $n=2,5/2$, by Eq. (146) is still coherent with the cascade of inverse power laws.

2. Perfect external screening

We stress again that the basic rule of perfect screening of an external infinitesimal charge δq is not destroyed by algebraic quantum screening: the integral of the induced charge in the bulk is exactly opposite to δq. According to Sec. V B 3 of Ref. [4], at quantum as well as at classical levels, both the static charge-charge correlation $C(\mathbf{r}) = \Sigma_{\alpha,\gamma} e_\alpha e_\gamma S_{\alpha\gamma}(\mathbf{r})$ and the response function $\beta\int_0^1 ds C_T(\mathbf{r},s)$ obey the charge and dipole (internal) sum rules (144) and (145) satisfied by $\Sigma_\alpha e_\alpha S_{\alpha\gamma}(\mathbf{r})/\rho_\gamma$ (which describes the charge e_γ surrounded by its polarization cloud). $C_T(\mathbf{r},s)$ is the time-ordered charge-charge correlation function in imaginary time, and the quantum linear response reads

$$\frac{\sum_\gamma e_\gamma \rho^{\rm ind,L}_\gamma(\mathbf{k};\delta q(\mathbf{k}))}{\delta q(\mathbf{k})} = -\frac{4\pi}{\mathbf{k}^2}\beta\int_0^1 ds C_T(\mathbf{k},s). \tag{149}$$

Thus the perfect external screening provides a sum rule that determines the Fourier transform of the response function up to order $|\mathbf{k}|^2$, whenever $\mathbf{B}_0$ is switched on or not,

$$\int_0^1 ds C_T(\mathbf{k},s) \underset{|\mathbf{k}|\to 0}{\sim} \frac{1}{4\pi\beta}\mathbf{k}^2. \tag{150}$$

We stress that in the quantum case $\int_0^1 ds C_T(\mathbf{r},s)$ is different from the charge-charge correlation $C(\mathbf{r})$, and there is no sum rule analogous to Eq. (150) for $C(\mathbf{r})$. However, the classical version of Eq. (150), the so-called Stillinger-Lovett sum rule, determines the second moment of $C^{\rm cl,reg}(\mathbf{r}) = \Sigma_{\alpha,\gamma} e_\alpha e_\gamma S^{\rm cl,reg}_{\alpha\gamma}(\mathbf{r})$,

$$C^{\rm cl,reg}(\mathbf{k}) \underset{|\mathbf{k}|\to 0}{\sim} \frac{1}{4\pi\beta}\mathbf{k}^2. \tag{151}$$

Moreover, $S^{\rm cl}_{D,\alpha\gamma}$ also satisfies the classical external sum rule. Indeed, the diagrammatic relation (147) and the fact that $\Sigma_\gamma e_\gamma S^{\rm cl}_{D,\alpha\gamma}(\mathbf{k})$ starts as $|\mathbf{k}|^2$ when $|\mathbf{k}|$ goes to zero allow $S^{\rm cl}_{D,\alpha\gamma}$ itself to saturate Eq. (151). More precisely, the Debye charge-charge correlation $\Sigma_{\alpha,\gamma} e_\alpha e_\gamma S^{\rm cl}_{D,\alpha\gamma}(\mathbf{k})$ does obey Eq. (151) because the Debye polarization cloud $\Sigma^{\rm cl}_{D,\alpha\gamma}$ is such that

$$\sum_\gamma e_\gamma S^{\rm cl}_{D,\alpha\gamma}(\mathbf{k}) \underset{|\mathbf{k}|\to 0}{\sim} \frac{e_\alpha\rho_\alpha}{\kappa_D^2}\mathbf{k}^2. \tag{152}$$

A consequence of Eq. (152) is that, when $\mathbf{B}_0=\mathbf{0}$, since κ_D^2 is of order ρ, the orders in density of the leading coefficients $A_{\alpha\gamma}$, B_α, and C in the zero-density limit also undergo a cascade. $A_{\alpha\gamma}$, B_α, and C start at order ρ^2, ρ, and ρ^0, respectively. As a consequence, the coefficient of the charge-charge correlation does not vanish in the zero-density limit. This reflects the fact that results obtained in the limit of an infinitely dilute plasma do not coincide with calculations performed for particles in the vacuum, where no screening effect takes place.

When $\mathbf{B}_0\neq\mathbf{0}$, there is no cascade of power laws and the low-density limits of tails D_α/r^5 and D/r^5 of the particle-charge and charge-charge correlations are just given by the $D_{\alpha\gamma}/r^5$ tail of the particle-particle correlation with adequate summation $\Sigma_\gamma e_\gamma$ and $\Sigma_\alpha e_\alpha$, as it should. This property holds at higher orders in density, because the diagrammatic structures of the various $1/r^5$ tails merely involve fewer and fewer diagrams as charges are summed over, according to Sec. VIII B of Paper I. The coefficients of all these $1/r^5$ tails start at order ρ^2. The $1/r^5$ tail at order ρ^2 does disappear after integration over angles. (This was shown at any finite density by the use of general analyticity arguments in Fourier space given in Sec. VI C of Paper I.) The latter result is compatible with Eq. (150).

For the OCP, as in the case of multicomponent plasmas, the quantum response function does satisfy the above perfect screening sum rule (150) (see page 1122 of Ref. [4]). Moreover, in this system, the extra exact sum rule (16) determines the coefficient of the term of order $|\mathbf{k}|^2$ in the charge-charge correlation $C_{\rm OCP}(\mathbf{k})=e^2 S_{\rm OCP}(\mathbf{k})$ itself, because the motion of the center of mass happens to move independently in the harmonic force created by the background. We notice that there exists another constraint in the absence of $\mathbf{B}_0$ (see page 166 of Ref. [6]): the mechanical balance enforces the value of the $|\mathbf{k}|^2$ term in $\Sigma_\gamma e_\gamma \rho^{\rm ind,L}_\gamma(\mathbf{k};\delta q(\mathbf{k}))|_{\mathbf{B}_0=\mathbf{0}}/\delta q(\mathbf{k})$, and subsequently, the value of the $|\mathbf{k}|^4$ term in $\int_0^1 ds C_{\rm OCP}|_{\mathbf{B}_0=\mathbf{0}}$, by virtue of Eq. (149). The latter terms are analytic, which is in agreement with the fact that the charge density $\Sigma_\gamma e_\gamma \rho^{\rm ind,L}_\gamma(\mathbf{r};\delta q)$ induced by a point charge δq decays faster than $1/r^5$ when $\mathbf{B}_0=\mathbf{0}$, in fact as $1/r^8$ [13].

C. Comparison with $\hbar$ expansions

In the case of a multicomponent plasma, $\hbar$ expansions correspond to MB statistics and they are allowed only if the Coulomb potential is regularized at the origin (in order to avoid the macroscopic collapse of the multicomponent plasma, whatever the latter is in a classical or in a quantum dynamical regime). For instance, point particles may be replaced by spheres described by a repulsive hard- or soft-core potential $v_{\rm SR}(r)$ (see Sec. VIII C of Paper I). In fact, $\hbar$ expansions for correlations exist in the literature only in the case $\mathbf{B}_0=\mathbf{0}$ [12]. (However, some attempts to investigate the case $\mathbf{B}_0\neq\mathbf{0}$ are in progress [16].)

When $\mathbf{B}_0=\mathbf{0}$, at the first order in $\hbar$, namely, $\hbar^4$, the $1/r^6$ tail comes from the convolution (146). Indeed, the semiclassical expansions take the form [11,12]

$$\rho_{\alpha\gamma}^{(2)T}(r)=\rho_{\alpha\gamma}^{(2)T}(r)|^{\mathrm{cl}}+\hbar^2 R_{\alpha\gamma}^{(2)}(r)+\hbar^4 R_{\alpha\gamma}^{(4)}(r)+O(\hbar^6). \tag{153}$$

The classical contribution and the first quantum correction $\hbar^2 R_{\alpha\gamma}^{(2)}$ decay faster than any inverse power law whereas the corrections of orders higher than $\hbar^2$ decrease as $1/r^6$. More precisely, the tail of the semiclassical correlation $\hbar^4 R_{\alpha\gamma}^{(4)}(r)$ at order $\hbar^4$ comes in fact from Eq. (146) as the exact low-density tails $A_{\alpha\gamma}^{\{2\}}/r^6$ and $A_{\alpha\gamma}^{\{5/2\}}/r^6$. Henceforth, the cascade of power laws in the semiclassical limit [12] is induced by the same mechanism as in the low-density limit of the fully quantum regime (see Sec. VIII B 1).

Moreover, as argued in Sec. VI B of Paper II, there exists a regime of physical parameters where both semiclassical and low-density expansions may be performed. Since the effective potential $V_{\alpha\gamma}^{\mathrm{eff}(6)}|^{\{0\}}$ is the only quantum term in Eq. (146) and is exactly proportional to $\hbar^4$, the first two terms $A_{\alpha\gamma}^{\{2\}}/r^6$ and $A_{\alpha\gamma}^{\{5/2\}}/r^6$ in the low-density expansion of the exact $A_{\alpha\gamma}/r^6$ tail of the correlation coincide with the first two terms $\hbar^4 T_{\alpha\gamma}^{(4)}|^{\{2\}}/r^6$ and $\hbar^4 T_{\alpha\gamma}^{(4)}|^{\{5/2\}}/r^6$ in the low-density expansion of the $\hbar^4 T_{\alpha\gamma}^{(4)}/r^6$ tail of the semiclassical correlation at order $\hbar^4$.

In the case of the OCP, the low-density limit (138) coincides with the first term in the Wigner-Kirkwood expansion of the coefficient of the $1/r^{10}$ tail [11–13,17], namely, the term of order $\hbar^4$. We recall that this expansion is legitimate for the OCP, because the latter system has a well-behaved thermodynamic limit even in the Maxwell-Boltzmann approximation. As in the case of multicomponent plasmas, $\hbar$ and ρ expansions may be performed simultaneously in a low-degeneracy regime with weakly quantum dynamics and weak Coulomb coupling (see Sec. VI B of Paper II). We notice that this coincidence between the ρ^2 term in the low-density expansion and the $\hbar^4$ term in the semiclassical expansion turn out both for the coefficient of the $1/r^{10}$ tail of the correlation as well as for the free-energy expression. Since the system handled in the semiclassical derivation is a OCP from the beginning of calculations, the agreement between formulas obtained from different descriptions of the OCP is another argument for the validity of the procedure used to obtain results for the OCP from expressions derived for a TCP.

D. Intrinsic quantum nature of tails induced only by $\mathbf{B}_0$

1. Comparison with a simple model

A deeper insight into the structures of the $1/r^6$ tail (2) and the $1/r^5$ tail (10) of $\rho_{\alpha\gamma}^{(2)T}(\mathbf{r})$ with or without $\mathbf{B}_0$ may be obtained from the comparison with the simple model of two quantum charges embedded in a classical plasma. This model has been discussed in Sec. V of Paper I. The effective interaction $U_{12}^{\mathrm{eff}}(\mathbf{r})$ between the two quantum charges in the model decays algebraically at large distances r. According to Sec. V B of Paper I,

$$U_{12}^{\mathrm{eff}}(r)|_{\mathbf{B}_0=\mathbf{0}} \underset{r\to\infty}{\sim} \bar{V}_{\alpha_1\alpha_2}^{\mathrm{eff}(6)}(r)|^{\{0\}}, \tag{154}$$

where $\bar{V}_{\alpha_1\alpha_2}^{\mathrm{eff}(6)}|^{\{0\}}$ is given by Eq. (5) with the measure $\bar{D}(\boldsymbol{\xi}_i)$ in place of $D(\boldsymbol{\xi}_i)$. $\bar{D}(\boldsymbol{\xi}_i)$ involves the electrostatic free energy $F_{i,\mathrm{elect}}^{(1)}(\boldsymbol{\xi}_i)$ for the immersion of a single closed curve $\lambda_i\boldsymbol{\xi}_i$ into the classical gas. In the same way,

$$U_{12}^{\mathrm{eff}}(\mathbf{r})|_{\mathbf{B}_0} \underset{r\to\infty}{\sim} \bar{V}_{\alpha_1\alpha_2}^{\mathrm{eff}(5)}(\mathbf{r})|^{\{0\}}, \tag{155}$$

where $\bar{V}_{\alpha_1\alpha_2}^{\mathrm{eff}(5)}|^{\{0\}}$ is given by Eq. (11) with the measure $\overline{D_{\mathbf{B}_0}}(\boldsymbol{\xi}_i)$ in place of $D_{\mathbf{B}_0}(\boldsymbol{\xi}_i)$.

In a weak-coupling regime for the classical plasma, namely, at sufficiently low density or high temperature, $F_{i,\mathrm{elect}}^{(1)}(\boldsymbol{\xi}_i)$ tends to its Debye expression [12]

$$F_{i,D}(\boldsymbol{\xi}_i)=\frac{e_i^2}{2}\int_0^1 ds\int_0^1 ds'\frac{\exp[-\kappa_D\lambda_i|\boldsymbol{\xi}_i(s)-\boldsymbol{\xi}_i(s')|]-1}{\lambda_i|\boldsymbol{\xi}_i(s)-\boldsymbol{\xi}_i(s')|}. \tag{156}$$

In the zero-coupling limit, κ_D vanishes. Then $F_{i,D}(\boldsymbol{\xi}_i)$ tends to the value $-\kappa_D e_i^2/2$, which is independent from the loop shape $\boldsymbol{\xi}_i$, so that

$$\overline{D_{\mathbf{B}_0}}(\boldsymbol{\xi}_i) \underset{\kappa_D\to 0}{\sim} D_{\mathbf{B}_0}(\boldsymbol{\xi}_i). \tag{157}$$

(This result would also be obtained by considering a semiclassical limit for the two quantum charges, in which case λ_i goes to zero.)

When $\mathbf{B}_0=\mathbf{0}$, comparison of Eqs. (154) and (157) with Eq. (2) shows that the $1/r^6$ leading tail of $-\beta U_{12}^{\mathrm{eff}}(r)|_{\mathbf{B}_0=\mathbf{0}}$ in the weak-coupling limit for the classical plasma does not coincide with the exact low-density $[A_{\alpha_1\alpha_2}/\rho_{\alpha_1}\rho_{\alpha_2}]/r^6$ tail of the Ursell function $h_{\alpha_1\alpha_2}(r)|_{\mathbf{B}_0=\mathbf{0}}$ in the fully quantum many-body problem. Indeed, $A_{\alpha_1\alpha_2}/r^6$ contains extra contributions with respect to $-\rho_{\alpha_1}\rho_{\alpha_2}\beta\bar{V}_{\alpha_1\alpha_2}^{\mathrm{eff}(6)}(r)|^{\{0\}}$ arising from bonds F^{cc}; these contributions make $A_{\alpha\gamma}^{\{2\}}/\rho_\alpha\rho_\gamma$ independent from species α or γ, according to Eq. (70). However, when $\mathbf{B}_0\neq\mathbf{0}$, the large-distance behavior (155) of $-\beta U_{12}^{\mathrm{eff}}(\mathbf{r})|_{\mathbf{B}_0}$ in the same regime (157) happens to coincide with the $[D_{\alpha_1\alpha_2}/\rho_{\alpha_1}\rho_{\alpha_2}]/r^5$ tail of the exact quantum Ursell function $h_{\alpha_1\alpha_2}(\mathbf{r})|_{\mathbf{B}_0}$ at low density, as a result of several cancellations in screening contributions.

The interpretation is the following. At the first order in density, there are indirect interactions between the two charges e_α and e_γ, because the latter ones interact through Debye screening (which gives contributions of order ρ^0) with other charges of the quantum medium and these charges have an algebraic interaction between each other. At first orders in density the involved Debye interaction is either purely classical (for any value of B_0) or of diffraction type (only when $\mathbf{B}_0\neq\mathbf{0}$). In technical words, classical Debye interactions correspond to F^{cc} bonds and diffraction Debye contributions to F^{cm} bonds, more precisely to the part of F^{cm} which is a monopole-dipole Debye interaction. When $\mathbf{B}_0=\mathbf{0}$, the quantum interaction is some kind of squared dipole-dipole potential possibly screened by one or two F^{cc} bonds. When $\mathbf{B}_0\neq\mathbf{0}$, the interaction is either a dipole-dipole interaction W_3 screened by two F^{cm} bonds, or a dipole-quadrupole interaction W_4 screened by one F^{cm} bond and possibly by a F^{cc} bond, or a quadrupole-quadrupole bond W_5, which is possibly screened by one or two F^{cc} bonds.

The indirect interactions cannot be taken into account in the simple model where all particles of the medium are classical. Besides, generically they do not cancel each other in a true quantum plasma, and it is indeed the case even in the low-density limit when $\mathbf{B}_0=\mathbf{0}$. This can be seen in another way by inspection of the intermediate formula (66). In a limit procedure where only particles e_α and e_γ located at $\mathbf{0}$ and $\mathbf{r}$ remain quantum, a purely quantum interaction involving $[W_3]^2$ survives only between them, so that only $\delta_{\alpha,\alpha'}$ and $\delta_{\gamma,\gamma'}$ do contribute in Eq. (66); then we retrieve the result of the model in the weak-coupling limit. [We notice as a curiosity that for a symmetric two-component plasma, where $e_+=-e_-$ and $m_+=m_-$, we find $\kappa_{e/m}^2=0$ so that, according to Eq. (1), the expression of the model happens to coincide with the quantum many-body result at low density, though the charges that are involved in screening are indeed quantum.]

When $\mathbf{B}_0\neq\mathbf{0}$, it turns out that, as a result of a cancellation mechanism which takes place only when there is no exchange effect, two given quantum charges interact directly by a quantum interaction W_5 at order ρ^2 and, at order $\rho^{5/2}$, W_5 is merely screened by a squared Debye interaction possibly convoluted with a single Debye term. The two quantum charges are not affected by the direct interactions W_3, W_4, and W_5 involving charges of the medium at the first order in density. Thus there is no indirect quantum interaction arising only from $\mathbf{B}_0$ at order ρ^2 and the low-density result for the true quantum many-body problem turns out to coincide with the simple result for two quantum charges in a classical bath.

An interpretation of the above cancellation is that, at the first order in density, which coincides with the low-density limit of the first order in $\hbar$, only semiclassical effects appear and the effective interaction arising from purely quantum dynamics and statistics in the presence of $\mathbf{B}_0$ cannot be conveyed by quantum charges of the medium which are only semiclassically screened from the two given quantum particles.

2. Linear and nonlinear effects

Now, we turn to the large-distance behavior of the induced charge density. The order in density at which the leading tail of $\Sigma_\gamma e_\gamma \rho_\gamma^{\mathrm{ind,L}}(\mathbf{r};\delta q)$ starts is determined by the lowest order in ρ_γ for which the coefficient of the decay of $\Sigma_\gamma e_\gamma \rho_{\alpha\gamma}^{(2)T}(\mathbf{r})$ is linear in e_α. Indeed, as already mentioned in Paper I, at finite density and at any point $\mathbf{r}$, the induced charge density is related to the particle-charge correlation through Eq. (75). We stress that Eq. (75) is valid even for a finite charge e_α and is not restricted to linear effects. At finite density, the induced charge density given by the linear response theory decays with the same inverse power law as the particle-charge correlation, as also argued in Sec. VIII C of Paper I.

When $\mathbf{B}_0=\mathbf{0}$, at first order in density, according to Eq. (2), two particles with charges e_α and e_γ have both direct and indirect squared dipolar interactions with each other. The indirect interaction is conveyed by quantum charges e_{α_i} of the medium which interact with the considered particles e_α and e_γ by the Debye potential, which is linear in the charge, while the quantum interaction is quadratic in each charge e_{α_i}. As a consequence, the $B_\alpha^{\{1\}}/r^8$ tail of the particle-charge correlation $\Sigma_\gamma e_\gamma \rho_{\alpha\gamma}^{(2)T}(r)|_{\mathbf{B}_0=\mathbf{0}}$ at first order in density does involve linear terms in e_α. Thus the B/r^8 decay of the linearly induced charge density $\Sigma_\gamma e_\gamma \rho_\gamma^{\mathrm{ind,L}}(r;\delta q)|_{\mathbf{B}_0=\mathbf{0}}$ indeed starts at order ρ^0, according to Eq. (75).

However, when $\mathbf{B}_0\neq\mathbf{0}$, at first order in density, quantum particles interact only through a direct effective quadrupole-quadrupole potential $V_{\alpha\gamma}^{\mathrm{eff}(5)}|^{\{0\}}$ [see Eq. (11)]. $V_{\alpha\gamma}^{\mathrm{eff}(5)}|^{\{0\}}$ arises from derivatives of the Coulomb interaction, which is proportional to $e_\alpha e_\gamma$, and from quadrupolar moments, each of which is controlled by the magnetic coupling constant $u_{C\alpha}=(\beta\hbar/2m_\alpha c)\times e_\alpha B_0$. In the limit of an infinitesimal e_α at B_0 fixed, the quadrupolar moment becomes quadratic in e_α. Thus the effective direct quadrupolar interaction is cubic in e_α when e_α goes to zero, and so is the $D_\alpha^{\{2\}}/r^5$ tail of the particle-charge correlation $\Sigma_\gamma e_\gamma \rho_{\alpha\gamma}^{(2)T}(\mathbf{r})|_{\mathbf{B}_0}$ at the first order ρ^2 in the same limit. As a consequence, according to Eq. (75), the $D^\star/r^5$ tail of the induced charge density $\Sigma_\gamma e_\gamma \rho_\gamma^{\mathrm{ind,L}}(\mathbf{r};\delta q)|_{\mathbf{B}_0}$ vanishes at first order ρ, $D^{\star\{1\}}=0$. However, screening of Debye type is no longer completely canceled at next order in ρ [see Eq. (108)]. Subsequently, a linear contribution in e_α for infinitesimal e_α shows up in $D_\alpha^{\{5/2\}}$ and the $D^\star/r^5$ tail of the induced charge given by the linear response theory starts at order $\rho^{3/2}$ according to Eq. (129).

E. Classical time-displaced correlations

In the presence as well as in the absence of $\mathbf{B}_0$, the classical time-displaced correlations have algebraic leading behaviors with the same exponents as the corresponding quantum static correlations. The origin of these algebraic decays is the mass inertia which prevents any polarization cloud from following instantaneously the motion of the charge which it surrrounds, so that the average classical polarization cloud around any set of given particles does not have the symmetry properties (absence of multipolar moments) that would ensure the perfect screening [4], namely, the exponential clustering of particle distributions.

When $\mathbf{B}_0=\mathbf{0}$, as shown in Ref. [12], the classical time-displaced particle correlation decays at least as $1/r^6$ in a multicomponent plasma (according to an analysis of hierarchy equations). In the very special case of the OCP, where the particle-particle correlation coincides with the charge-charge correlation, the falloff of the time-displaced classical correlation is even faster: it decays as $C(t)/r^{10}$, according to the behavior of the t^8 term in a small-time expansion [11].

In the presence of $\mathbf{B}_0$, only the case of the OCP has been studied, to our knowledge. The charge-charge correlation decays as $D(t)/r^5$. Indeed, the first-order term in the small-$\mathbf{k}$ expansion of the Fourier transform $e^2 S_{\mathrm{OCP}}(\mathbf{k},t)|_{\mathbf{B}_0}$ starts at order $|\mathbf{k}|^2$ by a term given by formula (2.20) in Ref. [1]. This exact result may be obtained from the microscopic Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy as well as from linear response and macroscopic electrodynamics. This term has the same structure as in the quantum sum rule (16) with $\cos(\omega_\pm t)$ in place of

$$\frac{\beta\hbar\omega_{\pm}}{2}\coth\left(\frac{\beta\hbar\omega_{\pm}}{2}\right). \tag{158}$$

Thus the term of order $|\mathbf{k}|^2$ in $e^2 S_{\mathrm{OCP},\mathbf{B}_0}(\mathbf{k},t)$, $e^2 S^{(2)}_{\mathrm{OCP},\mathbf{B}_0}(\mathbf{k},t)$ oscillates with time with well-defined frequencies which themselves oscillate with the angle between $\mathbf{k}$ and $\mathbf{B}_0$ as in Eq. (17). If this exact result is expanded in powers of t, the nonanalytic term in $e^2 S^{(2)}_{\mathrm{OCP},\mathbf{B}_0=\mathbf{0}}(\mathbf{k},t)$ appears only in the t^8 term. We recall that, in the case $\mathbf{B}=\mathbf{0}$, the $1/r^{10}$ tail of $e^2 S_{\mathrm{OCP}}$ derived from a direct t expansion of the correlation also appears only from order t^8 [11].

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APPENDIX A

In this appendix we investigate the structure of the $1/r^6$ tail of h^{nn} before loop expansions when $\mathbf{B}_0=\mathbf{0}$. We introduce the notation $f^{n-[q,q']}(\chi,\chi')$ for a function which is of parity $(-1)^q$ and $(-1)^{q'}$ under inversion of $\mathbf{X}$ and $\mathbf{X}'$, respectively, and which is the sum of graphs where χ is a non-Coulomb-root point while χ' is any kind of root point.

According to Ref. [10] and Sec. VII of Paper I, the $1/r^6$ tail $S^{(6)[0,0]}_{h^{nn}}(\mathbf{r},\chi_1,\chi_2)$ of h^{nn} which is involved in Eq. (36) comes either from the term $[W_3]^2/2$ in the asymptotic behavior of one bond F_R and has the structure

$$\int d\chi_1'\int d\chi_2'\left(\int d\mathbf{x}\mathcal{G}_2^{n-[0,0]}(\mathbf{x};\chi_1,\chi_1')\right)\times\frac{1}{2}[W_3(\mathbf{r},\chi_1',\chi_2')]^2\left(\int d\mathbf{y}\mathcal{G}_2^{-n[0,0]}(\mathbf{y};\chi_2',\chi_2)\right), \tag{A1}$$

or it comes from the product of the leading $1/r^3$ asymptotic behaviors $S^{(3)}_{\mathcal{C}}$ of two convolutions, each of which involves at least one bond F_R. According to the general analysis of Appendix A in Paper I, the $1/r^3$ tail before integration over loop shapes is odd under inversion of each loop shape, $S^{(3)}_{\mathcal{C}}(\mathbf{r},\chi_1,\chi_2)=S^{(3)[1,1]}_{\mathcal{C}}(\mathbf{r},\chi_1,\chi_2)$. These two functions $S^{(3)}_{\mathcal{C}}$ link either exactly three points and $S^{(6)[0,0]}_{h^{nn}}(\mathbf{r},\chi_1,\chi_2)$ is of the form

$$\int d\chi_1'\int d\chi_2'\int d\chi_2''\left(\int d\mathbf{x}\mathcal{G}_2^{n-[0,0]}(\mathbf{x};\chi_1,\chi_1')\right)\times S^{(3)[1,1]}_{\mathcal{C}}(\mathbf{r},\chi_1',\chi_2')S^{(3)[1,1]}_{\mathcal{C}}(\mathbf{r},\chi_1',\chi_2'')\times\left(\int d\mathbf{y}\int d\mathbf{y}'\mathcal{G}_3^{--n[1,1,0]}(\mathbf{y},\mathbf{y}';\chi_2',\chi_2'',\chi_2)\right) \tag{A2}$$

or they link exactly four points, with the result

$$\int d\chi_1'\int d\chi_1''\int d\chi_2'\int d\chi_2''\times\left(\int d\mathbf{x}\int d\mathbf{x}'\mathcal{G}_3^{n--[0,1,1]}(\mathbf{x},\mathbf{x}';\chi_1,\chi_1',\chi_1'')\right)\times S^{(3)[1,1]}_{\mathcal{C}}(\mathbf{r},\chi_1',\chi_2')S^{(3)[1,1]}_{\mathcal{C}}(\mathbf{r},\chi_1'',\chi_2'')\times\left(\int d\mathbf{y}\int d\mathbf{y}'\mathcal{G}_3^{--n[1,1,0]}(\mathbf{y},\mathbf{y}';\chi_2',\chi_2'',\chi_2)\right). \tag{A3}$$

Equations (A1)–(A3) are the three allowed structures for $S^{(6)[0,0]}_{h^{nn}}(\mathbf{r},\chi_a,\chi_b)$.

For the sake of pedagogy, we give examples of diagrams for each kind of structures of $1/r^6$ tails. For instance, some contributions to $S^{(6)[0,0]}_{h^{nn}}(\mathbf{r},\chi_1,\chi_2)$ of the form (A1) come from the following diagrams in h^{nn}. The contribution from F_R-W is $[W_3]^2/2$ and then $\mathcal{G}_2^{n-[0,0]}(\mathbf{x};\chi_1,\chi_1')=\delta(\mathbf{x})\delta_{\chi_1,\chi_1'}$. The contribution to $S^{(6)[0,0]}_{h^{nn}}(\mathbf{r},\chi_1,\chi_2)$ from the diagram Π equal to $F_R*\rho F_R$ is given by the $1/r^6$ tail of the diagram $\tilde{\Pi}$ equal to $[F_R-W]*\rho[F_R-W]$. This tail is the sum of two terms,

$$\int d\chi_2'\frac{1}{2}[W_3(\mathbf{r},\chi_1,\chi_2')]^2\left(\int d\mathbf{y}\rho(\chi_2')[F_R-W](\mathbf{y},\chi_2',\chi_2)\right) \tag{A4}$$

and the symmetric term obtained by exchanging the roles of root points 1 and 2. This tail corresponds to a $\mathcal{G}_2^{n-[0,0]}(\mathbf{x},\chi_1,\chi_1')$ which is identical to the previous one, while $\mathcal{G}_2^{-n}(\mathbf{y},\chi_2',\chi_2)=\rho(\chi_2')[F_R-W](\mathbf{y},\chi_2',\chi_2)$. According to Eq. (31), the contribution to $\int d\mathbf{y}\mathcal{G}_2^{-n}$ from W reduces to that from W_3. According to Eq. (34), $\int d\mathbf{y}\mathcal{G}_2^{-n[0,0]}(\mathbf{y};\chi_2',\chi_2)=\rho(\chi_2')\int d\mathbf{y}F_R(\mathbf{y};\chi_2',\chi_2)$.

A contribution to $S^{(6)[0,0]}_{h^{nn}}(\mathbf{r},\chi_1,\chi_2)$ of the form (A2) arises from the diagram Π that reads $F_R[F_R^{*}\rho F_R]$. It is equal to those among the $1/r^6$ tails of diagrams $W[W*\rho(F_R-W)]$ and $W[(F_R-W)*\rho W]$ that are even in $\mathbf{X}_1$ and $\mathbf{X}_2$. The tail of the former diagram may be written as

$$\int d\chi_2''W_3(\mathbf{r},\chi_1,\chi_2)W_3(\mathbf{r},\chi_1,\chi_2'')\times\left(\int d\mathbf{y}\rho(\chi_2'')[F_R-W_3](\mathbf{y},\chi_2'',\chi_2)\right), \tag{A5}$$

where we have used property (31).

An $S^{(6)[0,0]}_{h^{nn}}(\mathbf{r},\chi_1,\chi_2)$ tail of the form (A3)—together with contributions of the form (A2)—originates from the diagram $[F_R*\rho F_R][F_R*\rho F_R]$. More precisely, the tail with structure (A3) is given by the diagram $\tilde{\Pi}$ equal to

$$[(F_R-W)*\rho W][W*\rho(F_R-W)].$$

It reads

$$\int d\chi_1' \int d\chi_2' \left(\int d\mathbf{x} \rho(\chi_1')[F_R - W_3](\mathbf{x},\chi_1,\chi_1') \right) \times W_3(\mathbf{r},\chi_1',\chi_2) W_3(\mathbf{r},\chi_1,\chi_2') \times \left(\int d\mathbf{y} \rho(\chi_2')[F_R - W_3](\mathbf{y},\chi_2',\chi_2) \right). \quad (A6)$$

Equation (A6) corresponds to

$$\mathcal{G}_3^{n--[0,1,1]}(\mathbf{x},\mathbf{x}';\chi_1,\chi_1',\chi_1'') = \delta(\mathbf{x}')\,\delta_{\chi_1,\chi_1''}\rho(\chi_1')[F_R - W_3](\mathbf{x},\chi_1,\chi_1')$$

while $\mathcal{G}_3^{--n[1,1,0]}(\mathbf{y},\mathbf{y}';\chi_2',\chi_2'',\chi_2)$ has a similar expression.

APPENDIX B

In this appendix we turn to the derivation of the loop-density expansions of the $1/r^6$ tail of h^{nn}. We use the structures of this tail with two, three, or four intermediate points given in Appendix A. Loop-density expansions are performed according to the principles of Sec. IV A.

As already sketched in Sec. IV A, diagrams in h^{nn} that contribute to the coefficient of $S_{h^{nn}}^{(6)}$ at a given order in ρ_{loop} are readily determined as follows. Diagrams $\tilde{\Pi}$ that appear in functions $\mathcal{G}_2$ and $\mathcal{G}_3$ are replaced by diagrams $\tilde{\Pi}_T$ built with bonds F^{cc}, F^{cm}, F^{mc}, $[F^{cc}]^2/2$, W, and $F_{RT}-W=F_R-[F^{cc}]^2/2-W$. The analysis in loop density performed for analogous bonds in Sec. IV A of Paper II is used again with the following results.

$\int d\mathbf{x}\mathcal{G}_2^{n-[0,0]}(\mathbf{x};\chi_1,\chi_1')$ starts at order ρ_{loop}^0, with

$$\mathcal{G}_2^{n-[0,0]}|_{\text{loop}}^{\{0\}}(\mathbf{x};\chi_1,\chi_1') = \delta(\mathbf{x})\,\delta_{\chi_1,\chi_1'} \quad (B1)$$

whereas $\int d\mathbf{x}\int d\mathbf{x}'\mathcal{G}_3^{n--[0,1,1]}(\mathbf{x},\mathbf{x}';\chi_1,\chi_1',\chi_1'')$ starts only at order ρ_{loop}, with

$$\mathcal{G}_3^{n--[0,1,1]}|_{\text{loop}}^{\{1\}}(\mathbf{x},\mathbf{x}';\chi_1,\chi_1',\chi_1'') = \delta(\mathbf{x}')\,\delta_{\chi_1,\chi_1''}\rho(\chi_1')F_{RT}(\mathbf{x},\chi_1,\chi_1'). \quad (B2)$$

In the following, the notation $\mathcal{G}|_{\text{loop}}^{\{n\}}$ is to be understood as the contribution of $\mathcal{G}$ of order ρ_{loop}^n *after* integration over position variables. We notice, though this formula is not used in the present paper, that

$$\mathcal{G}_3^{n--[0,1,1]}|_{\text{loop}}^{\{3/2\}}(\mathbf{x},\mathbf{x}';\chi_1,\chi_1',\chi_1'') = \frac{1}{2}\rho(\chi_1')\rho(\chi_1'')\{\cdots\}, \quad (B3)$$

with

$$\{\cdots\} = F^{cm}(1,1')F^{cm}(1,1'')F^{cc}(1',1'') + F^{cm}(1,1')F^{cc}(1,1'')F^{cm}(1',1'') + F^{cc}(1,1')F^{cm}(1,1'')F^{mc}(1',1'') + F^{cc}(1,1')F^{cc}(1,1'')F_{RT}(1',1''). \quad (B4)$$

In Eq. (B4) the contributions from W and $F_{RT}-W$ have been summed and 1/2 is the symmetry factor of diagrams. (The first three diagrams are such that one and only one m is associated with $1'$ as well as with $1''$.)

Moreover, contributions to $\int d\mathbf{x}\mathcal{G}_2^{n-[0,0]}(\mathbf{x};\chi_1,\chi_1')$ at orders $\rho_{\text{loop}}^{1/2}$ and ρ_{loop} arise from diagrams with the structure $\mathcal{G}_2^{nn[0,0]}*\Sigma_D$. Indeed, the $\rho_{\text{loop}}^{1/2}$ term corresponds to

$$\mathcal{G}_2^{n-[0,0]}|_{\text{loop}}^{\{1/2\}} = \frac{1}{2}[F^{cc}]^2\rho*\Sigma_D. \quad (B5)$$

We notice that the contribution from diagram $F^{cm}(\mathbf{x},\chi_1,\chi_1')$ to $\int d\mathbf{x}\mathcal{G}_2(\mathbf{x},\chi_1,\chi_1')$ at any order ρ_{loop}^n, with $n\leq 1/2$, vanishes according to Eq. (30). Thus the convolution of an algebraic tail with F^{cm} increases the exponent of the falloff after integration over orientation of $\mathbf{x}$. The term of order ρ in $\int d\mathbf{x}\mathcal{G}_2^{n-[0,0]}(\mathbf{x};\chi_1,\chi_1')$ is given by a $\mathcal{G}_2^{nn[0,0]}|_{\text{loop}}^{\{1\}}*\Sigma_D$ with

$$\mathcal{G}_2^{nn[0,0]}|_{\text{loop}}^{\{1\}} \equiv F_{RT} + \frac{1}{2}[F^{cc}]^2*\rho\frac{1}{2}[F^{cc}]^2 + \frac{1}{2}[F^{cc}]^2*\rho F^{cc}*\rho\frac{1}{2}[F^{cc}]^2 + F^{cc}\left\{ F^{cc}*\rho\frac{1}{2}[F^{cc}]^2 + \frac{1}{2}[F^{cc}]^2*\rho F^{cc} + F^{cc}*\rho\frac{1}{2}[F^{cc}]^2*\rho F^{cc} \right\} + \text{bridge}_5. \quad (B6)$$

In this writing we have summed W and $F_{RT}-W$ while Bridge$_5$ denotes the value of a bridge diagram with five bonds F^{cc} and two root points,

$$\text{bridge}_5(\mathbf{r}_1-\mathbf{r}_1',\chi_1,\chi_1') \equiv \frac{1}{2}\int d\mathcal{P}\rho(\chi)\int d\mathcal{P}'\rho(\chi')F^{cc}(\mathcal{L}_1,\mathcal{P}) \times F^{cc}(\mathcal{L}_1,\mathcal{P}')F^{cc}(\mathcal{P},\mathcal{P}')F^{cc}(\mathcal{P},\mathcal{L}_1')F^{cc}(\mathcal{P}',\mathcal{L}_1'). \quad (B7)$$

Bridge$_5$ is analogous to the bridge diagram I_{bridge_6} with six bonds F^{cc} and one root point, which is introduced in Sec. V D of Paper II for the calculation of the free energy from the diagrammatic representation of the ratio $\rho/\rho_\alpha^{\text{id}\star,\text{MB}}$. Diagrams at next order in ρ_{loop} are too numerous to be presented here.

As a conclusion, the structures (A1), (A2), and (A3) of the tail $S_{h^{nn}}^{(6)}$ with two, three, or four intermediate points start at orders ρ_{loop}^0, ρ_{loop}, and ρ_{loop}^2, respectively. Therefore the $1/r^6$ tail of the particle-particle correlation starts at order ρ^2. Indeed, in its contribution to the particle-particle correlation, h^{nn} is convoluted by dressings that start at order ρ_{loop}^0, and the root points of h are multiplied by weights ρ_{loop}. Thus, up to order $\rho_{\rm loop}^{5/2}$, $A_{\alpha_a\alpha_b}/r^6$ is given by Eq. (44) where $S_{h^{nn}}^{(6)[0,0]}$ has the structure with two intermediate points (A1). Then the results of Sec. V B are derived from Eqs. (B1) and (B5).

APPENDIX C

In this appendix we explicitly check identities (81) and (82) at the first order in density. The D_α/r^5 tail of the particle-charge correlation in the presence of $\mathbf{B_0}$ reads

$$\frac{D_\alpha}{r^5}=\int d\chi_a\rho(\chi_a)\delta_{e_{\alpha_a},\alpha}\times\int d\chi_b\rho(\chi_b)p_b e_{\alpha_b}S_{pc}^{(5)[0,0]}(\mathbf{r},\chi_a,\chi_b), \quad \text{(C1)}$$

where, according to Sec. VIII,B of Paper I, $S_{pc}^{(5)[0,0]}(\mathbf{r},\chi_a,\chi_b)$ is the $1/r^5$ tail of $\Sigma_D^* * h^{nn} * \rho F^{mc}$ which is even under inversion of each root point. According to Eq. (28), the small-$\mathbf{k}$ expansion of F^{cm} starts at order $|\mathbf{k}|$ so that $S_{pc}^{(2)[0,0]}(\mathbf{k},\chi_a,\chi_b)$ may be decomposed as the sum of two contributions,

$$\int d\chi_1\int d\chi_2\rho(\chi_2)F^{cm(1)}(\mathbf{k},\chi_a,\chi_1)\times S_{h^{nn}}^{(0)[1,1]}F^{cm(1)}\times(\mathbf{k},\chi_2,\chi_b) \quad \text{(C2a)}$$

$$+\int d\chi_1\int d\chi_2\rho(\chi_2)\Sigma_D^{(0)}(\mathbf{k},\chi_1;\chi_a)S_{h^{nn}}^{(1)[0,1]}\times F^{cm(1)}(\mathbf{k},\chi_2,\chi_b). \quad \text{(C2b)}$$

We have not included the third term

$$\int d\chi_1\int d\chi_2\rho(\chi_2)\Sigma_D^{(0)}(\mathbf{k},\chi_1;\chi_a)\times S_{h^{nn}}^{(0)[0,0]}F^{mc(2)}(\mathbf{k},\chi_2,\chi_b)=0. \quad \text{(C3)}$$

It vanishes for the same reason as Eq. (84).

Since $\rho F^{cm(1)}$ and $\Sigma_D^{(0)}$ are exactly of order $\rho_{\rm loop}^0$, $S_{pc}^{(2)[0,0]}$ starts at order $\rho_{\rm loop}^0$ where it involves Eqs. (88) and (89). At the first order ρ^0 in particle density the same compensation mechanism as in the case of $S_{pp}^{(5)[0,0]}$ operates for $S_{pc}^{(5)[0,0]}$. Indeed, similarly to Eq. (99), the effective contribution from $\Sigma_D^* * h^{nn} * \rho F^{mc}$ to the tail $S_{pc}^{(5)}$ at the lowest order in $\rho_{\rm loop}$ may be reorganized in Fourier space as

$$\left[F^{cm(1)}\rho w_3^{[1,1]}+\frac{1}{2}F^{cc(0)}\rho w_4^{[2,1]}\right]F^{mc(1)} \quad \text{(C4a)}$$

$$+\frac{1}{2}w_4^{[2,1]}\rho F^{mc(1)}. \quad \text{(C4b)}$$

According to Eq. (94), only Eq. (C4b) contributes to the inverse Fourier transform

$$S_{pc}^{(5)[0,0]}(\mathbf{r},\boldsymbol{\xi}_a,\boldsymbol{\xi}_b)|^{\{0\}}=\mathscr{F}^{-1}\left[\frac{1}{2}\sum_{\alpha_1}\int D_{\alpha_1,B_0}(\boldsymbol{\xi}_1)w_4^{[2,1]}(\mathbf{k},\boldsymbol{\xi}_a,\boldsymbol{\xi}_1;\alpha_a,\alpha_1)\times\rho_{\alpha_1}F^{mc(1)\{-1\}}(\mathbf{k},\boldsymbol{\xi}_1;\alpha_1,\alpha_b)\right](\mathbf{r}). \quad \text{(C5)}$$

Subsequently, the $1/r^5$ tail of the particle-charge correlation, given by Eqs. (C1) and (C5), has the same structure as Eq. (101) with $F_{\alpha\gamma}^{pp}(\mathbf{k},s_1,s_2)$ replaced by $\Sigma_\gamma e_\gamma F_{\alpha\gamma}^{pc}(\mathbf{k},s_1,s_2)$, where $F_{\alpha\gamma}^{pc}(\mathbf{k},s_1,s_2)$ has an expression similar to Eq. (102) with the term in braces equal to

$$\frac{1}{2}\int_0^1 ds_2'\int D_{\alpha,B_0}(\boldsymbol{\xi}_1)\int D_{\gamma,B_0}(\boldsymbol{\xi}_2)[\mathbf{k}\cdot\boldsymbol{\xi}_1(s_1)]^2[\mathbf{k}\cdot\boldsymbol{\xi}_2(s_2)]\times[\mathbf{k}\cdot\boldsymbol{\xi}_2(s_2')]\frac{4\pi}{\mathbf{k}^2}. \quad \text{(C6)}$$

The nonanalytic term in Eq. (C6) has the same structure in $\mathbf{k}$ as Eq. (105) with the coefficient $\delta C_\alpha(s_1,s_1)\delta C_\gamma(s_2,s_2)$ replaced by $2\delta C_\alpha(s_1,s_1)\int_0^1 ds_2'\delta C_\gamma(s_2,s_2')$. The factor 2 is compensated by the factor 1/2 arising from Eq. (97), and the D_α/r^5 tail of $\Sigma_\gamma e_\gamma\rho_{\alpha\gamma}^{(2)T}(\mathbf{r})$ is equal to $\Sigma_\gamma e_\gamma$ times the expression (110), namely, we get Eq. (81).

The origin of the D/r^5 tail of the charge-charge correlation is reduced to

$$\frac{D}{r^5}=\int d\chi_a\rho(\chi_a)p_a e_{\alpha_a}\int d\chi_b\rho(\chi_b)\times p_b e_{\alpha_b}S_{cc}^{(5)[0,0]}(\mathbf{r},\chi_a,\chi_b), \quad \text{(C7)}$$

where $S_{cc}^{(5)[0,0]}(\mathbf{r},\chi_a,\chi_b)$ is the $1/r^5$ tail of $F^{cm}\rho * h^{nn} * \rho F^{mc}$. More precisely,

$$S_{cc}^{(2)[0,0]}(\mathbf{k},\chi_a,\chi_b)=F^{cm(1)}(\mathbf{k},\chi_a,\chi_1)S_{h^{nn}}^{(0)[1,1]}(\mathbf{k},\chi_1,\chi_2)\times F^{mc(1)}(\mathbf{k},\chi_2,\chi_b). \quad \text{(C8)}$$

Since $\rho F^{cm(1)}$ is exactly of order $\rho_{\rm loop}^0$, by inserting Eqs. (88) and (28) into Eq. (C8), we obtain that at the lowest order in particle density

$$\mathscr{F}\left[\frac{D^{\{2\}}}{r^5}\right](\mathbf{k})=\sum_{\alpha_1}\rho_{\alpha_1}e_{\alpha_1}\sum_{\alpha_2}\rho_{\alpha_2}e_{\alpha_2}\int D_{\alpha_1,B_0}(\boldsymbol{\xi}_1)\times\int D_{\alpha_2,B_0}(\boldsymbol{\xi}_2)\int_0^1 ds\int_0^1 ds'[\mathbf{k}\cdot\lambda_{\alpha_1}\boldsymbol{\xi}_1(s)]\times[\mathbf{k}\cdot\lambda_{\alpha_1}\boldsymbol{\xi}_2(s')]W_3(\mathbf{k},\boldsymbol{\xi}_1,\boldsymbol{\xi}_2;\alpha_1,\alpha_2). \quad \text{(C9)}$$

Equation (C9) has the same structure as Eq. (101) with $F_{\alpha\gamma}^{pp}(\mathbf{k},s_1,s_2)$ replaced by $\Sigma_{\alpha,\gamma}e_\alpha e_\gamma F_{\alpha\gamma}^{cc}(\mathbf{k},s_1,s_2)$ where $F_{\alpha\gamma}^{cc}(\mathbf{k},s_1,s_2)$ is given by an expression similar to Eq. (102) with the term in braces replaced by

$$\int_0^1 ds_1' \int_0^1 ds_2' \int D_{\alpha,B_0}(\boldsymbol{\xi}_1) \int D_{\gamma,B_0}(\boldsymbol{\xi}_2)[\mathbf{k}\cdot\boldsymbol{\xi}_1(s_1)] \times [\mathbf{k}\cdot\boldsymbol{\xi}_1(s_1')][\mathbf{k}\cdot\boldsymbol{\xi}_2(s_2)][\mathbf{k}\cdot\boldsymbol{\xi}_2(s_2')]\frac{4\pi}{\mathbf{k}^2}. \tag{C10}$$

The nonanalytic term in Eq. (C10) has the structure (105) with $\delta C_\alpha(s_1,s_1)\delta C_\gamma(s_2,s_2)$ replaced by $4\int_0^1 ds_1' \delta C_\alpha(s_1,s_1')\int_0^1 ds_2' \delta C_\gamma(s_2,s_2')$. According to Eq. (97), the factor 4 is compensated by a factor 1/4 and we obtain Eq. (82).

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