

Thermal Contact. II. A Solvable Toy Model

F. Cornu

Laboratoire de Physique Théorique, UMR 8627 du CNRS
Université Paris-Sud, Bât. 210
F-91405 Orsay, France

M. Bauer

Institut de Physique Théorique de Saclay*, CEA Saclay
F-91191 Gif-sur-Yvette Cedex, France

May 2, 2015

Most results contained in the joined papers arXiv:1302.4538 and arXiv:1302.4540 put on cond-mat.stat-mech on February 19 2013 have now appeared in print as

- [a] Thermal Contact through a Diathermal Wall: A Solvable Toy Model,
F. Cornu and M. Bauer, *J. Stat. Mech.* (2013) P10009
- [b] Part of Affinity and Fluctuations in a Mesoscopic Noria,
M. Bauer and F. Cornu, *J. Stat. Phys.* **155** (2014) 703, [arXiv:1402.2422]
- [c] Local detailed balance : a microscopic derivation,
M. Bauer and F. Cornu, *J. Phys. A: Math. Theor.* **48** (2015) 015008, [arXiv:1412.8179]

The present revised version takes into account the minor corrections made in the published articles but the bibliography is that of the first version (not updated). In reference [c] the denomination local detailed balance has been used in place of modified detailed balance in order to fit the terminology that seems to be most commonly used nowadays.

Abstract

A diathermal wall between two heat baths at different temperatures can be mimicked by a layer of independent spin pairs with some internal energy and where each spin σ_a is flipped by thermostat a ($a = 1, 2$). The transition rates are determined from the modified detailed balance discussed in Ref.[1]. Generalized heat capacities, excess heats, the housekeeping entropy flow and the thermal conductivity in the steady state are calculated. The joint probability distribution of the heat cumulated exchanges at any time is computed explicitly. We obtain the large deviation function of heat transfer via a variety of approaches. In particular, by a saddle-point method performed accurately, we obtain the explicit expressions not only of the large deviation function, but also of the amplitude prefactor, in the long-time probability density for the heat current. The following physical properties are discussed : the effects of typical time scales of the mesoscopic dynamics which do not appear in equilibrium statistical averages and the limit of strict energy dissipation towards a thermostat when its temperature goes to zero. We also derive some properties of the fluctuations in the two-spin system viewed as a thermal machine performing cycles.

PACS : 05.70.Ln, 02.50.Ga, 05.60.Cd

*CEA/DSM/IPhT, Unité de recherche associée au CNRS

KEYWORDS : thermal contact; master equation; solvable model; excess heat; thermal conductivity; large deviation function ; kinetic mean-field effect; strict dissipation ; thermal cycles.

Corresponding author : CORNU Françoise, E-mail: Francoise.Cornu@u-psud.fr

Contents

1	Introduction	3
1.1	Issues at stake	3
1.2	Contents of the paper	6
2	Model	7
2.1	Constraints upon transition rates arising from microscopic discrete ergodic energy-conserving dynamics	8
2.2	Determination of transition rates	8
3	Non Equilibrium Stationary State (NESS) as a canonical distribution with an effective temperature	9
3.1	Stationary state distribution	9
3.2	Linear static response to a variation of some external parameter	11
3.2.1	Relation with static correlations for a “canonical” NESS	11
3.2.2	Dependence of the mean energy upon the time scales of the microscopic dynamics	11
3.2.3	Stationary mean energy and generalized heat capacities	11
3.2.4	Stationary heat current and linear thermal conductivity	13
3.3	Non-linear static response in the NESS	14
3.3.1	Non-linear thermal conductivity	14
3.3.2	Housekeeping entropy flow and mean excess heats	15
4	Joint probability distribution for heat cumulated exchanges at finite time in the model	16
4.1	Explicit calculations	16
4.1.1	Constraint from energy conservation	16
4.1.2	Generating function method	17
4.1.3	Diagonalization	18
4.1.4	Results for the generating function	19
4.1.5	Results for the joint probability	19
4.2	Various explicit probabilities	20
4.3	Excess heats	22
4.4	Symmetry property for reversed heat transfers (when $T_1 \neq 0$) specific to the model	22
4.4.1	Symmetry arising from modified detailed balance	22
4.4.2	A relation specific to the model	24
4.5	Decaying property of joint probabilities for large heat exchanges	25
5	Heat amount cumulants for any T_1 and T_2	26
5.1	Generic properties for a system with a finite number of configurations	26
5.1.1	Characteristic function for the heat amount Q_2	26
5.1.2	Relation between long-time cumulants per unit time for Q_1 and Q_2	26
5.2	Explicit formulæ for the cumulants per unit time	27
6	Large deviation function for the cumulated heat current Q_2/t	30
6.1	Derivation from Gärtner-Ellis theorem	31
6.1.1	Method	31
6.1.2	Various expressions for f_{Q_2} and its properties	31
6.2	Derivation from a saddle-point method	34
6.2.1	Method	34
6.2.2	Deformation of contours	36
6.2.3	Large τ behavior	39
6.3	Derivation by Laplace’s method on a discrete sum	40

7	Dependence upon typical time scales of the thermostats	41
7.1	Stationary mean values in the infinite ν_2/ν_1 limit	41
7.2	Various probabilities in the infinite ν_2/ν_1 limit	41
7.3	Interpretation: mean-field regime	43
7.4	Symmetry property specific to the probability of $\mathcal{Q}_1$	43
7.5	Cumulants per unit time for $\mathcal{Q}_1^d$ and $\mathcal{Q}_2$	43
7.6	Long-time current distribution in the infinite ν_2/ν_1 limit	44
8	Case where $T_1 = 0$: pure energy dissipation towards thermal bath 1	46
8.1	Microscopic irreversibility	46
8.2	Long-time behavior	47
8.3	Limit where ν_2/ν_1 becomes infinite	47
8.3.1	Finite-time behaviors	47
8.3.2	Long-time behavior	48
8.4	Limit where ν_2/ν_1 tends to zero	49
9	Thermal cycles	49
9.1	Markov processes with a cyclic transition graph	50
9.1.1	Forward and backward thermal cycles	50
9.1.2	Fluctuations in the time it takes to make a thermal cycle	52
9.1.3	Degeneration to random walks	53
9.2	The case of the two-spin system	54

1 Introduction

1.1 Issues at stake

Though a rather specialized topic within non-equilibrium physics, the problematic of thermal contact is in itself a vast subject, and one of foremost theoretical and practical interest. A theoretic, microscopic, understanding of non-equilibrium dynamics is still lacking today, even in the particular case of heat exchanges. When compared with the most general non-equilibrium physics world, certain specific settings have some advantages and this is the case of thermal contact: when only quantities conserved by dynamics (energy for a thermal contact) are exchanged between baths and an out-of-equilibrium system, and different baths are in contact with different parts of this out-of-equilibrium system, the fact that the baths remain at equilibrium all along the experiment allows to keep a simple, thermodynamical interpretation of various physical quantities (usually entropy variations) related to the exchanges.

There are at least two important microscopic versions of thermal contact one can have in mind.

One is as an interface between two thermal baths, generically at different temperatures. In our real (three-dimensional) world, the most natural model geometry for the interface is a real or fictitious (two-dimensional) surface. Each side of the surface consists of atoms of one bath. Heat flows from the high temperature bath to the low temperature bath either via interactions among atoms sitting on each side of an immaterial interface (a case relevant when the baths consist of solid materials), or via interactions of the atoms with a thin or structureless material interface (a case relevant for instance when the baths are gaseous, no matter is allowed to be exchanged, and the interface is a diathermal wall). One can of course generalize to more than two baths, in contact via an appropriate number of interfaces.

Another version of thermal contact is as an extended piece of material, with specified physical properties, and such that two (or more) parts of its boundary are in contact with thermal baths. A typical example is a bar of metal whose extremities are maintained at different temperatures. Though the piece of material is not in thermodynamic equilibrium *stricto sensu*, it is often of practical value to assign a local temperature at each point of the bar, and the famous Fourier law states that, for an isotropic material sustaining small temperature gradients, the heat current is proportional to the temperature gradient at each point. This law is of course approximate and phenomenological. But finding a physically motivated model, even a crude one, for which

the relation between local temperature and heat current can be derived from first principles, and compared with Fourier's law, is a major challenge in the field. This is true even when the piece of material is a homogeneous bar and only its longitudinal dimension has to be taken into account, its section being homogeneous with a good accuracy.

Our aims in this article can be considered as very modest, especially when compared to the general issues raised by both versions of thermal contact alluded above. But before describing in detail our model, let us put it briefly in context and compare it to other, more or less similar approaches.

A significant trend in out-of-equilibrium statistical mechanics has been the search for solvable models which could give some hints in the comprehension of out-of-equilibrium phenomena in the absence of any theoretical framework which would play the role of Gibbs' statistical ensemble theory for equilibrium states. Several kinds of models for heat conduction have been introduced. In some of them the two heat baths are connected by a system with deterministic dynamics, such as an anharmonic chain (Fermi-Pasta-Ulam model) [2, 3, 4, 5, 6] or a one-dimensional hard particle gas [7, 8]. In other models the system which ensures the energy transfer from one energy reservoir to the other has a stochastic dynamics: it may be an Ising spin system (see for instance among others [9, 10, 11, 12]) or a particle with Langevin stochastic dynamics [13, 14]. In the latter case the heat exchanges are described as the work performed by a force including a friction as well as a random noise component. The latter interpretation of heat has been proposed and investigated by Sekimoto [15] for ratchets models, and has been used again in the interpretation of the Hatano-Sasa identity [16] as well as in the investigation of heat fluctuations in Brownian transducers [17].

In the first part of the present work [1], referred to as paper I in the sequel, we have investigated the generic statistical properties for experimentally measurable quantities in the case of thermal contact models with the following features: the system has a finite number of possible configurations, the heat exchanges are described as changes in the populations of energy levels, and the configurations evolve under a stochastic master equation with transition rates bound to obey the modified detailed balance restated in (2.3). We have shown how the latter relation arises from the existence of an underlying ergodic deterministic microscopic dynamics which conserves energy.

In the present article, we concentrate on the crudest possible version of a thermal contact: two heat baths are connected via the smallest possible contact. So in the "thermal contact at an interface" image, we would replace a surface by a point. And in the "heat flow through a piece of material" image, the bar we consider has microscopic length, cutting any hope of understanding temperature gradients. We believe that the model has some interest with regard to these interpretations despite its disarming simplicity. Especially for the "thermal contact at an interface", we could take a contact surface made of a collection of spin pairs, the two spins in a pair being coupled as above to allow for heat transfer, but the different pairs being independent. Of course in a more realistic thermal contact, some interactions between pairs, reflecting the two-dimensional geometry, would be present, but there is no obvious reason to believe that these interactions would change qualitatively the physics of heat transfer. For instance, within this model, the law of large numbers allows to quantify how fluctuations of the heat flow are suppressed when the size of the interface goes from microscopic to mesoscopic and macroscopic. We shall not embark on this study in the present paper, but we shall give in a moment a third view of thermal contact for which our model is relevant.

Before that, we make yet another simplifying assumption on the way heat is exchanged between the baths: heat bath 1 (resp. 2) can flip a dynamical variable σ_1 (resp. σ_2) which can take only two distinct values. The energy $\mathcal{E}(\sigma_1, \sigma_2)$ changes when a contact dynamical variable is flipped, and assuming an energy conserving dynamics, this means that some energy comes from, or is given to, the heat bath responsible for the flip. Without loss of generality, we may assume that the values taken by σ_1 and σ_2 belong to $\{-1, 1\}$, and we shall use the name *spins* for σ_1 and σ_2 . In general, the energy $\mathcal{E}(\sigma_1, \sigma_2)$ for the contact dynamical variables could take 4 distinct values, but we even concentrate on the case $\mathcal{E}(\sigma_1, \sigma_2) = \frac{1}{2}(1 - \sigma_1\sigma_2)\Delta e$, where $\Delta e > 0$ is the energy gap. In the language of spins, this means the absence of external magnetic fields. More abstractly, it

implies a twofold symmetry.

We shall concentrate on a description of the time evolution of interface states (σ_1, σ_2) by a Markov process, i.e. by a probabilistic description. But other approaches are possible. For instance, as explained in paper I, the motivation for our choices of transition rates comes from invoking an ergodicity argument for a deterministic discrete time evolution of the compound “heat bath 1 plus interface plus heat bath 2”.

As the toy model has only 4 states, solving it can be reduced in some sense to the diagonalization of a 4×4 matrix, and the twofold symmetry of the energy functional allows to reduce this task to the diagonalization of a pair of 2×2 matrices. However, this is not the end of the story, and this takes us to the third interpretation of the model.

The third view of a thermal contact mentioned above is not microscopic but mesoscopic. We regard σ_1 and σ_2 as some relevant collective variables and $\mathcal{E}(\sigma_1, \sigma_2)$ as an effective energy. Then the system can be viewed, and analyzed, as a thermal machine. That is, our crude model keeps track of one (and maybe only one) interesting feature: the system can make cycles. Consider a sequence of flips in the interface, starting from the state (σ_1, σ_2) :

$$\begin{array}{ccc} (\sigma_1, \sigma_2) & \rightarrow & (-\sigma_1, \sigma_2) \\ \uparrow & & \downarrow \\ (\sigma_1, -\sigma_2) & \leftarrow & (-\sigma_1, -\sigma_2) \end{array}$$

after which the interface has returned to its original state. Writing E_1, E_2 for the initial energies in heat bath 1 and 2, the sequence translates into

$$\begin{array}{ccc} (E_1, E_2) & \rightarrow & (E_1 - \sigma_1 \sigma_2 \Delta e, E_2) \\ & & \downarrow \\ (E_1 - 2\sigma_1 \sigma_2 \Delta e, E_2 + \sigma_1 \sigma_2 \Delta e) & \leftarrow & (E_1 - \sigma_1 \sigma_2 \Delta e, E_2 + \sigma_1 \sigma_2 \Delta e) \\ \downarrow & & \\ (E_1 - 2\sigma_1 \sigma_2 \Delta e, E_2 + 2\sigma_1 \sigma_2 \Delta e) & & \end{array}$$

i.e. an amount of heat $2\sigma_1 \sigma_2 \Delta e$ has been transferred from heat bath 1 to heat bath 2. We use the term mesoscopic (as opposed to macroscopic) for two reasons: first the dynamics at the interface is not deterministic, i.e. knowing (σ_1, σ_2) at some time does not allow to know its value in the future and second (this is somehow a consequence though) there may be portions of time in which the net flow of heat is from the cold bath to the hot bath.

When seen in this light, the model is already more interesting: the time evolution of the heat bath energies is a random walk in continuous time¹, a subject known to lead to a number of nontrivial mathematical problems, some of them having a direct physical relevance. And indeed we shall concentrate mainly on the physics, with the aim of performing detailed analytical computations.

Another interest of our specific solvable model is that it plays the role of a pedagogical example where the general statements are made very explicit. For instance, though the fluctuation relations entail a constraint upon large deviation functions, they do not allow to determine them. The analytical calculation of the large deviation functions may provide a deeper understanding in the information which they contain. We shall see that, within the model, the computation of large deviation functions for the energy variations in the baths can be remarkably simple or tricky, depending on the kind of techniques one uses.

Finally, let us note that in the absence of any general framework for out-of-equilibrium statistical mechanics, the formulæ obtained for the solvable model can give a flavor of the physical effects. For instance, the time scales of the microscopic dynamics, which do not show off in equilibrium averages, play a role in out-of-equilibrium properties even at the macroscopic level. Moreover the model can be considered in the limit where the temperature of the cold thermostat vanishes ; then the strict dissipation of energy towards the zero-temperature gives rise to specific phenomena.

¹And in two spatial dimensions. The sum of the two coordinates can only take a finite number of values, but the waiting times and some correlations prevent from concentrating only on one component.

1.2 Contents of the paper

The results of the explicit analytical calculations for the solvable model where the system is reduced to two spins are the following.

In the case where the spin system involves only two spins, the transition rates are determined by the modified detailed balance (2.3) up to the typical inverse times ν_a of spin flips by each thermal bath a , characterized by its temperature T_a . For an Ising interaction between the two spins, the transition rates for the energy exchanges with one bath take a form similar to that introduced by Glauber [18] in his investigation of the time-dependent statistics of the Ising chain in contact with a single thermal bath. Most of the time, our results will hold whatever the values of T_1 and T_2 are. However, it is sometimes convenient to know in which direction heat flows from one reservoir to the other on the average, and then we shall always assume that $T_1 \leq T_2$. By symmetry, this induces no loss of generality anyway: the results for $T_1 \geq T_2$ can be retrieved by permuting T_1 with T_2 and ν_1 with ν_2 .

The Non-Equilibrium Stationary State (NESS) of the model happens to have a very specific property (subsection 3.1) : since the transition rates are invariant under the simultaneous flips of both spins, the configuration probability distribution in the NESS coincides with an equilibrium canonical distribution at some inverse temperature β_* .

The linear and non-linear static responses are explicitly calculated (subsections 3.2 and 3.3). The expressions for the generalized heat capacities involve not only the temperatures of the energy reservoirs $a = 1, 2$ but also the typical inverse time scales ν_a 's of the heat exchange dynamics with each reservoir. The ν_a 's are also called kinetic parameters in the following. In the vicinity of equilibrium the mean heat current is proportional to the difference $T_2 - T_1$ between the bath temperatures; then a linear thermal conductivity can be defined. When the system is far from equilibrium the mean heat current is a bounded function of the thermostat temperatures (saturation phenomenon); one can introduce a non-linear thermal conductivity which vanishes in the limit where the relative temperature difference goes to infinity. The expression of the housekeeping entropy flow is given, and the excess mean heats, which are defined in terms of the measurable averages of the cumulative heats [19], are explicitly calculated, for the static response protocol, from the average heat amounts received by the system from each bath during a finite time t , and which are determined in subsection 4.3.

The joint probability distribution for received cumulative heats Q_1 and Q_2 is determined at any finite time and for any initial distribution probability through a generating function method (see section 4). Other distribution probabilities are then derived from its expression (4.40), and the explicit results are summarized in subsection 4.2. The results are given in terms of two integrals in the complex plane. The system obeys the finite-time symmetry (4.68) enforced by the modified detailed balance for the ratio of the probabilities to measure some given heat amounts Q_1 and Q_2 or their opposite values when the system is initially prepared in an equilibrium state. But it also satisfies another finite-time symmetry specific to the model for the ratio of the same probabilities when the system has any initial distribution probability. The latter fluctuation relation (4.72) is more subtle as it involves the initial probability distribution for the product of the spins (or equivalently for the energy of the spin pair).

The cumulants for the cumulative heat Q_2 are studied in section 5 from the characteristic function of the probability density for Q_2 . The relation between the characteristic function of a probability density $\Pi(Q; t)$ with the generating function for the probability function $P(Q; t)$ when the variable Q can take only discrete values is recalled in subsubsection 5.1.1. The explicit formulæ for the first four cumulants per unit time in the infinite-time limit are given in (5.15). Even at equilibrium the cumulants are not those of a Gaussian.

The large deviation function for the cumulative heat current Q_2/t is calculated by three different methods (section 6): from the Gärtner-Ellis theorem (subsection 6.1), from a saddle-point method (subsection 6.2) and from Laplace's method on a discrete sum (subsection 6.3). The second and third methods rely explicitly on the discrete nature of heat exchanges in the model and on an ad hoc definition of large deviation functions discussed in paper I, but they allow to compute subdominant contributions as well. The first method is straightforward, one just has to

check that the general applicability hypotheses (recalled in detail below) are fulfilled and this is easy in our case. The third method is also simple because it deals with a sum of nonnegative terms, so no compensation is possible. The saddle point method however is remarkably tricky in our case, for reasons that we shall detail below. The expressions in terms of various parameter sets are given in (6.10) and (6.16). In order to readily obtain the large deviation function in the case where the temperature of the colder bath vanishes, its expressions for positive and negative currents are explicitly distinguished in (6.13)-(6.14).

The limit where the kinetic parameter of one thermostat becomes infinitely large with respect to the kinetic parameter of the other thermostat is studied in section 7. In this limit the stationary distribution of the spins is the equilibrium canonical probability at the temperature of the “fast” heat bath while the typical inverse time scale in the mean instantaneous heat current is the kinetic parameter ν_{slow} of the “slow” heat bath. The probability distribution for the heat amount received from the slow thermostat, Q_{slow} , at any finite time t is that of an asymmetric random walk with the inverse time scale ν_{slow} . As a consequence the probability distribution for Q_{slow} obeys a fluctuation relation at any finite time (see (7.20)). The probability distributions of $\sigma_1\sigma_2$ and Q_{slow} are independent from each other, and this mean-field property is interpreted as a kinetic effect in the considered limit. The very simple forms of the infinite-time cumulants per unit time are given. The long-time distribution of the cumulative heat current is exhibited : it vanishes exponentially fast over a time-scale given by the inverse of the large-deviation function (7.32) with an amplitude which is explicitly calculated.

In the limit where the temperature of the colder thermostat vanishes (section 8) the microreversibility is broken, but the system still reaches a stationary state where all configurations have a non-vanishing weight, because the Markov matrix is still irreducible. The large deviation function is expressed in (8.9). In the limit where the kinetic parameter of one thermostat becomes infinitely large with respect to the kinetic parameter of the other thermostat, the probability distribution for the heat amount $\pm Q_{\text{slow}}$, with sign $- (+)$ if the slow thermostat is the cold (hot) one, becomes a Poisson process at any finite time t , because the zero-temperature thermostat can only absorb energy (strict dissipation towards the zero-temperature bath). Again the very simple forms of the infinite-time cumulants per unit time are given, as well as the large deviation function (8.21).

The last section is devoted to a probabilistic study of the system seen as a mesoscopic engineless thermal wheel, with an average heat flow from the hot reservoir to the cold reservoir, but also fluctuations around the average which we try to quantify. We compute the probability for the thermal machine to work backwards, and the law of the fluctuations of the time it takes to the machine to do one cycle. We present the argument for a system slightly more general than the two-spin system, because the computations and their meaning are more transparent this way, and then apply the formulæ to the two-spin system.

2 Model

The physical system we deal with in this article is a toy model of thermal contact, consisting of two heat baths, generically at different temperatures, put indirectly in contact via a small subsystem made of two interacting Ising spins σ_1 and σ_2 . Each spin σ_a , $a = 1, 2$ is in contact with a single bath denoted by a . We aim at a statistical description, where the details of what happens in the heat baths is not observed, but only the evolution of the two spins, i.e. of the configuration $\mathcal{C} \equiv (\sigma_1, \sigma_2)$. We assume that this evolution is described by a Markov process (in continuous time) with transition rate $(\mathcal{C}'|\mathbb{W}|\mathcal{C})$ from configuration $\mathcal{C}$ to configuration $\mathcal{C}'$.

As the system is out of equilibrium, the form of $\mathbb{W}$ is not a direct consequence of known physical laws, and it is unclear whether a nature-given preferred choice exists. So we start with a purely technical and down to earth description of our choice for the transition rates, that we shall use for all later explicit computations. The general principles and steps that guided us to the modified detailed balance that the transition rates must obey have been given in paper I. The main ideas are the following.

2.1 Constraints upon transition rates arising from microscopic discrete ergodic energy-conserving dynamics

As usual, we view a heat bath as an ideal limit of some large but finite system. So the system we describe is obtained via a limiting procedure from a large system made of two large parts and a small one, which is reduced to the two Ising spins σ_1 and σ_2 , each one directly in contact with one of the large parts.

We expect that in this limit many details become irrelevant, so we assume for the sake of the argument that the degrees of freedom in the large parts are discrete.

As in classical statistical mechanics, we take the viewpoint that the statistical description of σ_1, σ_2 is an effective mesoscopic description arising from a deterministic, energy-conserving dynamics for the whole system. With discrete variables, there is no general definition of time reversal invariance, but we impose that the dynamics is ergodic.

We also want the dynamics to reflect the fact that the two large parts interact only indirectly: there is an interaction energy $\mathcal{E}(\sigma_1, \sigma_2)$ between the two spins and the spin σ_a is flipped thanks to energy exchanges with the large part a ($a = 1, 2$). Defining the operator $\mathbb{F}_a$ as the operator flipping the spin σ_a while leaving the other spin unchanged (e.g. $\mathbb{F}_1(\sigma_1, \sigma_2) = (-\sigma_1, \sigma_2)$), in this process the energy of the large part a is changed from E_a to E'_a according to the energy conservation law

$$E'_a - E_a = \begin{cases} -[\mathcal{E}(\mathcal{C}') - \mathcal{E}(\mathcal{C})] & \text{if } \mathcal{C}' = \mathbb{F}_a \mathcal{C} \\ 0 & \text{otherwise,} \end{cases} \quad (2.1)$$

while the energy of the other large part is unchanged.

As shown in paper I, when the large parts are described at a statistical level and in a transient regime where the large parts are described in the thermodynamic limit, the transition rate $(\mathcal{C}'|\mathbb{W}|\mathcal{C})$ from configuration $\mathcal{C}$ to configuration $\mathcal{C}'$ obeys three constraints: first the graph associated with the transition rates is connected; second there is microscopic reversibility for any couple of configurations $(\mathcal{C}, \mathcal{C}')$,

$$(\mathcal{C}'|\mathbb{W}|\mathcal{C}) \neq 0 \quad \Leftrightarrow \quad (\mathcal{C}|\mathbb{W}|\mathcal{C}') \neq 0; \quad (2.2)$$

third the ratio of transition rates obeys the so-called modified detailed balance (MDB),

$$\text{for } \mathcal{C}' = \mathbb{F}_a \mathcal{C} \quad \frac{(\mathcal{C}'|\mathbb{W}|\mathcal{C})}{(\mathcal{C}|\mathbb{W}|\mathcal{C}')} = e^{-\beta_a [\mathcal{E}(\mathcal{C}') - \mathcal{E}(\mathcal{C})]}. \quad (2.3)$$

We remind the reader that the latter relation is also referred to in the literature as the “generalized detailed balance”.

2.2 Determination of transition rates

The transition rates are non-zero only if the initial and final configurations $\mathcal{C}$ and $\mathcal{C}'$ differ only by the flip of one spin : $(\mathcal{C}'|\mathbb{W}|\mathcal{C}) = 0$ unless either $\mathcal{C}' = \mathbb{F}_1 \mathcal{C}$ or $\mathcal{C}' = \mathbb{F}_2 \mathcal{C}$. Since σ_a can take only the two values $+1$ and -1 , the transition rate where σ_a is flipped takes the generic form

$$(\mathbb{F}_a \mathcal{C}|\mathbb{W}|\mathcal{C}) = \frac{\nu_a(\sigma_b)}{2} [1 - \sigma_a \Gamma_a(\sigma_b)]. \quad (2.4)$$

The four parameters ν_a, Γ_a , $a = 1, 2$, are a priori arbitrary except that the ν 's are > 0 and the Γ 's are of absolute value ≤ 1 .

Taking for simplicity an interaction energy between the spins

$$\mathcal{E}(\sigma_1, \sigma_2) = \frac{1 - \sigma_1 \sigma_2}{2} \Delta e, \quad (2.5)$$

where $\Delta e > 0$ is the energy gap between the two energy levels, one gets from the modified detailed balance in the form (2.3) that

$$\text{for } \mathcal{C}' = \mathbb{F}_a \mathcal{C} \quad \frac{(\mathcal{C}'|\mathbb{W}|\mathcal{C})}{(\mathcal{C}|\mathbb{W}|\mathcal{C}')} = e^{-\sigma_1 \sigma_2 \beta_a \Delta e}, \quad (2.6)$$

a condition similar to the one obtained by Glauber [18] in the equilibrium case. As $e^{2x} = \frac{1+\tanh x}{1-\tanh x}$ the generic form (2.4) of $(\mathbb{F}_a \mathcal{C} | \mathbb{W} | \mathcal{C})$ has to satisfy

$$(\mathbb{F}_a \mathcal{C} | \mathbb{W} | \mathcal{C}) = \frac{\nu_a(\sigma_b)}{2} [1 - \sigma_1 \sigma_2 \gamma_a] \quad (2.7)$$

with

$$\gamma_a \equiv \tanh \left(\beta_a \frac{\Delta e}{2} \right). \quad (2.8)$$

If β_1 and β_2 are finite, $0 \leq \gamma_1 < 1$ and $0 \leq \gamma_2 < 1$, and the microscopic reversibility condition (2.2) is also satisfied. Without loss of generality we could, and will sometimes, assume that $T_1 \leq T_2$. Then $\gamma_1 \geq \gamma_2$.

For the sake of simplicity, in the following we assume that ν_a depends only on the properties of the thermostat and not on the value of σ_b . (This choice enforces the equality between the transition rate from (σ_1, σ_2) and that from $(-\sigma_1, -\sigma_2)$, which are two configurations with the same energy.) Apart from simplicity, we have no convincing argument that this should be THE nature-given preferred choice. Anyway, we write

$$(\mathbb{F}_a \mathcal{C} | \mathbb{W} | \mathcal{C}) = \frac{\nu_a}{2} [1 - \sigma_1 \sigma_2 \gamma_a]. \quad (2.9)$$

This ends the argument explaining our choice of transition rates and gives a physical interpretation of the parameters: γ_a is formed with the energy scale in the two-spin system and the temperature of bath a , while ν_a describes a rate at which bath a attempts to flip spin σ_a .

We notice that, though the transition rate expressions have been derived from hypotheses implying the microscopic reversibility (2.2), these expressions still make sense if $\beta_2 < \beta_1 = +\infty$. (The limit $\beta_1 \rightarrow +\infty$ where microscopic reversibility is broken is discussed in section 8.)

Moreover, even if $\beta_1 = +\infty$, the Markov matrix $\mathbb{M}$ defined by

$$(\mathcal{C}' | \mathbb{M} | \mathcal{C}) = \begin{cases} (\mathcal{C}' | \mathbb{W} | \mathcal{C}) & \text{if } \mathcal{C}' \neq \mathcal{C} \\ -\sum_{\mathcal{C}''} (\mathcal{C}'' | \mathbb{W} | \mathcal{C}) & \text{if } \mathcal{C}' = \mathcal{C} \end{cases} \quad (2.10)$$

is irreducible, namely any configuration $\mathcal{C}'$ can be reached by a succession of jumps with non-zero transition rates from any configuration $\mathcal{C}$.

3 Non Equilibrium Stationary State (NESS) as a canonical distribution with an effective temperature

3.1 Stationary state distribution

The master equation which rules the evolution of the probability $P(\mathcal{C}; t)$ can be written in terms of the Markov matrix $\mathbb{M}$ defined in (2.10) as

$$\frac{dP(\mathcal{C}; t)}{dt} = \sum_{\mathcal{C}'} (\mathcal{C} | \mathbb{M} | \mathcal{C}') P(\mathcal{C}'; t). \quad (3.1)$$

In the basis where the probability $P(\sigma_1, \sigma_2; t)$ is represented by the column vector

$$|P(t)\rangle = \begin{pmatrix} P(++; t) \\ P(--; t) \\ P(+--; t) \\ P(-+; t) \end{pmatrix} \quad (3.2)$$

the matrix $\mathbb{M}$ takes the form

$$\mathbb{M} = \frac{\nu_1 + \nu_2}{2} \begin{pmatrix} -1 + \gamma_\star & 0 & \bar{\nu}_2(1 + \gamma_2) & \bar{\nu}_1(1 + \gamma_1) \\ 0 & -1 + \gamma_\star & \bar{\nu}_1(1 + \gamma_1) & \bar{\nu}_2(1 + \gamma_2) \\ \bar{\nu}_2(1 - \gamma_2) & \bar{\nu}_1(1 - \gamma_1) & -1 - \gamma_\star & 0 \\ \bar{\nu}_1(1 - \gamma_1) & \bar{\nu}_2(1 - \gamma_2) & 0 & -1 - \gamma_\star \end{pmatrix}. \quad (3.3)$$

In the latter equation we have introduced the dimensionless inverse time scales

$$\bar{\nu}_a = \frac{\nu_a}{\nu_1 + \nu_2} \quad \text{for } a = \{1, 2\}, \quad (3.4)$$

and we have set

$$\gamma_\star = \bar{\nu}_1 \gamma_1 + \bar{\nu}_2 \gamma_2. \quad (3.5)$$

The Markov matrix $\mathbb{M}$ is irreducible (even if $\gamma_1 = 1$, namely $T_1 \rightarrow 0$): for any pair of configurations $\mathcal{C}$ and $\mathcal{C}'$, there exists a succession of spin flips, with non-zero transition rates, which allows to make the system evolve from $\mathcal{C}$ to $\mathcal{C}'$. Henceforth, according to the Perron-Frobenius theorem there exists a single stationary state distribution $P_{\text{st}}(\mathcal{C})$ and it is nonzero for every configuration $\mathcal{C}$.

Moreover, since the system is made of two discrete variables which can take only the values ± 1 and since the transition rates are invariant under the simultaneous flips $\sigma_1 \rightarrow -\sigma_1$ and $\sigma_2 \rightarrow -\sigma_2$, the stationary distribution $P_{\text{st}}(\sigma_1, \sigma_2)$ takes the form $P_{\text{st}}(\sigma_1, \sigma_2) = a + d\sigma_1\sigma_2$. Indeed, the generic form of $P(\sigma_1, \sigma_2)$ reads $P(\sigma_1, \sigma_2) = a + b\sigma_1 + c\sigma_2 + d\sigma_1\sigma_2$. On the other hand, the invariance of the transition rates under the simultaneous flips $\sigma_1 \rightarrow -\sigma_1$ and $\sigma_2 \rightarrow -\sigma_2$ entails that if $a + b\sigma_1 + c\sigma_2 + d\sigma_1\sigma_2$ is a stationary solution, $a - b\sigma_1 - c\sigma_2 + d\sigma_1\sigma_2$ is also a stationary solution. But, since $\mathbb{M}$ is irreducible, there is only one stationary solution, so that $b = c = 0$. By solving explicitly the master equation (3.1) and using the normalization of a probability distribution, the stationary solution proves to be

$$P_{\text{st}}(\sigma_1, \sigma_2) = \frac{1}{4} [1 + \gamma_\star \sigma_1 \sigma_2] \quad (3.6)$$

where $\gamma_\star$ is defined in (3.5).

The stationary distribution of the model has the following remarkable property: it coincides with some equilibrium distribution. More precisely, the stationary state distribution is equal to the canonical state distribution at the effective inverse temperature $\beta_\star$

$$P_{\text{st}}(\sigma_1, \sigma_2) = P_{\text{can}}^{\beta_\star}(\sigma_1, \sigma_2), \quad (3.7)$$

where $\beta_\star$ is determined by the relation

$$\gamma_\star = \tanh\left(\beta_\star \frac{\Delta e}{2}\right), \quad (3.8)$$

and

$$P_{\text{can}}^\beta(\mathcal{C}) = \frac{e^{-\beta \mathcal{E}(\mathcal{C})}}{Z(\beta)}, \quad (3.9)$$

where $Z(\beta)$ is the canonical partition function at the inverse temperature β , $Z(\beta) = \sum_{\mathcal{C}} e^{-\beta \mathcal{E}(\mathcal{C})}$. We notice that the canonical form for the state distribution implies that $\beta_\star$ obeys the canonical ensemble relation which is equivalent to the definition of the inverse temperature in the micro-canonical ensemble, namely

$$\beta_\star = \frac{\partial S^{SG}[P_{\text{st}}]}{\partial \langle \mathcal{E} \rangle_{\text{st}}}, \quad (3.10)$$

where $\langle \mathcal{E} \rangle_{\text{st}} \equiv \sum_{\mathcal{C}} \mathcal{E}(\mathcal{C}) P_{\text{st}}(\mathcal{C})$ is the stationary mean value of the energy and $S^{SG}[P_{\text{st}}]$ is the value of the dimensionless Shannon-Gibbs entropy in the stationary state. The dimensionless Shannon-Gibbs entropy (where the Boltzmann constant is set equal to 1) is defined from the configuration probability distribution $P(\mathcal{C}; t)$ as

$$S^{SG}[P(t)] \equiv - \sum_{\mathcal{C}} P(\mathcal{C}; t) \ln P(\mathcal{C}; t). \quad (3.11)$$

Its evolution has been recalled in paper I.

3.2 Linear static response to a variation of some external parameter

In the present section we consider the static linear response of some observable $\mathcal{O}$ to a change of some external parameter, namely the inverse temperature β_a or the typical inverse time scale ν_a of bath a , with $a = 1, 2$.

In the protocols for the study of static linear response, the system is prepared in some stationary state at time $t_0 = 0^-$ and the external parameters are instantaneously changed by infinitesimal amounts at time $t = 0$. Then, in the infinite time limit, the system reaches another stationary state corresponding to the new values of the external parameters.

3.2.1 Relation with static correlations for a “canonical” NESS

Since the nonequilibrium stationary distribution given by (3.6) involves only one parameter, namely $\beta_\star$, the linear response coefficient $\partial\langle\mathcal{O}\rangle_{\text{st}}/\partial g_{\text{ext}}$ for the mean value of an observable $\mathcal{O}$ in the stationary distribution when some external parameter g_{ext} is varied is proportional to $\partial\langle\mathcal{O}\rangle_{\text{st}}/\partial\beta_\star$, namely $\partial\langle\mathcal{O}\rangle_{\text{st}}/\partial g_{\text{ext}} = (\partial\beta_\star/\partial g_{\text{ext}}) \times (\partial\langle\mathcal{O}\rangle_{\text{st}}/\partial\beta_\star)$. Moreover, by virtue of (3.7), the stationary distribution is the canonical distribution at the inverse temperature $\beta_\star$. Henceforth the coefficient $\partial\langle\mathcal{O}\rangle_{\text{st}}/\partial\beta_\star$ is merely opposite to the correlation between $\mathcal{O}$ and the energy $\mathcal{E}$ according to the canonical equilibrium identity

$$\frac{\partial\langle\mathcal{O}\rangle_{\text{can}}^{\beta_\star}}{\partial\beta_\star} = - [\langle\mathcal{O}\mathcal{E}\rangle_{\text{can}}^{\beta_\star} - \langle\mathcal{O}\rangle_{\text{can}}^{\beta_\star} \langle\mathcal{E}\rangle_{\text{can}}^{\beta_\star}], \quad (3.12)$$

where $\langle\mathcal{O}\rangle_{\text{can}}^{\beta_\star}$ denotes an average with respect to the canonical distribution $P_{\text{can}}^{\beta_\star}$. As a result, the relation valid for responses to the variation of any external parameter in the nonequilibrium stationary state reads

$$\frac{\partial\langle\mathcal{O}\rangle_{\text{st}}}{\partial g_{\text{ext}}} = - \frac{\partial\beta_\star}{\partial g_{\text{ext}}} [\langle\mathcal{O}\mathcal{E}\rangle_{\text{st}} - \langle\mathcal{O}\rangle_{\text{st}} \langle\mathcal{E}\rangle_{\text{st}}]. \quad (3.13)$$

3.2.2 Dependence of the mean energy upon the time scales of the microscopic dynamics

The main difference between the response of the mean energy in non-equilibrium and equilibrium states arises for the response to a variation of the time scales of the microscopic dynamics which rules the heat exchanges with the baths. When $\beta_1 = \beta_2$ the equilibrium mean energy $\langle\mathcal{E}\rangle_{\text{eq}} = \langle\mathcal{E}\rangle_{\text{can}}^{\beta_1}$ depends only on the thermodynamic temperature common to both baths. On the contrary, in the non-equilibrium case the stationary mean energy $\langle\mathcal{E}\rangle_{\text{st}}$ does also depend on both inverse time scales ν_1 and ν_2 . Indeed, since the stationary probability corresponds to the effective canonical distribution (3.7), the stationary mean energy reads

$$\langle\mathcal{E}\rangle_{\text{st}} = (1 - \gamma_\star) \frac{\Delta e}{2} = (1 - \bar{\nu}_1 \gamma_1 - \bar{\nu}_2 \gamma_2) \frac{\Delta e}{2}. \quad (3.14)$$

Changing ν_a means changing the physical connection between thermal bath a and the spin system. The linear response of the stationary energy associated with a variation of the inverse time scale ν_a is determined by the coefficient

$$\frac{\partial\langle\mathcal{E}\rangle_{\text{st}}}{\partial\nu_a} = - \frac{\nu_b}{(\nu_1 + \nu_2)^2} \gamma_a \frac{\Delta e}{2} \quad \text{for } \{a, b\} = \{1, 2\}. \quad (3.15)$$

3.2.3 Stationary mean energy and generalized heat capacities

The heat capacity C_{eq} is a measurable quantity defined as the ratio

$$C_{\text{eq}}(T) = \frac{\langle\delta\mathcal{Q}\rangle}{dT}, \quad (3.16)$$

where $\langle \delta Q \rangle$ is the mean heat amount received by the system in transformations which involve only heat transfers and make the system go from an equilibrium state at temperature T to another equilibrium state at temperature $T + dT$, while all other thermodynamic parameters which determine the equilibrium state are kept constant. ($\langle \delta Q \rangle = \lim_{t \rightarrow +\infty} \langle Q \rangle_t$ in the protocol mentioned in the introduction of the section.) According to the energy conservation, $\langle \delta Q \rangle = \langle \mathcal{E} \rangle_{\text{eq}}^{T+dT} - \langle \mathcal{E} \rangle_{\text{eq}}^T$ and the heat capacity is related to a partial derivative of the equilibrium mean energy,

$$C_{\text{eq}}(T) = \frac{\partial \langle \mathcal{E} \rangle_{\text{eq}}}{\partial T} = -\beta^2 \frac{\partial \langle \mathcal{E} \rangle_{\text{eq}}}{\partial \beta}. \quad (3.17)$$

When the system is in a stationary non-equilibrium state induced by thermal contact with two heat reservoirs at respective temperatures T_1 and T_2 , we can introduce measurable heat capacities by similar definitions. When the temperature T_1 of thermal bath 1 is changed by dT_1 , while the temperature T_2 of thermal bath 2 is kept fixed, and when the system evolves from a stationary state to another one only by heat transfers, then the generalized heat capacity $C_{\text{st}}^{[1]}$ is defined as

$$C_{\text{st}}^{[1]}(T_1, T_2) = \frac{\langle \delta (\mathcal{Q}_1 + \mathcal{Q}_2) \rangle}{dT_1}. \quad (3.18)$$

According to conservation energy, $\delta \langle \mathcal{Q}_1 + \mathcal{Q}_2 \rangle = \langle \mathcal{E} \rangle_{\text{st}}^{T_1+dT_1, T_2} - \langle \mathcal{E} \rangle_{\text{st}}^{T_1, T_2}$ and the heat capacity is related to a partial derivative of the stationary mean energy

$$C_{\text{st}}^{[1]}(T_1, T_2) = \left. \frac{\partial \langle \mathcal{E} \rangle_{\text{st}}^{T_1, T_2}}{\partial T_1} \right|_{T_2}. \quad (3.19)$$

In the present model the expression (3.14) of the stationary mean energy takes the very specific form

$$\langle \mathcal{E} \rangle_{\text{st}} = \bar{\nu}_1 \langle \mathcal{E} \rangle_{\text{eq}}^{T_1} + \bar{\nu}_2 \langle \mathcal{E} \rangle_{\text{eq}}^{T_2}. \quad (3.20)$$

Indeed the relation $\bar{\nu}_1 + \bar{\nu}_2 = 1$ and the expression of the equilibrium mean energy at the inverse temperature β ,

$$\langle \mathcal{E} \rangle_{\text{eq}}^T = (1 - \gamma) \frac{\Delta e}{2} \quad \text{when} \quad \beta_1 = \beta_2 = \beta, \quad (3.21)$$

allow to rewrite the mean energy expression (3.14) in the non-equilibrium stationary state in the form (3.20). By virtue of the specific decomposition (3.20) of the mean energy, the heat capacities $C_{\text{st}}^{[a]}(T_1, T_2)$'s read

$$C_{\text{st}}^{[a]}(T_1, T_2) = \bar{\nu}_a C_{\text{eq}}(T_a) \quad \text{with} \quad a = \{1, 2\}, \quad (3.22)$$

where, according to the relation (3.17) and the expression (3.21) of $\langle \mathcal{E} \rangle_{\text{eq}}^T$,

$$C_{\text{eq}}(T_a) = \left[1 - \tanh^2 \left(\frac{\beta_a \Delta e}{2} \right) \right] \left(\frac{\beta_a \Delta e}{2} \right)^2. \quad (3.23)$$

More generally, when the temperatures T_1 and T_2 of both thermostats are varied independently

$$\langle \mathcal{E} \rangle_{\text{st}}^{T_1+dT_1, T_2+dT_2} - \langle \mathcal{E} \rangle_{\text{st}}^{T_1, T_2} = C_{\text{st}}^{[1]}(T_1, T_2) dT_1 + C_{\text{st}}^{[2]}(T_1, T_2) dT_2. \quad (3.24)$$

If T_1 and T_2 are increased by the same infinitesimal quantity dT the corresponding heat capacity, defined as $C_{\text{st}}(T_1, T_2) \equiv \delta \langle \mathcal{Q}_1 + \mathcal{Q}_2 \rangle / dT$ is equal to the sum $C_{\text{st}}^{[1]}(T_1, T_2) + C_{\text{st}}^{[2]}(T_1, T_2)$. For the present model $C_{\text{st}}(T_1, T_2) = \bar{\nu}_1 C_{\text{eq}}(T_1) + \bar{\nu}_2 C_{\text{eq}}(T_2)$. In the limit where $T_1 = T_2 = T$, by virtue of the relation $\bar{\nu}_1 + \bar{\nu}_2 = 1$, we retrieve the equilibrium heat capacity $C_{\text{eq}}(T)$, as it should be.

3.2.4 Stationary heat current and linear thermal conductivity

The instantaneous heat current $j_a(\mathcal{C})$ received from heat bath a when the system jumps out of the configuration $\mathcal{C}$ has been defined in paper I as

$$j_a(\mathcal{C}) \equiv j_{\delta q_a}(\mathcal{C}) \equiv \sum_{\mathcal{C}'} \langle \mathcal{C}' | \mathbb{W} | \mathcal{C} \rangle \delta q_a(\mathcal{C}' \leftarrow \mathcal{C}), \quad (3.25)$$

where $\delta q_a(\mathcal{C}' \leftarrow \mathcal{C})$ is the heat received from thermal bath a when the system evolves from configuration $\mathcal{C}$ to configuration $\mathcal{C}' = \mathbb{F}_a \mathcal{C}$, where $\mathbb{F}_a$ is the flip caused by thermal bath a , namely

$$\begin{cases} \delta q_a(\mathcal{C}' \leftarrow \mathcal{C}) = [\mathcal{E}(\mathcal{C}') - \mathcal{E}(\mathcal{C})] & \text{if } \mathcal{C}' = \mathbb{F}_a \mathcal{C} \\ \delta q_a(\mathcal{C}' \leftarrow \mathcal{C}) = 0 & \text{otherwise.} \end{cases} \quad (3.26)$$

In the present model $j_2(\sigma_1, \sigma_2) = [\mathcal{E}(\sigma_1, -\sigma_2) - \mathcal{E}(\sigma_1, \sigma_2)] (\sigma_1, -\sigma_2 | \mathbb{W} | \sigma_1, \sigma_2) = \nu_2 [\sigma_1 \sigma_2 - \gamma_2] (\Delta e / 2)$.

In the stationary state the mean energy is constant so that the mean currents received from both baths cancel, $\langle j_1 \rangle_{\text{st}} + \langle j_2 \rangle_{\text{st}} = 0$. For the stationary state probability distribution (3.6), one has $\langle \sigma_1 \sigma_2 \rangle_{\text{st}} = \gamma_\star = \bar{\nu}_1 \gamma_1 + \bar{\nu}_2 \gamma_2$ and

$$\langle j_2 \rangle_{\text{st}} = \bar{\nu}_1 \bar{\nu}_2 (\gamma_1 - \gamma_2) \frac{(\nu_1 + \nu_2) \Delta e}{2}, \quad (3.27)$$

where $\gamma_1 - \gamma_2$ may be rewritten as

$$\gamma_1 - \gamma_2 = \tanh \left(\frac{(\beta_1 - \beta_2) \Delta e}{2} \right) \left[1 - \tanh \left(\frac{\beta_1 \Delta e}{2} \right) \tanh \left(\frac{\beta_2 \Delta e}{2} \right) \right]. \quad (3.28)$$

When $T_1 \leq T_2$, $\langle j_2 \rangle_{\text{st}} \geq 0$, as it should : the mean heat current flows from the hot bath to the cold bath. Note that $\langle j_2 \rangle_{\text{st}}$ is a bounded function of T_1 and T_2 . Thus, in the generic case $\langle j_2 \rangle_{\text{st}}$ is not proportional to the bath temperatures difference $T_2 - T_1$. As for any system, the linear dependence upon $T_2 - T_1$ (or $\beta_1 - \beta_2$) appears in the limit where $(\beta_1 - \beta_2) \Delta e \ll 1$. In the high temperature regime where both $\beta_1 \Delta e \ll 1$ and $\beta_2 \Delta e \ll 1$, the condition $(\beta_1 - \beta_2) \Delta e \ll 1$ is satisfied and $\langle j_2 \rangle_{\text{st}}$ is proportional to $T_2 - T_1$.

When $\beta_1 = \beta_2$ the system is at equilibrium and $\langle j_2 \rangle_{\text{eq}} = 0$. Moreover, as shown in paper I, the partial derivatives of the current obey the generic symmetry

$$\left. \frac{\partial \langle j_2 \rangle_{\text{st}}}{\partial \beta_1} \right|_{\beta_2} (\beta_1 = \beta, \beta_2 = \beta) = - \left. \frac{\partial \langle j_2 \rangle_{\text{st}}}{\partial \beta_2} \right|_{\beta_1} (\beta_1 = \beta, \beta_2 = \beta). \quad (3.29)$$

This property can also be checked from the expression (3.27) of $\langle j_2 \rangle_{\text{st}}$. It entails that

$$\langle j_2 \rangle_{\text{st}} \underset{(T_1, T_2) \rightarrow (T, T)}{\sim} (T_2 - T_1) \left. \frac{\partial \langle j_2 \rangle_{\text{st}}}{\partial T_2} \right|_{T_1} (T, T). \quad (3.30)$$

In other words, when T_1 and T_2 independently tend to the same value T , at first order in the independent variables $T_1 - T$ and $T_2 - T$ the ratio $\langle j_2 \rangle_{\text{st}} / (T_2 - T_1)$ depends on T but is independent of the ways $T_1 - T$ and $T_2 - T$ vanish.

As a consequence, for a non-equilibrium stationary state near equilibrium, namely when the temperature difference between the thermostats is such that $(\beta_1 - \beta_2) \Delta e \ll 1$, one can define the thermal conductivity as

$$\kappa_{\text{th}} \equiv \lim_{(T_1, T_2) \rightarrow (T, T)} \frac{\langle j_2 \rangle_{\text{st}}}{T_2 - T_1} = \left. \frac{\partial \langle j_2 \rangle_{\text{st}}}{\partial T_2} \right|_{T_1} (T, T). \quad (3.31)$$

From (3.27) we get the expression for the thermal conductivity,

$$\kappa_{\text{th}} = \frac{\nu_1 \nu_2}{\nu_1 + \nu_2} \left[1 - \tanh^2 \left(\frac{\beta \Delta e}{2} \right) \right] \left(\frac{\beta \Delta e}{2} \right)^2. \quad (3.32)$$

We remind the reader that the thermal conductivity, which is a positive transport coefficient, is related to the kinetic coefficient (also called Onsager coefficient) introduced in phenomenological irreversible thermodynamics as

$$L \equiv \lim_{\mathcal{F} \rightarrow 0} \frac{\langle j_2 \rangle_{\text{st}}}{\mathcal{F}}, \quad (3.33)$$

where, as recalled in paper I, the thermodynamic force $\mathcal{F}$ can be defined from the stationary entropy production rate, which is opposite to the exchange entropy flow, $d_{\text{int}} S^{SG}/dt|_{\text{st}} = -d_{\text{exch}} S/dt|_{\text{st}}$, through the relation

$$\left. \frac{d_{\text{int}} S^{SG}}{dt} \right|_{\text{st}} = - \left. \frac{d_{\text{exch}} S}{dt} \right|_{\text{st}} = \mathcal{F} \langle j_2 \rangle_{\text{st}} \quad (3.34)$$

when there is only one independent mean instantaneous current. In the case of the thermal contact $\mathcal{F} = \beta_1 - \beta_2$. Therefore the relation between the kinetic coefficient and the thermal conductivity defined in (3.31) reads

$$L = \frac{\kappa_{\text{th}}}{\beta^2}. \quad (3.35)$$

Now we compare the results about the linear static response in non-equilibrium stationary states which are either in the vicinity of equilibrium or far away from equilibrium. When the system is far from equilibrium, namely when $(\beta_1 - \beta_2) \Delta e \gg 1$, (3.27) leads to

$$\left. \frac{\partial \langle j_2 \rangle_{\text{st}}}{\partial \beta_2} \right|_{\beta_1} = - \frac{\nu_1 \nu_2}{\nu_1 + \nu_2} \left[1 - \tanh \left(\frac{\beta_2 \Delta e}{2} \right)^2 \right] \left(\frac{\Delta e}{2} \right)^2 \quad (3.36)$$

$$\left. \frac{\partial \langle j_2 \rangle_{\text{st}}}{\partial \beta_1} \right|_{\beta_2} = \frac{\nu_1 \nu_2}{\nu_1 + \nu_2} \left[1 - \tanh \left(\frac{\beta_1 \Delta e}{2} \right)^2 \right] \left(\frac{\Delta e}{2} \right)^2. \quad (3.37)$$

The linear response coefficients $\partial \langle j_2 \rangle_{\text{st}} / \partial T_2|_{T_1}$ and $\partial \langle j_2 \rangle_{\text{st}} / \partial T_1|_{T_2}$ are no more opposite to each other. As a consequence, when T_1 and T_2 are varied independently, the corresponding variation of the stationary mean instantaneous current $\langle j_2 \rangle_{\text{st}}^{[\beta_1, \beta_2]}$ at first order reads

$$\langle j_2 \rangle_{\text{st}}^{[\beta'_1, \beta'_2]} - \langle j_2 \rangle_{\text{st}}^{[\beta_1, \beta_2]} \underset{(T'_1, T'_2) \rightarrow (T_1, T_2)}{\sim} (T'_1 - T_1) \left. \frac{\partial \langle j_2 \rangle_{\text{st}}^{[\beta_1, \beta_2]}}{\partial T_1} \right|_{T_2} + (T'_2 - T_2) \left. \frac{\partial \langle j_2 \rangle_{\text{st}}^{[\beta_1, \beta_2]}}{\partial T_2} \right|_{T_1}. \quad (3.38)$$

The latter variation depends not only on T_1 , T_2 and the variation of the temperature difference $(T'_1 - T'_2) - (T_1 - T_2)$ but also on the way in which T'_1 and T'_2 are varied around the given values T_1 and T_2 .

3.3 Non-linear static response in the NESS

3.3.1 Non-linear thermal conductivity

When the system is far from equilibrium, instead of introducing the linear response $\langle j_2 \rangle_{\text{st}}^{[\beta'_1, \beta'_2]} - \langle j_2 \rangle_{\text{st}}^{[\beta_1, \beta_2]}$ with $(\beta'_1 - \beta_1) \Delta e \ll 1$ and $(\beta'_2 - \beta_2) \Delta e \ll 1$ (and the associated linear response coefficients $\partial \langle j_2 \rangle_{\text{st}}^{[\beta_1, \beta_2]} / \partial \beta_a$), one may rather consider a non-linear thermal conductivity defined as

$$\kappa_{\text{th}}^{\text{nl}} = \frac{\langle j_2 \rangle_{\text{st}}^{[\beta_1, \beta_2]}}{T_2 - T_1}. \quad (3.39)$$

From (3.27)

$$\kappa_{\text{th}}^{\text{nl}} = \frac{1}{T_1 T_2} \frac{\nu_1 \nu_2}{\nu_1 + \nu_2} \frac{\tanh \left(\frac{(\beta_1 - \beta_2) \Delta e}{2} \right)}{\frac{(\beta_1 - \beta_2) \Delta e}{2}} \left[1 - \tanh \left(\frac{\beta_1 \Delta e}{2} \right) \tanh \left(\frac{\beta_2 \Delta e}{2} \right) \right] \left(\frac{\Delta e}{2} \right)^2. \quad (3.40)$$

According to the expression (3.27), $\langle j_2 \rangle_{\text{st}}$ is a bounded function of T_1 and T_2 , so that $\kappa_{\text{th}}^{\text{nlm}}$ vanishes when $T_2 - T_1$ becomes very large with respect to either T_1 or T_2 .

We also notice that when both thermostats are at very high temperature, namely when $\beta_1 \Delta e \ll 1$ and $\beta_2 \Delta e \ll 1$, $\langle j_2 \rangle_{\text{st}}$ is proportional to $\beta_1 - \beta_2$ with a coefficient independent of the temperatures. As a consequence, the partial derivatives $\partial \langle j_2 \rangle_{\text{st}}^{[\beta_1, \beta_2]} / \partial \beta_1$ and $\partial \langle j_2 \rangle_{\text{st}}^{[\beta_1, \beta_2]} / \partial \beta_2$ are opposite to each other, as in the symmetry property (3.29) in the very vicinity of the equilibrium limit. Then the difference (3.38) is proportional to the difference $(\beta'_1 - \beta_1) - (\beta'_2 - \beta_2)$,

$$\langle j_2 \rangle_{\text{st}}^{[\beta'_1, \beta'_2]} - \langle j_2 \rangle_{\text{st}}^{[\beta_1, \beta_2]} \underset{\substack{(\beta'_1, \beta'_2) \rightarrow (\beta_1, \beta_2) \\ \beta_1 \Delta e \rightarrow 0, \beta_2 \Delta e \rightarrow 0}}{\sim} [(\beta'_1 - \beta_1) - (\beta'_2 - \beta_2)] \left. \frac{\partial \langle j_2 \rangle_{\text{st}}}{\partial \beta_1} \right|_{\beta_2}. \quad (3.41)$$

Besides, the thermal conductivity (3.40) behaves as

$$\kappa_{\text{th}}^{\text{nlm}} \underset{\substack{\beta_1 \Delta e \rightarrow 0 \\ \beta_2 \Delta e \rightarrow 0}}{\sim} \frac{1}{T_1 T_2} \frac{\nu_1 \nu_2}{\nu_1 + \nu_2} \left(\frac{\Delta e}{2} \right)^2. \quad (3.42)$$

3.3.2 Housekeeping entropy flow and mean excess heats

In the long-time limit, whatever the initial configuration probability P_0 may be, the system reaches a stationary state where the Markovian stochastic dynamics enforces that the cumulated heats received from each thermostat, namely the random variables $\mathcal{Q}_1(t)$ and $\mathcal{Q}_2(t)$, have averages $\langle \mathcal{Q}_1(t) \rangle_{P_0}$ and $\langle \mathcal{Q}_2(t) \rangle_{P_0}$ which both grow linearly in time with opposite coefficients, $-\langle \mathcal{Q}_1(t) \rangle_{P_0} \underset{t \rightarrow +\infty}{\sim} \langle \mathcal{Q}_2(t) \rangle_{P_0} \underset{t \rightarrow +\infty}{\sim} t \langle j_2 \rangle_{\text{st}}^{[\beta_1, \beta_2]}$. Then

$$\lim_{t \rightarrow +\infty} \frac{\beta_1 \langle \mathcal{Q}_1(t) \rangle_{P_0} + \beta_2 \langle \mathcal{Q}_2(t) \rangle_{P_0}}{t} = -(\beta_1 - \beta_2) \langle j_2 \rangle_{\text{st}}^{[\beta_1, \beta_2]} = \left. \frac{d_{\text{exch}} S}{dt} \right|_{\text{st}}, \quad (3.43)$$

where the stationary exchange entropy flow appears by virtue of (3.34). Meanwhile the sum $\mathcal{Q}_1(t) + \mathcal{Q}_2(t)$ remains bounded at any time and its average tends to the heat amount corresponding to the mean energy difference between the final and initial stationary states,

$$\lim_{t \rightarrow +\infty} \langle \mathcal{Q}_1(t) + \mathcal{Q}_2(t) \rangle_{P_0} = \langle \mathcal{E} \rangle_{\text{st}}^{[\beta_1, \beta_2]} - \langle \mathcal{E} \rangle_{P_0}. \quad (3.44)$$

In the phenomenological framework of steady state thermodynamics [20], when work is supplied to the system, the total heat *given* to the system is usually expressed as the sum of an “excess” heat $\mathcal{Q}_{\text{exc}}$ associated with the energy exchange during transitions between two different steady states and a “housekeeping” heat $\mathcal{Q}_{\text{hk}}$ associated with the energy supplied to maintain the system in the NESS reached in the long-time limit. These two heat amounts have been discussed for a system in contact with only one thermal bath and submitted to a time-dependent external force which is described by Langevin dynamics [21, 16, 22].

By analogy, with the standard sign convention, we may introduce a “housekeeping” entropy flow supplied to the system which can be measured as the asymptotic behavior

$$\sigma_{\text{hk}}[P_{\text{st}}] \equiv - \lim_{t \rightarrow +\infty} \frac{\beta_1 \langle \mathcal{Q}_1(t) \rangle_{P_0} + \beta_2 \langle \mathcal{Q}_2(t) \rangle_{P_0}}{t}, \quad (3.45)$$

and which, by virtue of (3.43) coincides with the opposite of the stationary exchange entropy flow, namely with the stationary entropy production rate (see (3.34))

$$\sigma_{\text{hk}}[P_{\text{st}}] = - \left. \frac{d_{\text{exch}} S}{dt} \right|_{\text{st}} = \left. \frac{d_{\text{int}} S}{dt} \right|_{\text{st}}. \quad (3.46)$$

From the explicit expression (3.27) of the mean instantaneous heat current we obtain the expression for the housekeeping entropy flow (3.45)

$$\sigma_{\text{hk}}[P_{\text{st}}] = \frac{\nu_1 \nu_2}{\nu_1 + \nu_2} (\gamma_1 - \gamma_2) (\beta_1 - \beta_2) \frac{\Delta e}{2}. \quad (3.47)$$

When the system is prepared in a stationary state by thermal contact with heat reservoirs at the inverse temperatures β_1^0 and β_2^0 respectively, then $\langle \mathcal{E} \rangle_{P_0} = \langle \mathcal{E} \rangle_{\text{st}}^{[\beta_1^0, \beta_2^0]}$ and the difference in (3.44) becomes equal to $\langle \mathcal{E} \rangle_{\text{st}}^{[\beta_1, \beta_2]} - \langle \mathcal{E} \rangle_{\text{st}}^{[\beta_1^0, \beta_2^0]}$. With the standard convention, the “excess” heats *given* to the system $Q_{\text{exc},a}|_{[\beta_1^0, \beta_2^0]}^{[\beta_1, \beta_2]}$ with $a = 1, 2$ can be measured as

$$Q_{\text{exc},a}|_{[\beta_1^0, \beta_2^0]}^{[\beta_1, \beta_2]} \equiv - \lim_{t \rightarrow +\infty} \left[\langle \mathcal{Q}_a(t) \rangle_{P_{\text{st}}^{[\beta_1^0, \beta_2^0]}} - t \langle j_a \rangle_{\text{st}}^{[\beta_1, \beta_2]} \right]. \quad (3.48)$$

Then, by virtue of the stationary condition $\langle j_1 \rangle_{\text{st}} = -\langle j_2 \rangle_{\text{st}}$, the equality (3.44) becomes

$$-Q_{\text{exc},1}|_{[\beta_1^0, \beta_2^0]}^{[\beta_1, \beta_2]} - Q_{\text{exc},2}|_{[\beta_1^0, \beta_2^0]}^{[\beta_1, \beta_2]} = \langle \mathcal{E} \rangle_{\text{st}}^{[\beta_1, \beta_2]} - \langle \mathcal{E} \rangle_{\text{st}}^{[\beta_1^0, \beta_2^0]}. \quad (3.49)$$

The excess heats $Q_{\text{exc},a}|_{[\beta_1^0, \beta_2^0]}^{[\beta_1, \beta_2]}$ ’s defined in (3.48) are explicitly calculated in subsection (4.3) from the expressions of the average heat amounts $\langle \mathcal{Q}_a(t) \rangle_{P_0}$ ’s at any finite time t (for any initial distribution P_0 of the two-spin configuration) with the results given in (4.52).

In the linear response regime where the relative differences $(T_1 - T_1^0)/T_1^0$ and $(T_2 - T_2^0)/T_2^0$ are infinitesimal, by virtue of the definition (3.19) of the generalized heat capacities $C_{\text{st}}^{[a]}(T_1, T_2)$, with $a = 1, 2$,

$$-Q_{\text{exc},1}|_{[T_1, T_2]}^{[T_1+dT_1, T_2+dT_2]} - Q_{\text{exc},2}|_{[T_1, T_2]}^{[T_1+dT_1, T_2+dT_2]} \rightarrow C_{\text{st}}^{[1]}(T_1, T_2)dT_1 + C_{\text{st}}^{[2]}(T_1, T_2)dT_2. \quad (3.50)$$

We notice that the notion of heat capacity has been studied in the case of non equilibrium steady states where the system is submitted to a non-conservative force and is in contact with a single thermostat [23].

4 Joint probability distribution for heat cumulated exchanges at finite time in the model

Instead of studying the evolution of the probability distribution $P(\mathcal{C}; t)$ of the spins configuration $\mathcal{C} = (\sigma_1, \sigma_2)$, we address directly the evolution of the joint probability distribution $P(\mathcal{C}' | \mathcal{Q}_1, \mathcal{Q}_2, t | \mathcal{C})$ for the cumulated heats $\mathcal{Q}_1$ and $\mathcal{Q}_2$ received from the thermal baths 1 and 2 during a time t when the system is in configuration $\mathcal{C} = (\sigma_1, \sigma_2)$ at time $t_0 = 0$ and in configuration $\mathcal{C}' = (\sigma'_1, \sigma'_2)$ at time t . In order to obtain results which hold as generally as possible, the initial probability distribution for configurations is not assumed to have the same symmetry under simultaneous spin flips as the stationary distribution.

Since the two-spin system has only two energy levels separated by the energy gap Δe , the cumulated heats $\mathcal{Q}_a$ are integer multiples of Δe and we set

$$\mathcal{Q}_1 = -n_1 \Delta e \quad \text{and} \quad \mathcal{Q}_2 = n_2 \Delta e. \quad (4.1)$$

The minus sign in the definition of $\mathcal{Q}_1$ is introduced for the sake of conveniency, because the mean instantaneous heat currents $\langle j_1 \rangle_{\text{st}}$ and $\langle j_2 \rangle_{\text{st}}$ in the stationary state are opposite to each other. In other words, $n_1 \Delta e$ is the amount of heat dissipated towards heat bath 1, while $n_2 \Delta e$ is the amount of heat received from heat bath 2. With these notations $P(\mathcal{C}_f | \mathcal{Q}_1, \mathcal{Q}_2, t | \mathcal{C}_0)$ can be written as a matrix element of some evolution operator $\mathbb{U}(n_1, n_2; t)$

$$P(\mathcal{C}' | \mathcal{Q}_1, \mathcal{Q}_2, t | \mathcal{C}) = (\sigma'_1, \sigma'_2 | \mathbb{U}(n_1, n_2; t) | \sigma_1, \sigma_2). \quad (4.2)$$

4.1 Explicit calculations

4.1.1 Constraint from energy conservation

According to the expression (2.5) for the interaction energy between the two spins, the energy difference between the final and the initial configurations reads

$$\mathcal{E}(\sigma'_1, \sigma'_2) - \mathcal{E}(\sigma_1, \sigma_2) = \frac{\sigma_1 \sigma_2 - \sigma'_1 \sigma'_2}{2} \Delta e, \quad (4.3)$$

and it can take only three values 0, $+\Delta e$ and $-\Delta e$. On the other hand, according to (4.1), $\mathcal{Q}_1 + \mathcal{Q}_2 = (n_2 - n_1)\Delta e$, namely

$$\mathcal{Q}_1 + \mathcal{Q}_2 = \Delta n \times \Delta e \quad \text{where} \quad \Delta n \equiv n_2 - n_1. \quad (4.4)$$

Energy conservation entails that the energy variation of the two-spin system is equal to the sum of the heat amounts received from the thermostats: $\mathcal{E}(\sigma'_1, \sigma'_2) - \mathcal{E}(\sigma_1, \sigma_2) = \mathcal{Q}_1 + \mathcal{Q}_2$. As a consequence the correspondence between the total amount of received heat and the couple of initial and final states reads

$$\begin{aligned} \Delta n = 0 & \Leftrightarrow \sigma'_1 \sigma'_2 = \sigma_1 \sigma_2 \\ (\Delta n)^2 = 1 & \Leftrightarrow \sigma_1 \sigma_2 = \Delta n \quad \text{and} \quad \sigma'_1 \sigma'_2 = -\Delta n. \end{aligned} \quad (4.5)$$

Therefore it is convenient to introduce the decomposition

$$\mathbb{U}(n_1, n_2; t) = \sum_{\Delta n=0, \pm 1} \delta_{n_2, n_1 + \Delta n} \mathbb{U}(n_1, n_1 + \Delta n; t). \quad (4.6)$$

In the basis $\{(+, +), (-, -), (+, -), (-, +)\}$ already used in (3.2) the correspondence (4.5) enforces that $\mathbb{U}(n_1, n_2; t)$ can be decomposed into three 4×4 matrices

$$\mathbb{U} = \mathbb{U}_{[\Delta n=0]} + \mathbb{U}_{[\Delta n=+1]} + \mathbb{U}_{[\Delta n=-1]} \quad (4.7)$$

with

$$\mathbb{U}_{[\Delta n=0]} = \begin{pmatrix} \mathbb{A} & \mathbb{O} \\ \mathbb{O} & \mathbb{D} \end{pmatrix} \quad \mathbb{U}_{[\Delta n=+1]} = \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{C} & \mathbb{O} \end{pmatrix} \quad \mathbb{U}_{[\Delta n=-1]} = \begin{pmatrix} \mathbb{O} & \mathbb{B} \\ \mathbb{O} & \mathbb{O} \end{pmatrix}. \quad (4.8)$$

The subscript involving Δn indicates the unique value of Δn which is involved in a history where the initial and final states are (σ_1, σ_2) and (σ'_1, σ'_2) respectively. $\mathbb{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$ and $\mathbb{D}$ are 2×2 matrices.

4.1.2 Generating function method

The evolution equation for $\mathbb{U}(n_1, n_2; t)$ is easily derived by considering the probability

$$P(\sigma_1, \sigma_2, n_1, n_2; t) = \sum_{\sigma'_1, \sigma'_2} (\sigma_1, \sigma_2 | \mathbb{U}(n_1, n_2; t) | \sigma'_1, \sigma'_2) P(\sigma'_1, \sigma'_2; t=0). \quad (4.9)$$

$P(\sigma_1, \sigma_2, n_1, n_2; t)$ is the probability that the system is in configuration (σ_1, σ_2) a time t and has received the heat amounts $\mathcal{Q}_1 = -n_1 \Delta e$ and $\mathcal{Q}_2 = n_2 \Delta e$ during the time interval $[0, t]$ when the initial probability distribution for the spins is $P(\sigma'_1, \sigma'_2; t=0)$. The evolution equation for $P(\sigma_1, \sigma_2, n_1, n_2; t)$ is a generalization of the master equation (3.1) which governs the evolution of $P(\sigma_1, \sigma_2; t)$. By taking into account the explicit expression (2.9) for the transition rates we get

$$\begin{aligned} \frac{2}{\nu_1 + \nu_2} \frac{dP(\sigma_1, \sigma_2, n_1, n_2; t)}{dt} = & -[1 - \sigma_1 \sigma_2 (\bar{\nu}_1 \gamma_1 + \bar{\nu}_2 \gamma_2)] P(\sigma_1, \sigma_2, n_1, n_2; t) \\ & + \bar{\nu}_1 [1 + \sigma_1 \sigma_2 \gamma_1] P(-\sigma_1, \sigma_2, n_1 - \sigma_1 \sigma_2, n_2; t) \\ & + \bar{\nu}_2 [1 + \sigma_1 \sigma_2 \gamma_2] P(\sigma_1, -\sigma_2, n_1, n_2 + \sigma_1 \sigma_2; t) \end{aligned} \quad (4.10)$$

where the dimensionless inverse time scales $\bar{\nu}_a$'s are defined in (3.4).

The operator in the r.h.s. of the evolution equation (4.10) is partially diagonalized by considering the generating function $P(\sigma_1, \sigma_2, z_1, z_2; t) = \sum_{n_1=-\infty}^{+\infty} \sum_{n_2=-\infty}^{+\infty} z_1^{n_1} z_2^{n_2} P(\sigma_1, \sigma_2, n_1, n_2; t)$ which is absolutely convergent for z_1 and z_2 of modulus 1. Considering the latter generating function is equivalent to introducing

$$\widehat{\mathbb{U}}(z_1, z_2; t) \equiv \sum_{n_1=-\infty}^{+\infty} \sum_{n_2=-\infty}^{+\infty} z_1^{n_1} z_2^{n_2} \mathbb{U}(n_1, n_2; t). \quad (4.11)$$

Since $P(\sigma_1, \sigma_2, n_1, n_2; t = 0) = \delta_{n_1,0} \delta_{n_2,0} P(\sigma_1, \sigma_2; t = 0)$, we infer that $\widehat{\mathbb{U}}(z_1, z_2; t = 0) = \mathbb{I}_4$, where $\mathbb{I}_4$ denotes the identity 4×4 matrix. The inversion formula which allows to retrieve $\mathbb{U}(n_1, n_2; t)$ reads

$$\mathbb{U}(n_1, n_2; t) = \oint_{|z_1|=1} \frac{dz_1}{2\pi i} \oint_{|z_2|=1} \frac{dz_2}{2\pi i} \frac{\widehat{\mathbb{U}}(z_1, z_2; t)}{z_1^{n_1+1} z_2^{n_2+1}}. \quad (4.12)$$

The decomposition (4.6) of $\mathbb{U}(n_1, n_2; t)$ leads to a similar decomposition for $\widehat{\mathbb{U}}(z_1, z_2; t)$

$$\widehat{\mathbb{U}}(z_1, z_2; t) = \sum_{\Delta n=0,+1,-1} \widehat{\mathbb{U}}_{[\Delta n]}(z_1, z_2; t). \quad (4.13)$$

The decomposition (4.7) of $\mathbb{U}(n_1, n_2; t)$ into three 4×4 matrices (enforced by the constraint (4.5) due to energy conservation) is also valid for $\widehat{\mathbb{U}}(z_1, z_2; t)$. Moreover $\widehat{\mathbb{U}}_{[\Delta n]}(z_1, z_2; t)$ has necessarily the following dependence upon z_2 and $z_1 z_2$: $\widehat{\mathbb{U}}_{[\Delta n]}(z_1, z_2; t) = z_2^{\Delta n} \widehat{\mathbb{V}}_{[\Delta n]}(z_1 z_2; t)$. Therefore, by using the change of variable $z_1 \rightarrow z = z_1 z_2$ in (4.12) one gets

$$\mathbb{U}(n_1, n_1 + \Delta n; t) = \oint_{|z|=1} \frac{dz}{2\pi i} \frac{\widehat{\mathbb{U}}_{[\Delta n]}(z_1 = z, z_2 = 1; t)}{z^{n_1+1}}. \quad (4.14)$$

4.1.3 Diagonalization

The evolution of $\widehat{\mathbb{U}}(z_1, z_2; \tau)$ with the dimensionless time variable

$$\tau = \frac{\nu_1 + \nu_2}{2} t \quad (4.15)$$

reads

$$\frac{d\widehat{\mathbb{U}}(z_1, z_2; \tau)}{d\tau} = \mathbb{A}(z_1, z_2) \widehat{\mathbb{U}}(z_1, z_2; \tau), \quad (4.16)$$

where, from the evolution equation (4.10),

$$\mathbb{A}(z_1, z_2) = -\mathbb{I}_4 + \begin{pmatrix} \gamma_\star & 0 & b_2 & b_1 \\ 0 & \gamma_\star & b_1 & b_2 \\ c_2 & c_1 & -\gamma_\star & 0 \\ c_1 & c_2 & 0 & -\gamma_\star \end{pmatrix} \quad (4.17)$$

with the following notations: $b_1 = \bar{\nu}_1(1 + \gamma_1)z_1$, $b_2 = \bar{\nu}_2(1 + \gamma_2)\frac{1}{z_2}$, $c_1 = \bar{\nu}_1(1 - \gamma_1)\frac{1}{z_1}$ and $c_2 = \bar{\nu}_2(1 - \gamma_2)z_2$.

Since the transition rates are invariant under the simultaneous changes of both spin signs, it is convenient to consider the transformed matrix

$$\mathbb{A}'(z_1, z_2) = \mathbb{P}^{-1} \mathbb{A}(z_1, z_2) \mathbb{P} \quad (4.18)$$

with

$$\mathbb{P}^{-1} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}. \quad (4.19)$$

The matrix $\mathbb{A}'(z_1, z_2)$ corresponds to two sets of decoupled equations,

$$\mathbb{A}'(z_1, z_2) = -\mathbb{I}_4 + \begin{pmatrix} \mathbb{B}_+ & 0 \\ 0 & \mathbb{B}_- \end{pmatrix} \quad (4.20)$$

where $\mathbb{B}_\epsilon = \begin{pmatrix} \gamma_\star & \epsilon b_1 + b_2 \\ \epsilon c_1 + c_2 & -\gamma_\star \end{pmatrix}$ for $\epsilon = \pm$. As $\mathbb{B}_\epsilon$ is traceless, $\mathbb{B}_\epsilon^2(z_1, z_2)$ is proportional to $\mathbb{I}_2$. Explicitly

$$\mathbb{B}_\epsilon^2(z_1, z_2) = \Delta_\epsilon(z_1 z_2) \mathbb{I}_2 \quad (4.21)$$

with

$$\Delta_\epsilon(z) = 1 - 2A + \epsilon \left[(A + B)z + (A - B)\frac{1}{z} \right] \quad (4.22)$$

where

$$A = \bar{\nu}_1 \bar{\nu}_2 (1 - \gamma_1 \gamma_2) \quad \text{and} \quad B = \bar{\nu}_1 \bar{\nu}_2 (\gamma_1 - \gamma_2). \quad (4.23)$$

We notice that $A \geq |B|$. As a consequence,

$$e^{\tau \mathbb{A}'(z_1, z_2)} = e^{-\tau} \times \begin{pmatrix} e^{\tau \mathbb{B}_+(z_1, z_2)} & \mathbb{O} \\ \mathbb{O} & e^{\tau \mathbb{B}_-(z_1, z_2)} \end{pmatrix} \quad (4.24)$$

where

$$e^{\tau \mathbb{B}_\epsilon(z_1, z_2)} = \cosh(\tau \sqrt{\Delta_\epsilon(z_1 z_2)}) \mathbb{I}_2 + \frac{\sinh(\tau \sqrt{\Delta_\epsilon(z_1 z_2)})}{\sqrt{\Delta_\epsilon(z_1 z_2)}} \mathbb{B}_\epsilon(z_1, z_2). \quad (4.25)$$

Moreover the eigenvalues of the matrix $\frac{1}{2}(\nu_1 + \nu_2) \mathbb{A}(z_1, z_2)$ are, with the notations $\epsilon = \pm$ and $\eta = \pm$,

$$\mu^{(\epsilon, \eta)}(z) = \frac{\nu_1 + \nu_2}{2} \left[-1 + \eta \sqrt{\Delta_\epsilon(z)} \right]. \quad (4.26)$$

4.1.4 Results for the generating function

From (4.18) we can calculate $\widehat{\mathbb{U}}(z_1, z_2; \tau) = e^{\tau \mathbb{A}(z_1, z_2)} = \mathbb{P} e^{\tau \mathbb{A}'(z_1, z_2)} \mathbb{P}^{-1}$. From the explicit expressions (4.24) and (4.25) we get the matrices $\widehat{\mathbb{U}}_{[\Delta n]}(z_1, z_2; \tau)$ defined in (4.13). The 16 matrix elements can be written in the compact form

$$\begin{aligned} (\sigma_1, \sigma_2 | \widehat{\mathbb{U}}_{[\Delta n=0]}(z_1, z_2; \tau) | \sigma_1, \sigma_2) &= C^+(z_1 z_2; \tau) + \sigma_1 \sigma_2 \gamma_\star S^+(z_1 z_2; \tau) \\ (-\sigma_1, -\sigma_2 | \widehat{\mathbb{U}}_{[\Delta n=0]}(z_1, z_2; \tau) | \sigma_1, \sigma_2) &= C^-(z_1 z_2; \tau) + \sigma_1 \sigma_2 \gamma_\star S^-(z_1 z_2; \tau) \\ (-\sigma_1, \sigma_2 | \widehat{\mathbb{U}}_{[\Delta n]}(z_1, z_2; \tau) | \sigma_1, \sigma_2) &\stackrel{\Delta n=\pm 1}{=} \delta_{\sigma_1 \sigma_2, \Delta n} z_2^{\Delta n} F_{\Delta n}^+(z_1 z_2; \tau) \\ (\sigma_1, -\sigma_2 | \widehat{\mathbb{U}}_{[\Delta n]}(z_1, z_2; \tau) | \sigma_1, \sigma_2) &\stackrel{\Delta n=\pm 1}{=} \delta_{\sigma_1 \sigma_2, \Delta n} z_2^{\Delta n} F_{\Delta n}^-(z_1 z_2; \tau) \end{aligned} \quad (4.27)$$

with

$$C^\pm(z; \tau) = e^{-\tau} \frac{1}{2} \left[\cosh(\tau \sqrt{\Delta_+(z)}) \pm \cosh(\tau \sqrt{\Delta_-(z)}) \right] \quad (4.28)$$

$$S^\pm(z; \tau) = e^{-\tau} \frac{1}{2} \left[\frac{\sinh(\tau \sqrt{\Delta_+(z)})}{\sqrt{\Delta_+(z)}} \pm \frac{\sinh(\tau \sqrt{\Delta_-(z)})}{\sqrt{\Delta_-(z)}} \right] \quad (4.29)$$

and

$$F_{\Delta n}^\pm(z; \tau) = \frac{1}{z^{\Delta n}} \bar{\nu}_1 (1 - \Delta n \gamma_1) S^\pm(z; \tau) + \bar{\nu}_2 (1 - \Delta n \gamma_2) S^\mp(z; \tau). \quad (4.30)$$

4.1.5 Results for the joint probability

$\mathbb{U}(n_1, n_1 + \Delta n; t)$ is derived from $\widehat{\mathbb{U}}_{[\Delta n]}(z_1, z_2; t)$ through the unit circle integral in (4.14). In fact, because of the parity property of the cosh and sinh functions, the functions $C^\pm(z; \tau)$ and $S^\pm(z; \tau)$ are functions not of $\sqrt{\Delta_\epsilon(z)}$ but only of $\Delta_\epsilon(z)$. According to the definition (4.22) of $\Delta_\epsilon(z)$, the only singular points of $\Delta_\epsilon(z)$ are $z = 0$ and $z = +\infty$, and the same is true for the integrands in $\oint_{|z|=1} (dz/2\pi i) z^{-(m+1)} C^\pm(z; \tau)$ and $\oint_{|z|=1} (dz/2\pi i) z^{-(m+1)} S^\pm(z; \tau)$. Changing z into $-z$ we get that

$$\oint_{|z|=1} \frac{dz}{2\pi i} \frac{1}{z^{m+1}} \cosh(\tau \sqrt{\Delta_-(z)}) = (-1)^m \oint_{|z|=1} \frac{dz}{2\pi i} \frac{1}{z^{m+1}} \cosh(\tau \sqrt{\Delta_+(z)}). \quad (4.31)$$

As a consequence

$$\oint_{|z|=1} \frac{dz}{2\pi i} \frac{1}{z^{m+1}} C^\pm(z; \tau) = \frac{1 \pm (-1)^m}{2} e^{-\tau} c_m(\tau) \quad (4.32)$$

and

$$\oint_{|z|=1} \frac{dz}{2\pi i} \frac{1}{z^{m+1}} S^\pm(z; \tau) = \frac{1 \pm (-1)^m}{2} e^{-\tau} s_m(\tau), \quad (4.33)$$

where

$$c_m(\tau) \equiv \oint_{|z|=1} \frac{dz}{2\pi i} \frac{1}{z^{m+1}} \cosh\left(\tau \sqrt{\Delta_+(z)}\right) \quad (4.34)$$

and

$$s_m(\tau) \equiv \oint_{|z|=1} \frac{dz}{2\pi i} \frac{1}{z^{m+1}} \frac{\sinh\left(\tau \sqrt{\Delta_+(z)}\right)}{\sqrt{\Delta_+(z)}}. \quad (4.35)$$

Eventually, the matrix elements of $\mathbb{U}(n_1, n_1 + \Delta n; t)$ are derived from the expressions (4.27) for the matrix elements of $\widehat{\mathbb{U}}_{[\Delta n]}(z_1, z_2; t)$ with the result

$$\begin{aligned} (\sigma_1, \sigma_2 | \mathbb{U}(n, n; t) | \sigma_1, \sigma_2) &= \delta_{\text{even}}(n) \times U^{(0)}(n, \sigma_1 \sigma_2; \tau) \\ (-\sigma_1, -\sigma_2 | \mathbb{U}(n, n; t) | \sigma_1, \sigma_2) &= \delta_{\text{odd}}(n) \times U^{(0)}(n, \sigma_1 \sigma_2; \tau) \\ (-\sigma_1, \sigma_2 | \mathbb{U}(n, n + \Delta n; t) | \sigma_1, \sigma_2) &\underset{\Delta n = \pm 1}{=} \delta_{\sigma_1 \sigma_2, \Delta n} \times \delta_{\text{odd}}(n) \times U^{(1)}(n, \Delta n; \tau) \\ (\sigma_1, -\sigma_2 | \mathbb{U}(n, n + \Delta n; t) | \sigma_1, \sigma_2) &\underset{\Delta n = \pm 1}{=} \delta_{\sigma_1 \sigma_2, \Delta n} \times \delta_{\text{even}}(n) \times U^{(1)}(n, \Delta n; \tau), \end{aligned} \quad (4.36)$$

where $\delta_{\text{even}}(n) = \frac{1}{2} [1 + (-1)^n]$ and $\delta_{\text{odd}}(n) = \frac{1}{2} [1 - (-1)^n]$ while

$$\begin{aligned} U^{(0)}(n, \sigma_1 \sigma_2; \tau) &= e^{-\tau} [c_n(\tau) + \gamma_\star \sigma_1 \sigma_2 s_n(\tau)] \\ U^{(1)}(n, \Delta n; \tau) &= e^{-\tau} [\bar{\nu}_2 (1 - \gamma_2 \Delta n) s_n(\tau) + \bar{\nu}_1 (1 - \gamma_1 \Delta n) s_{n+\Delta n}(\tau)]. \end{aligned} \quad (4.37)$$

We notice that the parity condition factors $\frac{1}{2} [1 \pm (-1)^n]$ have a simple interpretation. During a history such that spin σ_1 is in the same state (in flipped states) in the initial and final configurations, thermal bath 1 has flipped spin σ an even (odd) number of times, so that the corresponding sum $n\Delta e$ of the successive amounts $\pm \Delta e$ dissipated towards thermal bath 1 is necessarily an even (odd) multiple of Δe .

4.2 Various explicit probabilities

The probability that the system is in configuration (σ_1, σ_2) at time t when the initial configuration is distributed according to the law P_0 can be calculated, by virtue of the definition (4.11), as

$$P_{P_0}(\sigma_1, \sigma_2; t) = \sum_{\Delta n = -1, 0, +1} \sum_{\sigma'_1, \sigma'_2} \left(\sigma_1, \sigma_2 | \widehat{\mathbb{U}}_{[\Delta n]}(z_1 = 1, z_2 = 1; t) | \sigma'_1, \sigma'_2 \right) P_0(\sigma'_1, \sigma'_2). \quad (4.38)$$

The matrix elements of $\widehat{\mathbb{U}}_{[\Delta n]}(z_1 = 1, z_2 = 1; t)$ are derived from (4.27) where, according to (4.22), $\Delta_+(z = 1) = 1$ while $\Delta_-(z = 1) = 1 - 4A$ with $A = \bar{\nu}_1 \bar{\nu}_2 (1 - \gamma_1 \gamma_2)$. Using the identities $P_0(\sigma_1, \sigma_2) + P_0(-\sigma_1, -\sigma_2) = \frac{1}{2} [1 + \sigma_1 \sigma_2 \gamma_0]$, where $\gamma_0 = \langle \sigma_1 \sigma_2 \rangle_{P_0}$, and $P_0(\sigma_1, \sigma_2) - P_0(-\sigma_1, -\sigma_2) = \frac{1}{2} [\sigma_1 \langle \sigma'_1 \rangle_{P_0} + \sigma_2 \langle \sigma'_2 \rangle_{P_0}]$, where $\langle \sigma'_1 \rangle_{P_0}$ (resp. $\langle \sigma'_2 \rangle_{P_0}$) is the average value of the first (resp. second) spin at time 0, a straightforward calculation leads to

$$\begin{aligned} P_{P_0}(\sigma_1, \sigma_2; t) &= \frac{1}{4} [1 + \sigma_1 \sigma_2 \gamma_\star] + \frac{1}{4} \sigma_1 \sigma_2 [\gamma_0 - \gamma_\star] e^{-2\tau} \\ &+ \frac{1}{8} [\sigma_1 \langle \sigma'_1 \rangle_{P_0} + \sigma_2 \langle \sigma'_2 \rangle_{P_0}] \left[\left(1 + \sigma_1 \sigma_2 \frac{\gamma_\star}{\alpha} \right) e^{-(1-\alpha)\tau} + \left(1 - \sigma_1 \sigma_2 \frac{\gamma_\star}{\alpha} \right) e^{-(1+\alpha)\tau} \right] \\ &+ \frac{1}{8} [-\sigma_1 \langle \sigma'_1 \rangle_{P_0} + \sigma_2 \langle \sigma'_2 \rangle_{P_0}] \frac{1}{\alpha} [\bar{\nu}_1 - \bar{\nu}_2 + \sigma_1 \sigma_2 (\bar{\nu}_1 \gamma_1 - \bar{\nu}_2 \gamma_2)] \left[e^{-(1-\alpha)\tau} - e^{-(1+\alpha)\tau} \right]. \end{aligned} \quad (4.39)$$

where $\alpha \equiv \sqrt{1 - 4A}$. When P_0 is invariant under the simultaneous reversal of the spins σ_1 and σ_2 , $P_0(\sigma_1, \sigma_2) = (1/4) [1 + \sigma_1 \sigma_2 \gamma_0]$ and only the terms in the first line of (4.39) do contribute. Then

the evolution of $P_{P_0}(\sigma_1, \sigma_2; t)$ towards the stationary distribution $P_{\text{st}}(\sigma_1, \sigma_2) = \frac{1}{4} [1 + \sigma_1 \sigma_2 \gamma_\star]$ involves only one time scale, namely $1/(\nu_1 + \nu_2)$ (recall that $\tau = \frac{1}{2}(\nu_1 + \nu_2)t$).

For any initial probability distribution P_0 of the spins, the probability $P_{P_0}(\sigma_1, \sigma_2, n_1, n_2; t)$ that at time t the system is in configuration (σ_1, σ_2) and has received a heat amount $\mathcal{Q}_1 = -n_1 \Delta e$ from bath 1 and $\mathcal{Q}_2 = n_2 \Delta e$ from bath 2 is calculated from (4.36) with the result

$$\begin{aligned} P_{P_0}(\sigma_1, \sigma_2, n, n; t) &= U^{(0)}(n, \sigma_1 \sigma_2; \tau) [\delta_{\text{even}}(n) P_0(\sigma_1, \sigma_2) + \delta_{\text{odd}}(n) P_0(-\sigma_1, -\sigma_2)] \\ P_{P_0}(\sigma_1, \sigma_2, n, n + \Delta n; t) &\underset{\Delta n = \pm 1}{=} \delta_{\sigma_1 \sigma_2, -\Delta n} U^{(1)}(n, \Delta n; \tau) [\delta_{\text{even}}(n) P_0(\sigma_1, -\sigma_2) + \delta_{\text{odd}}(n) P_0(-\sigma_1, \sigma_2)] \end{aligned} \quad (4.40)$$

Various joint probabilities can be derived from these expressions.

The joint probability $P_{P_0}(n_1, n_2; t)$ that at time t the system has received a heat amount $\mathcal{Q}_1 = -n_1 \Delta e$ from bath 1 and a heat amount $\mathcal{Q}_2 = n_2 \Delta e$ from bath 2 is

$$\begin{aligned} P_{P_0}(n, n; \tau) &= e^{-\tau} [c_n(\tau) + \gamma_\star \gamma_0 s_n(\tau)] \\ P_{P_0}(n, n + \Delta n; \tau) &\underset{\Delta n = \pm 1}{=} e^{-\tau} \frac{1}{2} [1 + \Delta n \gamma_0] [\bar{\nu}_2 (1 - \Delta n \gamma_2) s_n(\tau) + \bar{\nu}_1 (1 - \Delta n \gamma_1) s_{n+\Delta n}(\tau)], \end{aligned} \quad (4.41)$$

where γ_0 has been defined before (4.39).

The joint probability $P_{P_0}(\sigma_1 \sigma_2 = \pm 1, n_1; t)$ that at time t the system is in a configuration where $\sigma_1 \sigma_2$ is equal to ± 1 and has received a heat amount $\mathcal{Q}_1 = -n_1 \Delta e$ from bath 1 is seen via (4.40) to have value

$$\begin{aligned} P_{P_0}(\sigma_1 \sigma_2 = +1, n_1; \tau) &= \\ \frac{1}{2} e^{-\tau} \{ (1 + \gamma_0) c_{n_1}(\tau) + [2\gamma_\star + (\bar{\nu}_2 - \bar{\nu}_1 \gamma_1) (1 - \gamma_0)] s_{n_1}(\tau) + \bar{\nu}_1 (1 + \gamma_1) (1 - \gamma_0) s_{n_1-1}(\tau) \} \end{aligned} \quad (4.42)$$

and

$$\begin{aligned} P_{P_0}(\sigma_1 \sigma_2 = -1, n_1; \tau) &= \\ \frac{1}{2} e^{-\tau} \{ (1 - \gamma_0) c_{n_1}(\tau) + [-2\gamma_\star + (\bar{\nu}_2 + \bar{\nu}_1 \gamma_1) (1 + \gamma_0)] s_{n_1}(\tau) + \bar{\nu}_1 (1 - \gamma_1) (1 + \gamma_0) s_{n_1+1}(\tau) \}. \end{aligned} \quad (4.43)$$

The expressions for $P_{P_0}(\sigma_1 \sigma_2 = \pm 1, n_2; \tau)$ are obtained from the latter equations by making the exchanges $\bar{\nu}_1 \leftrightarrow \bar{\nu}_2$ and $\gamma_1 \leftrightarrow \gamma_2$ and the replacements $c_{n_1} \rightarrow c_{n_2}$, $s_{n_1} \rightarrow s_{n_2}$, and $s_{n_1-1} \rightarrow s_{n_2+1}$ for $\sigma_1 \sigma_2 = 1$, resp. $s_{n_1+1} \rightarrow s_{n_2-1}$ for $\sigma_1 \sigma_2 = -1$.

From these expressions we get the probability distribution for only one heat amount $\mathcal{Q}_1$ or $\mathcal{Q}_2$

$$\begin{aligned} P_{P_0}(n_1; \tau) &= e^{-\tau} \left\{ c_{n_1}(\tau) + [\bar{\nu}_2 + \bar{\nu}_1 \gamma_1 \gamma_0] s_{n_1}(\tau) \right. \\ &\quad \left. + \frac{1}{2} \bar{\nu}_1 (1 - \gamma_1) (1 + \gamma_0) s_{n_1+1}(\tau) + \frac{1}{2} \bar{\nu}_1 (1 + \gamma_1) (1 - \gamma_0) s_{n_1-1}(\tau) \right\} \end{aligned} \quad (4.44)$$

and similarly

$$\begin{aligned} P_{P_0}(n_2; \tau) &= e^{-\tau} \left\{ c_{n_2}(\tau) + [\bar{\nu}_1 + \bar{\nu}_2 \gamma_2 \gamma_0] s_{n_2}(\tau) \right. \\ &\quad \left. + \frac{1}{2} \bar{\nu}_2 (1 + \gamma_2) (1 - \gamma_0) s_{n_2+1}(\tau) + \frac{1}{2} \bar{\nu}_2 (1 - \gamma_2) (1 + \gamma_0) s_{n_2-1}(\tau) \right\}. \end{aligned} \quad (4.45)$$

From the identities $\sum_{n=-\infty}^{+\infty} c_n(\tau) = \cosh \tau$, and $\sum_{n=-\infty}^{+\infty} s_n(\tau) = \sinh \tau$, the probability that the total heat amount received from both thermostats is $\mathcal{Q}_1 + \mathcal{Q}_2 = (n_2 - n_1) \Delta e$ reads

$$\begin{aligned} P_{P_0}(n_2 - n_1 = 0; \tau) &= \frac{1}{2} [1 + \gamma_\star \gamma_0] + \frac{1}{2} [1 - \gamma_\star \gamma_0] e^{-2\tau} \\ P_{P_0}(n_2 - n_1 = \Delta n; \tau) &\underset{\Delta n = \pm 1}{=} \frac{1}{4} \{ 1 - \gamma_\star \gamma_0 + \Delta n [\gamma_0 - \gamma_\star] \} [1 - e^{-2\tau}]. \end{aligned} \quad (4.46)$$

As a consequence

$$\langle [\mathcal{Q}_1 + \mathcal{Q}_2]_\tau \rangle_{P_0} = \frac{1}{2} [\gamma_0 - \gamma_\star] [1 - e^{-2\tau}] \Delta e, \quad (4.47)$$

and we retrieve property (3.44).

We notice that all formulæ are still valid in the limit where T_1 vanishes, namely where $\beta_1 \Delta e$ goes to infinity.

4.3 Excess heats

The excess heats associated with the transition between two different steady states have been defined in (3.48). For the two-spin system they can be explicitly calculated. Indeed, when the system is initially prepared in the stationary state with distribution P_0 by contact with two thermostats at the inverse temperatures β_1^0 and β_2^0 and then is put at time $t = 0$ in contact with two heat baths at inverse temperatures β_1 and β_2 , the mean heat amount received from the thermostat 1 between time $t = 0$ and time t is given (with the convention (4.1)) by

$$\langle \mathcal{Q}_1(t) \rangle_{P_0} = -\Delta e \sum_{n_1=-\infty}^{+\infty} n_1 \sum_{\sigma_1, \sigma_2} \sum_{\sigma'_1, \sigma'_2} \sum_{\Delta n=0,1,-1} (\sigma'_1, \sigma'_2 | \mathbb{U}(n_1, n_1 + \Delta n; t) | \sigma_1, \sigma_2) P_0(\sigma_1, \sigma_2). \quad (4.48)$$

By virtue of the definition (4.11) of the relevant generating function and the decomposition (4.13), the latter expression can be rewritten as

$$\langle \mathcal{Q}_1(t) \rangle_{P_0} = -\Delta e \frac{\partial}{\partial z_1} \left(\sum_{\sigma_1, \sigma_2} \sum_{\sigma'_1, \sigma'_2} \sum_{\Delta n=0,1,-1} (\sigma'_1, \sigma'_2 | \widehat{\mathbb{U}}_{[\Delta n]}(z_1, z_2; t) | \sigma_1, \sigma_2) P_0(\sigma_1, \sigma_2) \right) \Big|_{z_1=z_2=1}. \quad (4.49)$$

From the explicit expressions (4.27) for the matrix elements and (4.22) for $\Delta_+(z)$, a straightforward calculation leads to

$$\langle \mathcal{Q}_1(t) \rangle_{P_0} = \langle j_1 \rangle_{\text{st}} t + [\bar{\nu}_1 \bar{\nu}_2 (\gamma_1 - \gamma_2) - \bar{\nu}_1 (\gamma_1 - \gamma_\star^0)] \frac{\Delta e}{2} [1 - e^{-(\nu_1 + \nu_2)t}] \quad (4.50)$$

where $\langle j_1 \rangle_{\text{st}} = -\langle j_2 \rangle_{\text{st}}$ is given in (3.27) and $\gamma_\star^0$ is a function of $\bar{\nu}_1$, $\bar{\nu}_2$, β_1^0 and β_2^0 written in (3.5). A similar calculation yields

$$\langle \mathcal{Q}_2(t) \rangle_{P_0} = \langle j_2 \rangle_{\text{st}} t + [-\bar{\nu}_1 \bar{\nu}_2 (\gamma_1 - \gamma_2) - \bar{\nu}_2 (\gamma_2 - \gamma_\star^0)] \frac{\Delta e}{2} [1 - e^{-(\nu_1 + \nu_2)t}]. \quad (4.51)$$

As a result, with the sign convention of definition (3.48), the excess heat given to the system by heat bath 1 in the present protocol reads

$$Q_{\text{exc},1} \Big|_{[\beta_1^0, \beta_2^0]}^{[\beta_1, \beta_2]} = -[\bar{\nu}_1 \bar{\nu}_2 (\gamma_1 - \gamma_2) - \bar{\nu}_1 (\gamma_1 - \gamma_\star^0)] \frac{\Delta e}{2}. \quad (4.52)$$

The excess heat given to the system by heat bath 2, $Q_{\text{exc},2} \Big|_{[\beta_1^0, \beta_2^0]}^{[\beta_1, \beta_2]}$, is obtained from the latter expression by the exchanges $\bar{\nu}_1 \leftrightarrow \bar{\nu}_2$ and $\gamma_1 \leftrightarrow \gamma_2$. These two excess heats are not opposite to each other, and comparison with the expressions (3.14) for the mean energies in the initial and final stationary states shows that the sum of the excess heats coming from both thermostats indeed satisfy the identity (3.49).

4.4 Symmetry property for reversed heat transfers (when $T_1 \neq 0$) specific to the model

4.4.1 Symmetry arising from modified detailed balance

The symmetry properties for reversed heat transfers when $T_1 \neq 0$ are more conveniently exhibited after a change of variable in the complex plane where the integrals involved in $P_{P_0}(\mathcal{Q}_1, \mathcal{Q}_2, t)$ are defined. The relevant functions $c_n(\tau)$ and $s_n(\tau)$ are defined in (4.34) and (4.35) while $\Delta_+(z)$ is given in (4.22). The origin $z = 0$ is a singular point in $\Delta_+(z)$ and $z\Delta_+(z) = (A+B)P(z)$ where the second order polynomial $P(z) = (z - z_+)(z - z_-)$ vanishes for two roots z_+ and z_- . The product of the roots is equal to

$$z_+ z_- = \frac{A-B}{A+B} = \frac{1}{\rho^2} \quad \text{with} \quad \rho \equiv e^{(\beta_1 - \beta_2)\Delta e/2} \quad (4.53)$$

and the sum of the roots is equal to $-(1 - 2A)/(A + B)$.

When $T_1 \neq 0$, $z_+ z_-$ does not vanish and by using the variable change $\zeta = z/\sqrt{z_- z_+}$, namely

$$\zeta = z\rho, \quad (4.54)$$

the unit circle is changed into a circle with radius ρ , while the roots z_+ and z_- are changed into ζ_- and ζ_+ with $\zeta_- \zeta_+ = 1$. Then, for a function such as $\cosh(\sqrt{\Delta_+(z)})$ or $\sinh(\tau\sqrt{\Delta_+(z)})/\sqrt{\Delta_+(z)}$, each of which is in fact a function of $\Delta_+(z)$ denoted by $f(\Delta_+(z))$,

$$\oint_{|z|=1} \frac{dz}{2\pi i} \frac{1}{z^{n+1}} f(\Delta_+(z)) = \rho^n \oint_{|\zeta|=\rho} \frac{d\zeta}{2\pi i} \frac{1}{\zeta^{n+1}} f(\tilde{\Delta}_+(\zeta)) \quad (4.55)$$

with $\tilde{\Delta}_+(\zeta) \equiv \Delta_+(\zeta/\rho)$. $\tilde{\Delta}_+(\zeta)$ is a symmetric function of ζ and $1/\zeta$,

$$\tilde{\Delta}_+(\zeta) = b + a \frac{\zeta + \zeta^{-1}}{2} \quad (4.56)$$

with $a = 2\sqrt{A^2 - B^2}$ and $b = 1 - 2A$, namely

$$a = 2\bar{\nu}_1 \bar{\nu}_2 \sqrt{(1 - \gamma_1^2)(1 - \gamma_2^2)} \quad (4.57)$$

$$b = 1 - 2\bar{\nu}_1 \bar{\nu}_2 (1 - \gamma_1 \gamma_2). \quad (4.58)$$

Since the only singular points in the integrand in the r.h.s. of (4.55) are $\zeta = 0$ and $\zeta = \infty$, the circle $|\zeta| = \rho$ can be deformed into the unit circle and we get the identity

$$\oint_{|z|=1} \frac{dz}{2\pi i} \frac{1}{z^{n+1}} f(\Delta_+(z)) = \rho^n \oint_{|z|=1} \frac{dz}{2\pi i} \frac{1}{z^{n+1}} f(\tilde{\Delta}_+(z)). \quad (4.59)$$

By inserting the latter identity in the definitions (4.34) and (4.35) we get the relations

$$\begin{aligned} c_n(\tau) &= \rho^n \tilde{c}_n(\tau) \\ s_n(\tau) &= \rho^n \tilde{s}_n(\tau) \end{aligned} \quad (4.60)$$

where

$$\tilde{c}_n(\tau) \equiv \oint_{|z|=1} \frac{dz}{2\pi i} \frac{1}{z^{n+1}} \cosh\left(\tau\sqrt{\tilde{\Delta}_+(z)}\right) \quad (4.61)$$

and

$$\tilde{s}_n(\tau) \equiv \oint_{|z|=1} \frac{dz}{2\pi i} \frac{1}{z^{n+1}} \frac{\sinh\left(\tau\sqrt{\tilde{\Delta}_+(z)}\right)}{\sqrt{\tilde{\Delta}_+(z)}}. \quad (4.62)$$

Since $\tilde{\Delta}_+(z)$ is invariant under the exchange of z and $1/z$,

$$\begin{aligned} \tilde{c}_n(\tau) &= \tilde{c}_{|n|}(\tau) \\ \tilde{s}_n(\tau) &= \tilde{s}_{|n|}(\tau). \end{aligned} \quad (4.63)$$

Therefore the functions involved in the matrix elements (4.36) of $\mathbb{U}(n_1, n_2; t)$ can be rewritten as

$$\begin{aligned} U^{(0)}(n, \sigma_1 \sigma_2; \tau) &= e^{-\tau} \rho^n [\tilde{c}_{|n|}(\tau) + \gamma_* \sigma_1 \sigma_2 \tilde{s}_{|n|}(\tau)] \\ U^{(1)}(n, \Delta n; \tau) &= e^{-\tau} \rho^n [\bar{\nu}_2 (1 - \gamma_2 \Delta n) \tilde{s}_{|n|}(\tau) + \bar{\nu}_1 (1 - \gamma_1 \Delta n) \rho^{\Delta n} \tilde{s}_{|n+\Delta n|}(\tau)]. \end{aligned} \quad (4.64)$$

where ρ , defined in (4.53), also reads $\rho = \sqrt{(1 + \gamma_1)(1 - \gamma_2)/(1 - \gamma_1)(1 + \gamma_2)}$.

According to paper I, the modified detailed balance implies some time-reversal symmetry property for histories, which itself entails some relation between probabilities of forward and backward evolutions where the given initial and final configurations are exchanged (and the heat amounts

are changed into their opposite values). In the spin model language, with the definitions (4.1), the symmetry exhibited in paper I for the probability that the system evolves from an initial configuration $\mathcal{C}_0 = (\sigma_1, \sigma_2)$ to a final configuration $\mathcal{C}_f = (\sigma'_1, \sigma'_2)$ while receiving the heat amount $\mathcal{Q}_1 = -n_1 \Delta e$ and $\mathcal{Q}_2 = n_2 \Delta e$ reads, for non-vanishing matrix elements,

$$\frac{(\sigma'_1, \sigma'_2 | \mathbb{U}(n_1, n_2; t) | \sigma_1, \sigma_2)}{(\sigma_1, \sigma_2 | \mathbb{U}(-n_1, -n_2; t) | \sigma'_1, \sigma'_2)} = e^{(n_1 \beta_1 - n_2 \beta_2) \Delta e}. \quad (4.65)$$

Comparison of the latter relation with the expressions (4.36) implies that

$$\frac{U^{(0)}(n, \sigma_1 \sigma_2; \tau)}{U^{(0)}(-n, \sigma_1 \sigma_2; \tau)} = e^{[n \beta_1 - n \beta_2] \Delta e} \quad (4.66)$$

and

$$\frac{U^{(1)}(n, \Delta n; \tau)}{U^{(1)}(-n, -\Delta n; \tau)} = e^{[n \beta_1 - (n + \Delta n) \beta_2] \Delta e}. \quad (4.67)$$

The latter relations can be checked from the explicit expressions (4.64).

We notice that the relation (4.65) can also be retrieved by noticing that the modified detailed balance entails that the matrix $\mathbb{A}(z_1, z_2)$, which rules the evolution of $\hat{\mathbb{U}}(z_1, z_2; \tau)$ according to (4.16), obeys the symmetry $\mathbb{A}(z_1, z_2) = \mathbb{A}^T(e^{-\beta_1 \Delta e}/z_1, e^{\beta_2 \Delta e}/z_2)$, where $\mathbb{A}^T$ denotes the transposed matrix of $\mathbb{A}$. Therefore, after the variable change $z_1 = \zeta_1/\rho_1$ and $z_2 = \zeta_2/\rho_2$ with $\rho_1 = \exp(\beta_1 \Delta e/2)$ and $\rho_2 = \exp(-\beta_2 \Delta e/2)$, the matrix $\mathbb{A}(z_1, z_2)$ becomes the matrix $\tilde{\mathbb{A}}(\zeta_1, \zeta_2) \equiv \mathbb{A}(z_1 = \zeta_1/\rho_1, z_2 = \zeta_2/\rho_2)$ which obeys the symmetry $\tilde{\mathbb{A}}^T(\zeta_1, \zeta_2) = \tilde{\mathbb{A}}(1/\zeta_1, 1/\zeta_2)$. Then the derivation of the symmetry (4.65) is the following. First we make the variable change $z_1 = \zeta_1/\rho_1$ and $z_2 = \zeta_2/\rho_2$ in the integral representation (4.12) for $\mathbb{U}(n_1, n_2; t)$. Since $\tilde{\mathbb{A}}(\zeta_1, \zeta_2)$ has no non-analyticity, apart from the $1/\zeta_1$ and $1/\zeta_2$ singular terms, the integrals on the circles with radii equal to ρ_1 and ρ_2 are equal to the integrals with the same integrands on the circles with radii equal to 1. If ζ is on the unit circle, $1/\zeta$ is also on this circle and we can make the variable change $\zeta_1 \rightarrow 1/\zeta_1$ and $\zeta_2 \rightarrow 1/\zeta_2$; then the symmetry of $\tilde{\mathbb{A}}$ leads to the symmetry (4.65).

As shown in paper I, the consequence (4.65) of the modified detailed balance (2.3) entails that, if the spin system is in an equilibrium state at inverse temperature β_0 at time $t = 0$ where it is put in contact with the two thermostats, then the ratio of the probabilities to measure some given heat amounts $\mathcal{Q}_1$ and $\mathcal{Q}_2$ or their opposite values obeys the fluctuation relation

$$\frac{P_{P_{\text{can}}^{\beta_0}}(\mathcal{Q}_1, \mathcal{Q}_2; t)}{P_{P_{\text{can}}^{\beta_0}}(-\mathcal{Q}_1, -\mathcal{Q}_2; t)} = e^{(\beta_0 - \beta_1) \mathcal{Q}_1 + (\beta_0 - \beta_2) \mathcal{Q}_2}. \quad (4.68)$$

4.4.2 A relation specific to the model

The present model happens to obey a very specific relation for reversed heat transfers when the initial state of the system has an arbitrary probability distribution P_0 . According to (4.36), after summation over the final configuration,

$$\begin{aligned} \sum_{\sigma'_1, \sigma'_2} (\sigma'_1, \sigma'_2 | \mathbb{U}(n, n; t) | \sigma_1, \sigma_2) &= U^{(0)}(n, \sigma_1 \sigma_2; \tau) \\ \sum_{\sigma'_1, \sigma'_2} (\sigma'_1, \sigma'_2 | \mathbb{U}(n, n + \Delta n; t) | \sigma_1, \sigma_2) &\underset{\Delta n = \pm 1}{=} U^{(1)}(n, \Delta n; \tau) \delta_{\sigma_1 \sigma_2, \Delta n}. \end{aligned} \quad (4.69)$$

Therefore, when the initial configurations are distributed with an arbitrary probability P_0

$$\sum_{\sigma'_1, \sigma'_2} (\sigma'_1, \sigma'_2 | \mathbb{U}(n, n; t) | P_0) = \sum_{\sigma_1, \sigma_2} U^{(0)}(n, \sigma_1 \sigma_2; \tau) P_0(\sigma_1, \sigma_2) \quad (4.70)$$

and

$$\sum_{\sigma'_1, \sigma'_2} (\sigma'_1, \sigma'_2 | \mathbb{U}(n, n + \Delta n; t) | P_0) \underset{\Delta n = \pm 1}{=} U^{(1)}(n, \Delta n; \tau) \times P_0(\sigma_1 \sigma_2 = \Delta n) \quad (4.71)$$

where $P_0(\sigma_1\sigma_2 = \Delta n)$ denotes the probability that the product $\sigma_1\sigma_2$ is equal to Δn in the initial configuration.

Then from the expressions (4.70) and (4.71) for $P_{P_0}(\mathcal{Q}_1, \mathcal{Q}_2, t)$ in the cases $\mathcal{Q}_1 = -\mathcal{Q}_2$ and $\mathcal{Q}_1 \neq -\mathcal{Q}_2$ respectively, and by virtue of the consequences (4.66)-(4.67) of the modified detailed balance, we get the property

$$\frac{P_{P_0}(\mathcal{Q}_1, \mathcal{Q}_2, t)}{P_{P_0}(-\mathcal{Q}_1, -\mathcal{Q}_2, t)} = e^{-\beta_1 \mathcal{Q}_1 - \beta_2 \mathcal{Q}_2} \left[\delta_{\mathcal{Q}_1 + \mathcal{Q}_2, 0} + \sum_{\epsilon = \pm 1} \delta_{\mathcal{Q}_1 + \mathcal{Q}_2, \epsilon \Delta} e^{\frac{P(\sigma_1\sigma_2 = \epsilon; t=0)}{1 - P(\sigma_1\sigma_2 = \epsilon; t=0)}} \right]. \quad (4.72)$$

The appearance of the initial probability for the sign of the spins product seemingly arises from the fact that, by virtue of energy conservation, the values of this sign in the final and initial states are related to the sum $\mathcal{Q}_2 + \mathcal{Q}_1$ by the constraint (4.5).

We stress that this relation is very specific to the present model. Its interest lies not in its precise form, but in that the right-hand side involves the initial distribution : it has sometimes been speculated that the (experimental) study of the ratio on the left-hand side for general systems could give some clues about the initial distribution. The above formula justifies this hope, but shows at the same time that even in the simple case at hand only partial information can be retrieved, and suggest that for more general systems even this partial information may be difficult to extract. In the cases where the initial distribution is equal either to an equilibrium state distribution at the inverse temperature β_0 or to the stationary state distribution which is a canonical distribution at the inverse temperature β_* , the relation allows to retrieve the generic relation (4.68).

4.5 Decaying property of joint probabilities for large heat exchanges

All quantities of interest involve the coefficients $c_m(\tau)$ and/or $s_m(\tau)$, computed via contour integrals in (4.34) (4.35). The integrands involve functions which are holomorphic for $z \in \mathbb{C}^*$, the pointed complex plane: the quantity $\Delta_+(z)$ defined in (4.22) has this property and the functions $\cosh(w)$ and $\sinh(w)/w$ are entire even functions, so that the square roots in the composed functions $\cosh(\tau\sqrt{\Delta_+(z)})$ and $\sinh(\tau\sqrt{\Delta_+(z)})/\sqrt{\Delta_+(z)}$ do no harm.

Hence, in the formulæ for $c_m(\tau)$ and $s_m(\tau)$, contours can be deformed. For $r \in]0, +\infty[$, let $C(\tau, r) \equiv \sup_{|z|=r} |\cosh(\tau\sqrt{\Delta_+(z)})| < +\infty$ and $S(\tau, r) \equiv \sup_{|z|=r} |\sinh(\tau\sqrt{\Delta_+(z)})/\sqrt{\Delta_+(z)}| < +\infty$.

Taking $|z| = r$ as integration contour, one gets immediately that, for each r , $c_m(\tau) \leq C(\tau, r)r^{-m}$ and $s_m(\tau) \leq S(\tau, r)r^{-m}$. This shows that $c_m(\tau)$ and $s_m(\tau)$ are $o(e^{-K|m|})$ at large $|m|$ for any K .

With some efforts, we could get some explicit upper bounds for $C(\tau, r)$ and $S(\tau, r)$. Then we could extremize over r to get a subexponential bound for $c_m(\tau)$ and $s_m(\tau)$, but we shall not need this refinement.

In the limit $T_1 = 0$, $\Delta_+(z)$ is in fact holomorphic for $z \in \mathbb{C}$ so that $c_m(\tau)$ and $s_m(\tau)$ vanish for $m = -1, -2, \dots$.

5 Heat amount cumulants for any T_1 and T_2

5.1 Generic properties for a system with a finite number of configurations

5.1.1 Characteristic function for the heat amount $\mathcal{Q}_2$

The random variable $\mathcal{Q}_2$ can take only discrete values $n_2 \Delta e$, where n_2 is a positive or negative integer. Therefore its probability density $\Pi(\mathcal{Q}; t)$, defined as $\Pi(\mathcal{Q}; t) d\mathcal{Q} = P(\mathcal{Q}_2 \in [\mathcal{Q}, \mathcal{Q} + d\mathcal{Q}]; t)$, reads

$$\Pi(\mathcal{Q}; t) = \frac{1}{\Delta e} \sum_{n_2=-\infty}^{+\infty} \delta\left(\frac{\mathcal{Q}}{\Delta e} - n_2\right) P(n_2; t), \quad (5.1)$$

where δ stands for the Dirac distribution. Since $P(n_2; t)$ decays faster than $\exp(-K|n_2|)$ for any $K > 0$ when $|n_2|$ goes to infinity (see subsection 4.5), the Laplace transform $\tilde{G}(\lambda; t)$ of $\Pi(\mathcal{Q})$, i.e. the characteristic function of the random variable $\mathcal{Q}_2$, is well defined for any λ ,

$$\tilde{G}(\lambda; t) \equiv \int_{-\infty}^{+\infty} d\mathcal{Q} e^{\lambda \mathcal{Q}} \Pi(\mathcal{Q}; t) = \langle e^{\lambda \mathcal{Q}_2(t)} \rangle. \quad (5.2)$$

As a consequence, the properties of the probability density $\Pi(\mathcal{Q}; t)$ can be investigated through those of its Laplace transform, thanks to the inversion formula

$$\Pi(\mathcal{Q}; t) = \int_{-i\infty}^{+i\infty} \frac{d\lambda}{2\pi i} e^{-\lambda \mathcal{Q}} \tilde{G}(\lambda; t). \quad (5.3)$$

According to the property (5.1), $\tilde{G}(\lambda)$ is a periodic function of λ with period equal to $i2\pi/\Delta e$, and the r. h. s. of the latter formula can be written as

$$\frac{1}{\Delta e} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-i\theta \mathcal{Q}/\Delta e} \tilde{G}\left(\frac{i\theta}{\Delta e}; t\right) \sum_{m=-\infty}^{+\infty} e^{-i2\pi m \mathcal{Q}/\Delta e}. \quad (5.4)$$

By virtue of Poisson equality $\sum_{m=-\infty}^{+\infty} e^{-i2\pi m \mathcal{Q}/\Delta e} = \sum_{n_2=-\infty}^{+\infty} \delta\left(\frac{\mathcal{Q}}{\Delta e} - n_2\right)$, comparison with (5.1) leads to

$$P(n_2; t) = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-i\theta n_2} \tilde{G}\left(\frac{i\theta}{\Delta e}; t\right). \quad (5.5)$$

The latter equality coincides with the inverse formula (analogous to (4.12)) in terms of the generating function, $G(z; t) = \sum_{n_2=-\infty}^{+\infty} z^{n_2} P(n_2; t)$,

$$P(n_2; t) = \oint_{|z|=1} \frac{dz}{2\pi i} \frac{1}{z^{n_2+1}} G(z; t) \quad (5.6)$$

where $G(z; t) \equiv \tilde{G}((1/\Delta e) \ln z; t)$.

5.1.2 Relation between long-time cumulants per unit time for $\mathcal{Q}_1$ and $\mathcal{Q}_2$

The generic properties of the cumulants of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ have been reviewed in paper I. We recall those which will be useful in the following. For a Markov process, the long-time behaviors of these cumulants are proportional to the time t elapsed from the beginning of the measurements. The asymptotic behavior of the cumulants per unit time are given by the derivatives of

$$\alpha_a(\lambda) \equiv \lim_{t \rightarrow +\infty} \frac{1}{t} \ln \langle e^{\lambda \mathcal{Q}_a(t)} \rangle \quad (5.7)$$

according to

$$\lim_{t \rightarrow +\infty} \frac{\kappa_{\mathcal{Q}_a}^{[p]}}{t} = \left. \frac{\partial^p \alpha_a(\lambda_a)}{\partial \lambda_a^p} \right|_{\lambda_a=0} \quad \text{for } a = \{1, 2\}. \quad (5.8)$$

Moreover, in the case of a system with a finite number of configurations, $\mathcal{Q}_1 + \mathcal{Q}_2 = \mathcal{E}(\mathcal{C}_f) - \mathcal{E}(\mathcal{C}_0)$ is restricted to some finite interval and

$$\alpha_1(\lambda) = \alpha_2(-\lambda). \quad (5.9)$$

As a consequence the long-time cumulants per unit time obey the following relations

$$\lim_{t \rightarrow \infty} \frac{\kappa_{\mathcal{Q}_1}^{[p]}}{t} = (-1)^p \lim_{t \rightarrow \infty} \frac{\kappa_{\mathcal{Q}_2}^{[p]}}{t}. \quad (5.10)$$

5.2 Explicit formulæ for the cumulants per unit time

According to the relation (5.10) between the long-time cumulants per unit time for $\mathcal{Q}_1$ and $\mathcal{Q}_2$, we have only to consider the cumulants for the heat amount $\mathcal{Q}_2$ received from bath 2. For the two-spin system, where $\mathcal{Q}_2 = n_2 \Delta e$, it is convenient to introduce the cumulants $\kappa_{n_2}^{[p]}$ for the dimensionless variable n_2 and the associated characteristic function $\langle e^{\bar{\lambda} n_2} \rangle$, where $\bar{\lambda}$ is a dimensionless variable. According to the relation (5.8) the long-time behavior of the cumulants per unit time are derived through the relation

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \kappa_{n_2}^{[p]} = \left. \frac{\partial^p \alpha_2(\bar{\lambda})}{\partial \bar{\lambda}^p} \right|_{\bar{\lambda}=0} \quad (5.11)$$

with

$$\alpha_2(\bar{\lambda}) = \lim_{t \rightarrow +\infty} \frac{1}{t} \ln \langle e^{\bar{\lambda} n_2} \rangle. \quad (5.12)$$

According to the definition of $\langle e^{\bar{\lambda} n_2} \rangle$, the relation (4.9) between the probability $P(\sigma_1, \sigma_2, n_1, n_2; t)$ and the operator $\mathbb{U}(n_1, n_2; t)$, together with the definition (4.11) of $\hat{\mathbb{U}}(z_1, z_2; t)$ and its evolution equation (4.16), the characteristic function may be expressed as

$$\langle e^{\bar{\lambda} n_2} \rangle = \sum_{\sigma_1, \sigma_2} \sum_{\sigma'_1, \sigma'_2} \left(\sigma_1, \sigma_2 | e^{\frac{\nu_1 + \nu_2}{2} t \mathbb{A}(z_1=1, z_2=e^{\bar{\lambda}})} | \sigma'_1, \sigma'_2 \right) P(\sigma'_1, \sigma'_2; t=0). \quad (5.13)$$

According to (4.17), $\mathbb{A}(z_1=1, z_2=e^{\bar{\lambda}}) + (1 + \gamma_*) \mathbb{I}_4$ is a real positive matrix and the Perron-Frobenius theorem can be applied. Henceforth $\alpha_2(\bar{\lambda})$ coincides with the eigenvalue of the matrix $[(\nu_1 + \nu_2)/2] \mathbb{A}(z_1=1, z_2=e^{\bar{\lambda}})$ with the largest modulus (and which is necessarily real). The four eigenvalues to consider are the $\mu^{(\epsilon, \eta)}(z=e^{\bar{\lambda}})$'s which are given by the expression (4.26), with $\epsilon = \pm$ and $\eta = \pm$. The one with the largest modulus corresponds to $(\epsilon, \eta) = (+, +)$ and reads

$$\alpha_2(\bar{\lambda}) = \frac{\nu_1 + \nu_2}{2} \left[-1 + \sqrt{1 - 2A + (A+B)e^{\bar{\lambda}} + (A-B)e^{-\bar{\lambda}}} \right] \quad (5.14)$$

where A and B are defined in (4.23).

It is plain to calculate a number of cumulants per unit time in the infinite-time limit from (5.11). Their behavior as a function of the kinetic parameter ν_2/ν_1 exhibits some interesting features. For large ν_2/ν_1 , the cumulants go to a limit which is the same for all odd and for all even cumulants, as will be explained in section 7. Fig.1 illustrates this convergence, which gets slower and slower for higher moments. The first six cumulants are represented. This figure also shows some oscillations at finite ν_2/ν_1 . These oscillations become more and more visible on higher cumulants. Fig.2 illustrates this phenomenon. Cumulants from the fifth to the ninth are represented. In both figures, the other model parameters are fixed to the sample values $\gamma_1 = 0.7$, $\gamma_2 = 0.4$, $\nu_1 = 2$.

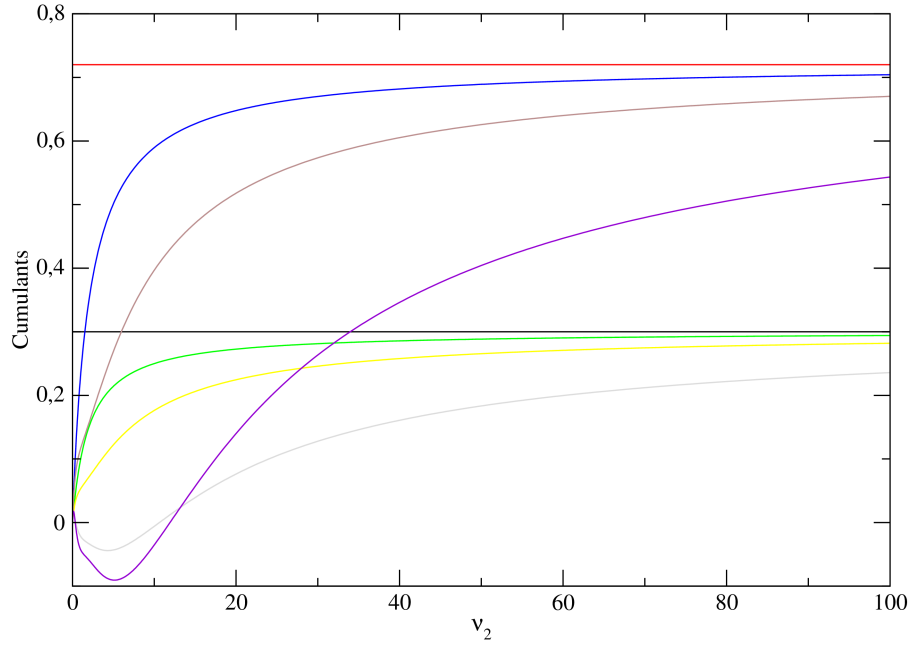


Figure 1: First infinite-time cumulants per unit time as functions of ν_2/ν_1 . Illustration of the large ν_2/ν_1 behavior with the two asymptotes, $\frac{\nu_1}{2}(\gamma_1 - \gamma_2)$ for odd cumulants and $\frac{\nu_1}{2}(1 - \gamma_1\gamma_2)$ for even cumulants. The other model parameters are fixed to the sample values $\gamma_1 = 0.7$, $\gamma_2 = 0.4$, $\nu_1 = 2$.

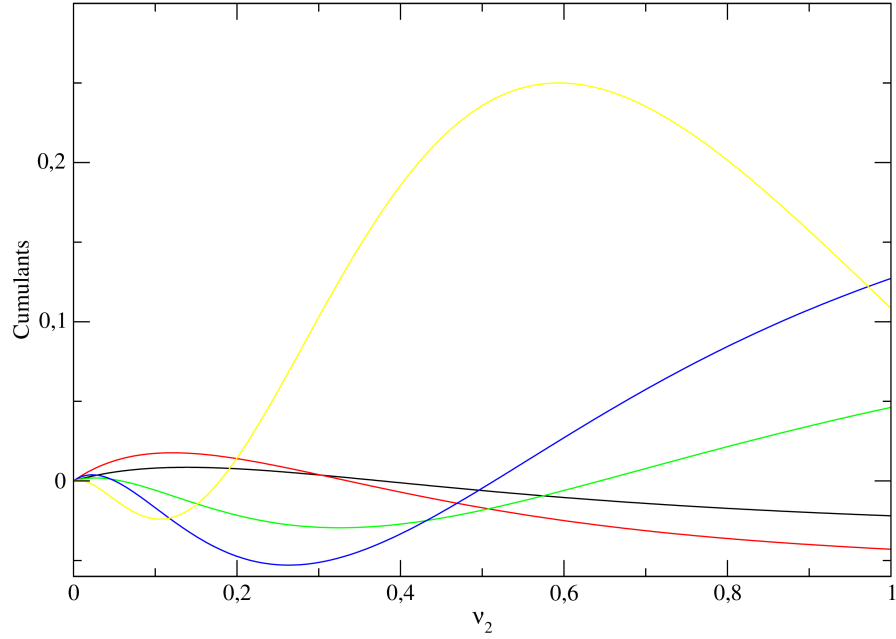


Figure 2: Higher cumulants. Illustration of the oscillations at finite ν_2/ν_1 . The other model parameters are fixed to the sample values $\gamma_1 = 0.7$, $\gamma_2 = 0.4$, $\nu_1 = 2$.

Only the first cumulants have analytic expressions simple enough to fit on a line. For the sake of conciseness, the results are first expressed in terms of the dimensionless time $\tau = \frac{1}{2}(\nu_1 + \nu_2)t$ as

$$\begin{aligned}\lim_{\tau \rightarrow +\infty} \frac{1}{\tau} \kappa_{n_2}^{[1]} &= B \\ \lim_{\tau \rightarrow +\infty} \frac{1}{\tau} \kappa_{n_2}^{[2]} &= A - B^2 \\ \lim_{\tau \rightarrow +\infty} \frac{1}{\tau} \kappa_{n_2}^{[3]} &= B [1 - 3A + 3B^2] \\ \lim_{\tau \rightarrow +\infty} \frac{1}{\tau} \kappa_{n_2}^{[4]} &= A - 3A^2 + B^2 [-4 + 18A - 15B^2].\end{aligned}\tag{5.15}$$

All odd cumulants are proportional to B , because all odd powers of $\bar{\lambda}$ in the expansion of the expression (5.14) for $\alpha_2(\bar{\lambda})$ are proportional to B . The first three cumulants are rewritten in terms of the model parameters as

$$\begin{aligned}\lim_{t \rightarrow +\infty} \frac{\langle Q_2 \rangle_{\text{st}}}{t} &= \bar{\nu}_1 \bar{\nu}_2 (\gamma_1 - \gamma_2) \frac{(\nu_1 + \nu_2) \Delta e}{2} = \langle j_2 \rangle_{\text{st}} \\ \lim_{t \rightarrow +\infty} \frac{\langle Q_2^2 \rangle_{\text{st}} - \langle Q_2 \rangle_{\text{st}}^2}{t} &= \bar{\nu}_1 \bar{\nu}_2 [1 - \gamma_1 \gamma_2 - \bar{\nu}_1 \bar{\nu}_2 (\gamma_1 - \gamma_2)^2] \frac{(\nu_1 + \nu_2) (\Delta e)^2}{2} \\ \lim_{t \rightarrow +\infty} \frac{\langle Q_2^3 \rangle_{\text{st}}^c}{t} &= \bar{\nu}_1 \bar{\nu}_2 (\gamma_1 - \gamma_2) \left[1 - 3\bar{\nu}_1 \bar{\nu}_2 (1 - \gamma_1 \gamma_2) + 3\bar{\nu}_1^2 \bar{\nu}_2^2 (\gamma_1 - \gamma_2)^2 \right] \frac{(\nu_1 + \nu_2) (\Delta e)^3}{2}.\end{aligned}\tag{5.16}$$

$\langle Q_2^3 \rangle^c$ is the third cumulant, which is equal to the third centered moment, namely $\langle Q_2^3 \rangle^c \equiv \langle [Q_2 - \langle Q_2 \rangle]^3 \rangle$.

At equilibrium $\gamma_1 = \gamma_2$, so that $B = 0$: then, by virtue of the remark after (5.15), the long-time behavior of all odd cumulants of Q_2 is subdominant with respect to the elapsed time t , and in the long-time limit $P(Q_2; t)$ becomes an even function of Q_2 at leading order in time t . The fourth cumulant of the cumulated heat Q_2 received from the thermostat 2 per unit time does not vanish: $\lim_{t \rightarrow +\infty} \frac{1}{t} \ln \langle e^{\lambda Q_2} \rangle_{\text{eq}}$ is not quadratic in λ , and even in the long time limit the variable Q_2 has a non-Gaussian distribution, contrarily to the variable $[Q_2 - \langle Q_2 \rangle] / \sqrt{t}$ (for which all cumulants of order larger than 3 vanish in the infinite time limit). The first two even cumulants per unit time read

$$\begin{aligned}\lim_{t \rightarrow +\infty} \frac{\langle Q_2^2 \rangle_{\text{eq}} - \langle Q_2 \rangle_{\text{eq}}^2}{t} &= \bar{\nu}_1 \bar{\nu}_2 (1 - \gamma^2) \frac{(\nu_1 + \nu_2) (\Delta e)^2}{2} \\ \lim_{t \rightarrow +\infty} \frac{\langle Q_2^4 \rangle_{\text{eq}}^c}{t} &= \bar{\nu}_1 \bar{\nu}_2 (1 - \gamma^2) [1 - 3\bar{\nu}_1 \bar{\nu}_2 (1 - \gamma^2)] \frac{(\nu_1 + \nu_2) (\Delta e)^4}{2}.\end{aligned}\tag{5.17}$$

$\langle Q_2^4 \rangle^c$ denotes the fourth cumulant, which can be expressed as $\langle Q_2^4 \rangle^c = \langle [Q_2 - \langle Q_2 \rangle]^4 \rangle - 3\langle [Q_2 - \langle Q_2 \rangle]^2 \rangle^2$.

For a system weakly out of equilibrium the Einstein-Green-Kubo relation, namely

$$\lim_{(\beta_1, \beta_2) \rightarrow (\beta, \beta)} \frac{\langle j_2 \rangle_{\text{st}}}{\beta_1 - \beta_2} = \frac{1}{2} \lim_{t \rightarrow +\infty} \frac{\langle Q_2^2 \rangle_{\text{eq}} - (\langle Q_2 \rangle_{\text{eq}})^2}{t},\tag{5.18}$$

is indeed obeyed by the system, as it should be. This can be checked by comparing the expression (5.17) with the limit obtained when $(\beta_1, \beta_2) \rightarrow (\beta, \beta)$ for the ratio $\langle j_2 \rangle_{\text{st}} / (\beta_1 - \beta_2)$ which, by virtue of (3.27), reads

$$\frac{\langle j_2 \rangle_{\text{st}}}{\beta_1 - \beta_2} = \bar{\nu}_1 \bar{\nu}_2 \frac{\gamma_1 - \gamma_2}{(\beta_1 - \beta_2) \Delta e} \frac{(\nu_1 + \nu_2) (\Delta e)^2}{2}.\tag{5.19}$$

When the system is far from equilibrium, comparison of the latter expression for $\langle j_2 \rangle_{\text{st}} / (\beta_1 - \beta_2)$ with the expression (5.16) for the long-time limit of the second cumulant per unit time shows that $\langle j_2 \rangle_{\text{st}} / (\beta_1 - \beta_2) \neq \lim_{t \rightarrow +\infty} [\langle Q_2^2 \rangle_{\text{st}} - \langle Q_2 \rangle_{\text{st}}^2] / t$, as it should be (see subsection 5.3 of paper I). Indeed, by virtue of equation (5.14), $\alpha_2(\bar{\lambda})$ obeys the symmetry relation $\alpha_2(\bar{\lambda}) = \alpha_2(-\bar{\mathcal{F}} - \bar{\lambda})$ with $\bar{\mathcal{F}} = \ln(A + B) / (A - B) = (\beta_1 - \beta_2) \Delta e$, but $\alpha_2(\bar{\lambda})$ is not a quadratic function of $\bar{\lambda}$, i.e. Q_2 has a non-Gaussian distribution in the long-time limit.

6 Large deviation function for the cumulated heat current $\mathcal{Q}_2/t$

In this section, we derive the large deviation function for the cumulative heat current $\mathcal{Q}_2/t$ by three methods. The first one is based on the general theory of large deviations for the definition of large deviation functions and uses one of its cornerstones, the Gärtner-Ellis theorem. The second and the third rely on the fact that $\mathcal{Q}_2$ takes discrete values in a t -independent set, and uses an ad-hoc definition of large deviation functions (see Appendix E of paper I). Though the general theory of large deviations and the ad-hoc definition for discrete exchanged quantities do not have to be the same, the ad-hoc definition is nevertheless a sensible definition of large deviations. Physically, the general and the ad-hoc definition are expected to yield the same result in a case as simple as the two-spin system, and our explicit computations can be seen as a proof of this fact. A natural tool to compute the ad-hoc large deviation function is via a contour integral representation, but as we shall see below, this method is surprisingly tricky even for the simple two-spin system at hand. In contrast with the general theory of large deviations, the contour integral method is the basis of a systematic expansion at large times. However, corrections are less universal than the dominant term.

The cumulative heat current received from heat bath 2 during the time interval t takes the values $\mathcal{J} = \mathcal{Q}_2/t$, with $\mathcal{Q}_2 = n\Delta e$, n integer. By dimensional analysis, the large deviation function $f_{\mathcal{Q}_2}(\mathcal{J})$, which has the dimension of an inverse time, must be a function of

$$\tilde{\mathcal{J}} = \frac{\mathcal{J}}{\Delta e} = \frac{n}{t}, \quad (6.1)$$

and we shall often consider the expressions of

$$\tilde{f}_{\mathcal{Q}_2}(\tilde{\mathcal{J}}) \equiv f_{\mathcal{Q}_2}(\mathcal{J}) \quad (6.2)$$

rather than those of $f_{\mathcal{Q}_2}(\mathcal{J}) = \tilde{f}_{\mathcal{Q}_2}(\mathcal{J}/\Delta e)$. Moreover, the explicit calculations are more conveniently dealt with if, instead of considering $\tilde{\mathcal{J}}$, we introduce the dimensionless current $\bar{\mathcal{J}}$ associated with the dimensionless time $\tau = [(\nu_1 + \nu_2)/2]t$,

$$\bar{\mathcal{J}} = \frac{n}{\tau} = \frac{2}{(\nu_1 + \nu_2)} \tilde{\mathcal{J}}. \quad (6.3)$$

The dimensionless large deviation function $\bar{f}_{\mathcal{Q}_2}$ of $\bar{\mathcal{J}}$ is such that $t\tilde{f}_{\mathcal{Q}_2}(\tilde{\mathcal{J}}) = \tau\bar{f}_{\mathcal{Q}_2}(\bar{\mathcal{J}})$, and the expression of $\tilde{f}_{\mathcal{Q}_2}(\tilde{\mathcal{J}})$ can be retrieved from that for $\bar{f}_{\mathcal{Q}_2}(\bar{\mathcal{J}})$ through

$$\tilde{f}_{\mathcal{Q}_2}(\tilde{\mathcal{J}}) = \frac{\nu_1 + \nu_2}{2} \bar{f}_{\mathcal{Q}_2} \left(\frac{2}{(\nu_1 + \nu_2)} \tilde{\mathcal{J}} \right). \quad (6.4)$$

We notice that large deviation functions for other cumulative quantities are related to $f_{\mathcal{Q}_2}$. Indeed, in a system with a finite number of configurations $\mathcal{Q}_1 + \mathcal{Q}_2$ is bounded and, as a consequence of the general theory of large deviations (see e.g. paper I),

$$f_{\mathcal{Q}_1}(\mathcal{J}) = f_{\mathcal{Q}_2}(-\mathcal{J}). \quad (6.5)$$

In the same vein, as $\Delta_{\text{exch}} S = \beta_1 \mathcal{Q}_1 + \beta_2 \mathcal{Q}_2 = -(\beta_1 - \beta_2) \mathcal{Q}_2 + \beta_1 (\mathcal{Q}_1 + \mathcal{Q}_2)$, with $\mathcal{Q}_1 + \mathcal{Q}_2$ bounded, the large deviation function for $\Delta_{\text{exch}} S$ and that for $\mathcal{Q}_2$ satisfy the simple relation

$$f_{\Delta_{\text{exch}} S}(\mathcal{J}) = f_{\mathcal{Q}_2} \left(-\frac{\mathcal{J}}{\beta_1 - \beta_2} \right). \quad (6.6)$$

6.1 Derivation from Gärtner-Ellis theorem

6.1.1 Method

By analogy with (5.12), we introduce the dimensionless function

$$\bar{\alpha}_2(\bar{\lambda}) \equiv \lim_{\tau \rightarrow +\infty} \frac{1}{\tau} \ln \langle e^{\bar{\lambda} n_2} \rangle. \quad (6.7)$$

A simplified version of the Gärtner–Ellis theorem (see e.g. the review for physicists [24] or the mathematical point of view [25]) states that, if $\bar{\alpha}_2(\bar{\lambda})$ exists and is differentiable for all $\bar{\lambda}$ in $\mathbb{R}$, then the large deviation function of the current $\bar{j} = n_2/\tau$ exists and it can be calculated as the Legendre-Fenchel transform of $\bar{\alpha}_2(\bar{\lambda})$, namely, with the signs chosen in the definitions used in the present paper,

$$\bar{f}_{\mathcal{Q}_2}(\bar{j}) = \min_{\bar{\lambda} \in \mathbb{R}} \{ \bar{\alpha}_2(\bar{\lambda}) - \bar{\lambda} \bar{j} \}. \quad (6.8)$$

As a consequence, since $\bar{\alpha}_2(\bar{\lambda})$ obeys the symmetry $\bar{\alpha}_2(\bar{\lambda}) = \bar{\alpha}_2(-(\beta_1 - \beta_2)\Delta e - \bar{\lambda})$ (as can be checked from (5.14)), $\bar{f}(\bar{j})$ obeys the fluctuation relation $\bar{f}(\bar{j}) - \bar{f}(-\bar{j}) = (\beta_1 - \beta_2)\Delta e \times \bar{j}$. Moreover, the cumulant generating function $\ln \langle e^{\bar{\lambda} n_2} \rangle$ is necessarily convex (downward). In the present case $\bar{\alpha}_2(\bar{\lambda})$ is strictly convex and continuously differentiable for all real $\bar{\lambda}$, so that the minimum in the definition of the Legendre-Fenchel transform can be readily calculated by using the Legendre transform,

$$\bar{f}_{\mathcal{Q}_2}(\bar{j}) = \bar{\alpha}_2(\bar{\lambda}_c(\bar{j})) - \bar{j} \bar{\lambda}_c(\bar{j}) \quad \text{with} \quad \frac{d\bar{\alpha}_2}{d\bar{\lambda}}(\bar{\lambda}_c) = \bar{j}. \quad (6.9)$$

6.1.2 Various expressions for $f_{\mathcal{Q}_2}$ and its properties

From the relation $\alpha_2(\bar{\lambda}) = [(\nu_1 + \nu_2)/2] \bar{\alpha}_2(\bar{\lambda})$ and the expression (5.14) for $\alpha_2(\bar{\lambda})$, when $T_1 \neq 0$ ($\gamma_1 \neq 1$), $A \neq B$ and we get

$$\bar{f}_{\mathcal{Q}_2}(\bar{j}) = \bar{j} \ln \sqrt{\frac{A+B}{A-B}} - |\bar{j}| \cosh^{-1} \left[\frac{Y(\bar{j})}{\sqrt{A^2 - B^2}} \right] - 1 + \sqrt{1 - 2A + 2Y(\bar{j})}. \quad (6.10)$$

$\cosh^{-1} x$ denotes the positive real whose hyperbolic cosine is equal to x , namely $\cosh^{-1} x = \ln [x + \sqrt{x^2 - 1}]$, and

$$Y(\bar{j}) = \bar{j}^2 + \sqrt{\bar{j}^4 + (1 - 2A)\bar{j}^2 + A^2 - B^2}. \quad (6.11)$$

The expression for $\bar{f}_{\mathcal{Q}_2}(\bar{j})$ involves the combinations of the model parameters

$$\ln \sqrt{\frac{A+B}{A-B}} = (\beta_1 - \beta_2) \frac{\Delta e}{2}, \quad A^2 - B^2 = \bar{\nu}_1^2 \bar{\nu}_2^2 (1 - \gamma_1^2) (1 - \gamma_2^2), \quad A = \bar{\nu}_1 \bar{\nu}_2 (1 - \gamma_1 \gamma_2). \quad (6.12)$$

The expression (6.10) for $\bar{f}_{\mathcal{Q}_2}(\bar{j})$ can be rewritten in two different forms according to the sign of $\bar{j}$. By using the identity $\cosh^{-1} x = \ln [x + \sqrt{x^2 - 1}]$, the $\cosh^{-1}$ term in (6.10) can be split into two contributions and, according to the sign of $\bar{j}$, we get

$$\bar{f}_{\mathcal{Q}_2}(\bar{j}) \underset{\bar{j} < 0}{=} -\bar{j} \ln(A - B) + \bar{j} \ln \left[Y(\bar{j}) + \sqrt{Y^2(\bar{j}) - (A^2 - B^2)} \right] - 1 + \sqrt{1 - 2A + 2Y(\bar{j})}, \quad (6.13)$$

while

$$\bar{f}_{\mathcal{Q}_2}(\bar{j}) \underset{\bar{j} > 0}{=} +\bar{j} \ln(A + B) - \bar{j} \ln \left[Y(\bar{j}) + \sqrt{Y^2(\bar{j}) - (A^2 - B^2)} \right] - 1 + \sqrt{1 - 2A + 2Y(\bar{j})}. \quad (6.14)$$

In the limit where T_1 vanishes ($A \rightarrow B$), the latter expressions yield the results discussed in section 8.

The thermodynamical and kinetic parameters of the heat baths are disentangled if, in place of A and B , we consider the parameters

$$p_+ = \frac{1}{2}(1 + \gamma_1)(1 - \gamma_2) \quad \text{and} \quad p_- = \frac{1}{2}(1 - \gamma_1)(1 + \gamma_2). \quad (6.15)$$

The relations with A and B are $A = \bar{\nu}_1 \bar{\nu}_2 (p_+ + p_-)$ and $B = \bar{\nu}_1 \bar{\nu}_2 (p_+ - p_-)$. Therefore, $(A + B)/(A - B) = p_+/p_-$, $\sqrt{A^2 - B^2} = 2\bar{\nu}_1 \bar{\nu}_2 \sqrt{p_+ p_-}$. Then, by virtue of the relation (6.4) and the expression (6.10) for $\bar{f}_{Q_2}(\bar{j})$, $\tilde{f}_{Q_2}(\tilde{j})$ reads

$$\tilde{f}_{Q_2}(\tilde{j}) = \tilde{j} \ln \sqrt{\frac{p_+}{p_-}} - |\tilde{j}| \cosh^{-1} \left[\frac{Z(\tilde{j})}{\sqrt{p_+ p_-}} \right] + \frac{\nu_1 + \nu_2}{2} \left[-1 + \sqrt{1 - 2\bar{\nu}_1 \bar{\nu}_2 [p_+ + p_- - 2Z(\tilde{j})]} \right]. \quad (6.16)$$

where, with the definition $Y(\bar{j})/\sqrt{A^2 - B^2} \equiv Z(\tilde{j})/\sqrt{p_+ p_-}$,

$$Z(\tilde{j}) = \frac{1}{\nu_1 \nu_2} \left[2\tilde{j}^2 + \sqrt{4\tilde{j}^4 + [(\nu_1 + \nu_2)^2 - 2\nu_1 \nu_2 (p_+ + p_-)] \tilde{j}^2 + (\nu_1 \nu_2)^2 p_+ p_-} \right]. \quad (6.17)$$

By virtue of the definitions (6.15) of p_+ and p_- , the thermodynamic parameters of the thermal baths appear in $\tilde{f}_{Q_2}(\tilde{j})$ through the following combinations

$$\sqrt{\frac{p_+}{p_-}} = e^{(\beta_1 - \beta_2) \frac{\Delta e}{2}}, \quad \sqrt{p_+ p_-} = \frac{1}{2} \sqrt{(1 - \gamma_1^2)(1 - \gamma_2^2)}, \quad p_+ + p_- = 1 - \gamma_1 \gamma_2. \quad (6.18)$$

At equilibrium the large deviation function is even. As $\gamma_1 - \gamma_2$ increases, the large deviation function becomes more and more asymmetric. In the zero temperature limit $\gamma_1 = 1$, the large deviation function becomes infinite for $\tilde{j} < 0$. Fig.3 illustrates the changes in the shape of the large deviation function, with increasing departure from equilibrium.

Some generic properties of a large deviation function can be checked in the case of the above explicit formulae. By virtue of (5.15) $\langle \bar{j} \rangle_{\text{st}} = B$ and one checks that $\bar{f}_{Q_2}(\langle \bar{j} \rangle_{\text{st}}) = 0$, $\bar{f}'_{Q_2}(\langle \bar{j} \rangle_{\text{st}}) = 0$ and $\bar{f}''_{Q_2}(\langle \bar{j} \rangle_{\text{st}}) = -1/(A - B^2)$, namely

$$\bar{f}''_{Q_2}(\langle \bar{j} \rangle_{\text{st}}) = - \left[\lim_{\tau \rightarrow +\infty} \frac{\kappa_{n_2}^{[2]}}{\tau} \right]^{-1} \quad (6.19)$$

The expression (6.10) for $\bar{f}_{Q_2}(\bar{j})$ is the sum of a term $\bar{j}(\beta_1 - \beta_2) \frac{\Delta e}{2}$ and an even function of $\bar{j}$. As a consequence, we check again that $\bar{f}_{Q_2}(\bar{j}) - \bar{f}_{Q_2}(-\bar{j}) = \bar{j}(\beta_1 - \beta_2) \Delta e$ namely, by virtue of (6.4), $f_{Q_2}(\mathcal{J})$ obeys the fluctuation relation $f_{Q_2}(\mathcal{J}) - f_{Q_2}(-\mathcal{J}) = (\beta_1 - \beta_2) \mathcal{J}$. Moreover the absolute value of $\bar{j}$ in the expression (6.10) for $\bar{f}_{Q_2}(\bar{j})$ is responsible for a (rather mild) singularity in the curve $f_{Q_2}(\mathcal{J})$ at $\mathcal{J} = 0$: a jump in the third derivative.

We notice that the large current behavior of $\bar{f}_{Q_2}(\bar{j})$ in the present model reads

$$\bar{f}_{Q_2}(\bar{j}) \underset{|\bar{j}| \rightarrow +\infty}{\sim} -2|\bar{j}| \ln |\bar{j}|. \quad (6.20)$$

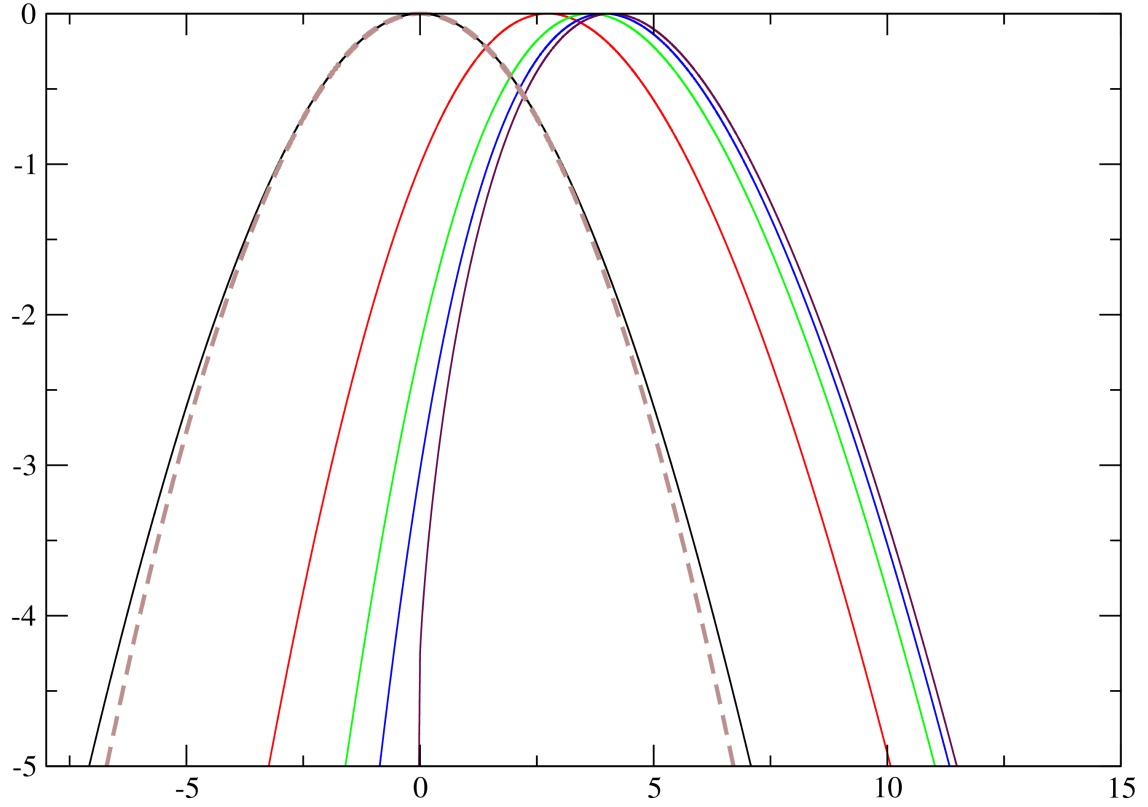


Figure 3: Effect of increasing the non-equilibrium driving parameter $\gamma_1 - \gamma_2$. A few large deviation functions are represented for various values of γ_1 , when the other parameters are fixed to the sample values $\gamma_2 = 0.1$, $\nu_1 = 100$, $\nu_2 = 10$. The leftmost curve (for $\gamma_1 = 0.1$) is the equilibrium large deviation function and the dashed curve is the quadratic with the same curvature at the origin. The rightmost curve is for $\gamma_1 = 1$ i.e. heat bath 1 at zero temperature. For the intermediate curves, from left to right, γ_1 takes the values 0.7, 0.9, 0.966667.

6.2 Derivation from a saddle-point method

Before embarking on the derivation, let us explain why the saddle point method is not straightforward for this model.

The saddle point approximation or expansion is well-suited for the asymptotic study of integrals of the form $\int_{\Gamma} dz \Psi(z) e^{\tau \Phi(z)}$ where τ is some large real parameter and the functions $\Psi(z)$, $\Phi(z)$ are holomorphic in a domain large enough that the initial integration contour Γ can be deformed to a steepest descent path while remaining within the holomorphicity domain during the deformation. One may also have to encircle some singularities when deforming the contour, and then one must keep track of their contributions, which may or may not dominate the saddle point contribution. This can of course be generalized to a finite sum $\sum_i \int_{\Gamma_i} dz \Psi_i(z) e^{\tau \Phi_i(z)}$ when each individual term satisfies the hypotheses above. Note however that to get the leading behavior one may have to take into account possible destructive interferences between different pieces, for instance if the real parts of saddle point values are the same for several Ψ_i 's, or if the saddle points for certain terms compete with encircled singularities for other terms.

In our case, we deal with an integral of the type $\int_{\Gamma} dz \sum_i \Psi_i(z) e^{\tau \Phi_i(z)}$ where there is a single integration contour, and the sum² $\sum_i \Psi_i(z) e^{\tau \Phi_i(z)}$ has nice holomorphicity properties that allow to deform contours (almost) freely, but each term in itself has singularities and cuts. So we have to face a kind of dilemma: either we want to keep holomorphicity, then the large parameter does not appear in an exponential – and to our knowledge no straightforward constant phase technique applies – or we look at each pure exponential piece individually, and then some branch cuts may prevent from deforming the contour purely as a constant phase steepest descent path: the steepest descent path is not closed, some parts of the original path are deformed along the cuts and they may dominate the saddle point. But also, the contribution of the pure exponential pieces may interfere. In our case, we have managed to show that in fact the interferences between contributions of one pure exponential and cut contributions from another pure exponential are destructive (with reminder terms controlled explicitly), leaving the contribution of only a single saddle point (not one saddle point for each pure exponential). But our argument relies on some tricks and features that appear to us at this stage as coincidences: we have not been able to identify a general framework avoiding our tedious analysis. And indeed, examples are known [26, 27] where (depending possibly on parameters) the cut contributions do or do not dominate the saddle point.

To conclude these comments, let us mention one general direction that seems worth pursuing, though we have not been able to use it to simplify significantly our argument even in our simple case. In physical problems, the functions $\Phi_i(z)$ will often be closely related to the different branches of a single algebraic function. For instance, the functions $\Phi_i(z)$ are often closely related to the eigenvalues of some z -dependent matrix. So a natural route would be to regard the integrals not in the z plane, but on the appropriate uniformizing Riemann surface, in our case an elliptic curve.

We now turn to the detailed analysis.

6.2.1 Method

The current probability density $\Pi(\bar{j}; \tau)$ is related to the probability $P(\mathcal{Q}_2/(\tau \Delta e) \in [\bar{j}, \bar{j} + d\bar{j}]; \tau)$ by the definition $P(\mathcal{Q}_2/(\tau \Delta e) \in [\bar{j}, \bar{j} + d\bar{j}]; \tau) \equiv \Pi(\bar{j}; \tau) d\bar{j}$. Since $\mathcal{Q}_2/\Delta e$ can take only integer values, the density distribution $\Pi(\bar{j}; \tau)$ is a sum of Dirac distributions

$$\Pi(\bar{j}; \tau) = \sum_{n=-\infty}^{+\infty} \delta\left(\bar{j} - \frac{n}{\tau}\right) \tau P\left(\frac{\mathcal{Q}_2}{\Delta e} = n; \tau\right). \quad (6.21)$$

In the long-time limit

$$\Pi(\bar{j}; \tau) \underset{\tau \rightarrow +\infty}{\sim} \sum_{n=-\infty}^{+\infty} \delta\left(\bar{j} - \frac{n}{\tau}\right) \tau P^{\text{as}}(\bar{j}; \tau) \quad (6.22)$$

²Which in our case consists of only two terms.

where $P^{\text{as}}(\bar{j}; \tau)$ is a function of the continuous parameter $\bar{j}$ that we shall compute below, and which is such that the following asymptotic behavior holds:

$$P\left(\frac{Q_2}{\Delta e} = \tau\bar{j}; \tau\right) \Big|_{\tau\bar{j} \text{ integer}} \underset{\tau \rightarrow +\infty}{\sim} P^{\text{as}}(\bar{j}; \tau). \quad (6.23)$$

The notation $g(\bar{j}, \tau)|_{\tau\bar{j} \text{ integer}}$ is a reminder of the rule that if the function $g(\bar{j}, \tau)$ is given by an integral representation, the latter must be calculated in the case where $\tau\bar{j}$ is an integer. By using one of the ad-hoc definitions of the large deviation function introduced in Appendix E.2 of paper I, the function $P^{\text{as}}(\bar{j}; \tau)$ can be rewritten as

$$P^{\text{as}}(\bar{j}; \tau) = A(\bar{j}, \tau) e^{\tau \bar{f}_{Q_2}(\bar{j})} \quad \text{with} \quad \lim_{\tau \rightarrow +\infty} \frac{1}{\tau} \ln A(\bar{j}, \tau) = 0. \quad (6.24)$$

When one is interested only in the large deviation function, the only information to be retained from the latter equation is merely

$$\bar{f}_{Q_2}(\bar{j}) = \lim_{\tau \rightarrow +\infty} \frac{1}{\tau} \ln P(Q_2 = \tau\bar{j}\Delta e; \tau) \Big|_{\tau\bar{j} \text{ integer}}. \quad (6.25)$$

Consequently, $\bar{f}_{Q_2}(\bar{j})$ can be investigated by means of a saddle-point method applied to the representation of $P(Q_2 = \tau\bar{j}\Delta e; \tau)|_{\tau\bar{j} \text{ integer}}$ in the complex z plane given by (4.45). In the latter expression $P(Q_2 = n\Delta e; \tau)$ is equal to $e^{-\tau}$ times a linear combination of $c_n(\tau)$, $s_n(\tau)$, $s_{n+1}(\tau)$ and $s_{n-1}(\tau)$. When $T_1 = 0$ the expressions (4.34) and (4.35) of the latter functions are convenient for studying the large τ behavior of $c_{n=\tau\bar{j}}(\tau)$, $s_{n=\tau\bar{j}}(\tau)$. When $T_1 \neq 0$ the study is slightly more complicated and it is more conveniently performed by considering the related coefficients defined by $c_n(\tau) = \rho^n \tilde{c}_n(\tau)$ and $s_n(\tau) = \rho^n \tilde{s}_n(\tau)$ where the expression (4.53) of ρ is finite when $T_1 \neq 0$. We present the details in the case where $T_1 \neq 0$.

When $T_1 \neq 0$, in the long-time limit, we have to consider the behaviors of the functions

$$K_c(\bar{j}; \tau) \equiv \tilde{c}_{n=\tau\bar{j}}(\tau) = \oint_{|z|=1} \frac{dz}{2\pi i} \frac{1}{z^{\tau\bar{j}+1}} \cosh\left(\tau \sqrt{\tilde{\Delta}_+(z)}\right) \quad (6.26)$$

and

$$K_s^{(\Delta n)}(\bar{j}; \tau) \equiv \tilde{s}_{\tau\bar{j}+\Delta n}(\tau) = \oint_{|z|=1} \frac{dz}{2\pi i} \frac{1}{z^{\Delta n}} \frac{1}{z^{\tau\bar{j}+1}} \frac{\sinh\left(\tau \sqrt{\tilde{\Delta}_+(z)}\right)}{\sqrt{\tilde{\Delta}_+(z)}} \quad (6.27)$$

with $\Delta n = 0, 1, -1$. It is sufficient to exhibit the derivation of the long-time behavior of $K_c(\bar{j}; \tau)$, because the calculation of the long-time behavior of $K_s^{(\Delta n)}(\bar{j}; \tau)$ follows the same lines. Moreover, according to the property $\tilde{c}_n(\tau) = \tilde{c}_{|n|}(\tau)$, we have to consider only the case where $\bar{j} > 0$.

For the study of the large τ limit, the cosh function in the integrand of $K_c(\bar{j}; \tau)$ is split into two exponentials, and $K_c(\bar{j}; \tau)$ appears as the sum of two integrals

$$2K_c(\bar{j}; \tau) = \oint_{|z|=1} \frac{dz}{2\pi i} \frac{1}{z} e^{\tau \Phi^{(+)}(z)} + \oint_{|z|=1} \frac{dz}{2\pi i} \frac{1}{z} e^{\tau \Phi^{(-)}(z)} \quad (6.28)$$

where

$$\Phi^{(\pm)}(z) = -\bar{j} \ln z \pm \sqrt{b + \frac{a}{2} \left(z + \frac{1}{z}\right)}. \quad (6.29)$$

We notice that, since $\tau\bar{j}$ is in fact an integer, $\exp(-\tau\bar{j} \ln z)$ is single valued and there is no cut in the complex plane z associated with the logarithmic function. However, since the cosh function has been split into two exponentials, we have to consider the two cuts associated with $\sqrt{\tilde{\Delta}_+(z)}$. These cuts are

$$]-\infty, -x_>] \quad \text{and} \quad [-x_<, 0] \quad (6.30)$$

where $-x_>$ and $-x_<$ are the two negative real roots of the second-order polynomial $z\tilde{\Delta}_+(z)$ where $\tilde{\Delta}_+(z)$ is given in (4.56). The roots are such that $0 \leq x_< < 1 < x_>$.

6.2.2 Deformation of contours

The large τ behavior of $K_c(\bar{j}; \tau)$ can be investigated by applying the saddle-point method to the contribution from the integral involving $\Phi^{(+)}(z)$. For that purpose we have to find a way to deform the unit circle into a contour that goes through a saddle point along a constant phase path where $\Phi^{(+)}(z)$ is maximum at the saddle point. It can be easily found that the function $\Phi^{(+)}(z)$ has two real saddle points where $\Phi^{(+)}(z)$ as well as its second derivative are real, but only one of them corresponds to a maximum of $\Phi^{(+)}(z)$ when the real axis is crossed perpendicularly. The latter saddle point is $x_c = \exp[\cosh^{-1}(x_+(\bar{j}))]$, namely by using $\cosh^{-1}(x) = \ln[x + \sqrt{x^2 - 1}]$,

$$x_c = x_+(\bar{j}) + \sqrt{x_+^2(\bar{j}) - 1} \quad (6.31)$$

with

$$x_+(\bar{j}) \equiv \frac{2}{a} \left[\bar{j}^2 + \sqrt{\bar{j}^4 + b\bar{j}^2 + \frac{a^2}{4}} \right]. \quad (6.32)$$

The constant phase contour Γ^* which crosses the real axis perpendicularly at x_c can be looked for in the form $z_*(\theta) = e^{\lambda_*(\theta) + i\theta}$. It proves to be

$$z_*(\theta) = e^{\cosh^{-1}(y_*(\theta; \bar{j})) + i\theta} \quad (6.33)$$

where $\theta \in]-\pi, \pi[$ and

$$y_*(\theta; \bar{j}) = u(\theta; \bar{j}) \cos \theta + \sqrt{1 + \frac{2b}{a} u(\theta; \bar{j}) + u^2(\theta; \bar{j})} \quad (6.34)$$

with

$$u(\theta; \bar{j}) = \frac{\theta^2}{\sin^2 \theta} \frac{2\bar{j}^2}{a}. \quad (6.35)$$

The contour Γ^* crosses the negative real axis at the point $z_*(\theta = \pi) \equiv -x_*(\bar{j})$ with

$$-x_*(\bar{j}) = -\exp \left[\cosh^{-1} \left(\frac{b}{a} + \pi^2 \frac{\bar{j}^2}{a} \right) \right] \quad (6.36)$$

which lies on the cut $] -\infty, -x_>]$, because $-x_> = -\exp[-\cosh^{-1}(\frac{b}{a})]$. As a consequence, the unit circle can be deformed into the contour Γ^* and a contour $\mathcal{C}_{[-x_*, -x_>]}^{(+)}$ that goes around the cut $] -\infty, -x_>]$ between the points $-x_*(\bar{j})$ and $-x_>$ in the clockwise sense (see Fig.4)

$$\oint_{|z|=1} \frac{dz}{2\pi i z} e^{\tau \Phi^{(+)}(z)} = \oint_{\Gamma^*} \frac{dz}{2\pi i z} e^{\tau \Phi^{(+)}(z)} + \oint_{\mathcal{C}_{[-x_*, -x_>]}^{(+)}} \frac{dz}{2\pi i z} e^{\tau \Phi^{(+)}(z)}. \quad (6.37)$$

On the other hand, in the integral involving $\Phi^{(-)}(z)$ the unit circle can be deformed into a circle, minus the point on the negative real axis, with radius R that goes to infinity and a path around the cut $] -\infty, -x_>]$. By using the parametrization $z = R e^{i\theta}$, with $\theta \neq \pi$, we get the following large $|z|$ behavior: $|\exp[\tau \Phi^{(-)}(z)]| \underset{|z| \rightarrow +\infty}{\sim} \exp[-\tau \sqrt{aR} \cos(\theta/2)]$, so that the contribution of the integral along a circle of radius R vanishes in the limit where R goes to infinity. Consequently, (see Fig.5),

$$\oint_{|z|=1} \frac{dz}{2\pi i z} e^{\tau \Phi^{(-)}(z)} = \oint_{\mathcal{C}_{[-\infty, -x_>]}^{(+)}} \frac{dz}{2\pi i z} e^{\tau \Phi^{(-)}(z)}. \quad (6.38)$$

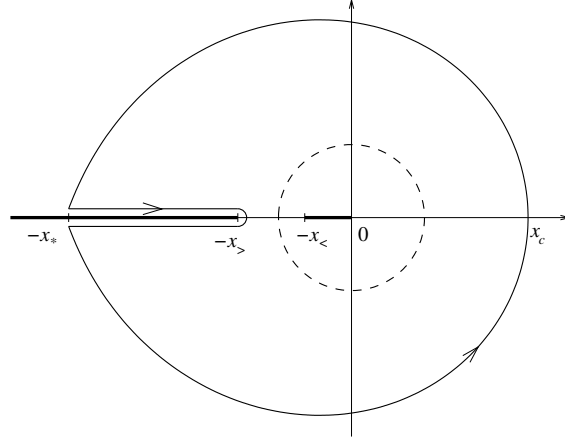


Figure 4: Deformed contour for $e^{\tau\Phi^{(+)}(z)}$ which decomposes as a steepest-descent contour Γ^* which goes through the saddle-point x_c in the anti clockwise sense and a piece which circumvents part of the cut $]-\infty, -x_>]$ in the clockwise sense. The other cut $[-x_<, 0]$ lies inside the unit circle represented by a dotted line.

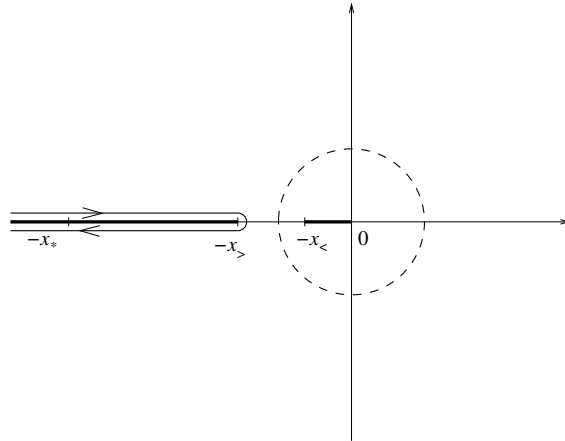


Figure 5: Deformed contour for $e^{\tau\Phi^{(-)}(z)}$ consisting of the circle at infinity, not represented here, and a path which circumvents the whole cut $]-\infty, -x_>]$ in the clockwise sense.

The crucial point is then to notice that the sum $e^{\tau\Phi^{(+)}(z)} + e^{\tau\Phi^{(-)}(z)}$ is an analytic function of z , which has no cut along the interval $] -\infty, -x_{>}]$. As a consequence, the integral along the contour $\mathcal{C}_{]-\infty, -x_{>}]^{(+)}$ with $\Phi^{(-)}(z)$ can be replaced by the opposite of the the same integral with $\Phi^{(+)}(z)$ in place of $\Phi^{(-)}(z)$, and the equality (6.38) becomes

$$\oint_{|z|=1} \frac{dz}{2\pi i z} e^{\tau\Phi^{(-)}(z)} = \oint_{\mathcal{C}_{]-\infty, -x_{>}]^{(-)}} \frac{dz}{2\pi i z} e^{\tau\Phi^{(+)}(z)}, \quad (6.39)$$

where $\mathcal{C}_{]-\infty, -x_{>}]^{(-)}$ is the contour which goes around the cut $] -\infty, -x_{>}]$ in the anti clockwise sense.

When the contributions (6.37) et (6.39) from the integrals involving respectively $\Phi^{(+)}$ and $\Phi^{(-)}$ are summed according to the definition (6.28) of $K_c(\bar{j}; \tau)$, we get

$$2K_c(\bar{j}; \tau) = \oint_{\Gamma_{\star}} \frac{dz}{2\pi i z} e^{\tau\Phi^{(+)}(z)} + \oint_{\mathcal{C}_{]-\infty, -x_{\star}]^{(-)}} \frac{dz}{2\pi i z} e^{\tau\Phi^{(+)}(z)} \quad (6.40)$$

We stress that $\exp[\tau\Phi^{(+)}(z)]$ diverges when $|z|$ goes to infinity, except on the negative real axis, so that the contour integral along the cut $] -\infty, -x_{\star}]$ cannot be closed at the point $z = -\infty$. The expression (6.40) corresponds to integrate $e^{\tau\Phi^{(+)}(z)}/(2\pi i z)$ along the contour in Fig.6.

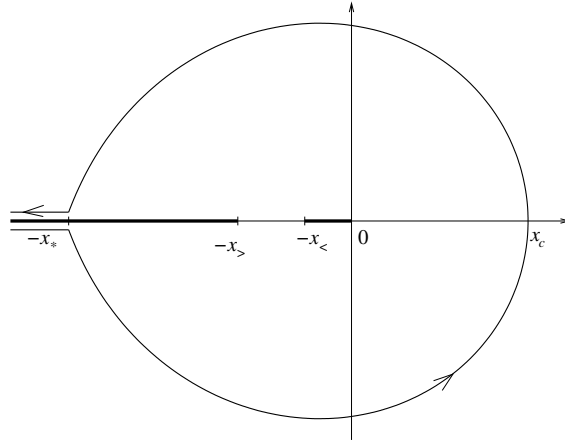


Figure 6: Contour of integration for $e^{\tau\Phi^{(+)}(z)}$ in the integral representation (6.40) of $K_c(\bar{j}; \tau)$.

On the contour $\mathcal{C}_{]-\infty, -x_{\star}]}$, $z = e^{\lambda + i\sigma\pi}$ where $\sigma = +1$ if z is above the cut and $\sigma = -1$ otherwise. Since $\tau\bar{j} = n$ where n is an integer

$$\oint_{\mathcal{C}_{]-\infty, -x_{\star}]^{(-)}} \frac{dz}{2\pi i z} e^{\tau\Phi^{(+)}(z)} = (-1)^n \int_{\ln x_{\star}}^{+\infty} \frac{d\lambda}{\pi} e^{-\tau\bar{j}\lambda} \sin\left(\tau\sqrt{a \cosh \lambda - b}\right). \quad (6.41)$$

The sign of this contribution changes for two consecutive values of $\bar{j}$, but its absolute value is bounded,

$$\left| \oint_{\mathcal{C}_{]-\infty, -x_{\star}]^{(-)}} \frac{dz}{2\pi i z} e^{\tau\Phi^{(+)}(z)} \right| \leq \frac{1}{\pi\tau\bar{j}} e^{-\tau\bar{j} \ln x_{\star}}. \quad (6.42)$$

6.2.3 Large τ behavior

According to the saddle-point formula

$$\oint_{\Gamma^*} \frac{dz}{2\pi i z} e^{\tau \Phi^{(+)}(z)} \underset{\tau \rightarrow +\infty}{\sim} \frac{1}{\sqrt{\tau}} \frac{1}{x_c \sqrt{2\pi \times d^2 \Phi^{(+)} / dz^2|_{z=x_c}}} e^{\tau \Phi^{(+)}(x_c)}. \quad (6.43)$$

By using the inequalities, $-\ln x_* < -\ln x_c$, derived from the expression (6.31)-(6.32) and (6.36), and $-\bar{j} \ln x_c \leq \Phi^{(+)}(x_c)$, derived from (6.29), the bound exhibited in (6.42) implies that

$$\oint_{\mathcal{C}_{[-\infty, -x_*]}^{(-)}} \frac{dz}{2\pi i z} e^{\tau \Phi^{(+)}(z)} = o\left(e^{\tau \Phi^{(+)}(x_c)}\right) \quad (6.44)$$

where $o(e^{\tau g(j)})$ denotes a function which decays faster than $e^{\tau g(j)}$ when τ goes to $+\infty$.

Eventually, the definition (6.26) of $K_c(\bar{j}; \tau)$ and the decomposition (6.40) together with the behaviors (6.43) et (6.44) lead to

$$\tilde{c}_{\tau \bar{j}}(\tau)|_{\tau \bar{j} \text{ integer}} \underset{\tau \rightarrow +\infty}{\sim} A_{\tilde{c}} \times e^{\tau \Phi^{(+)}(x_c)} \quad (6.45)$$

where

$$\Phi^{(+)}(x_c) = -\bar{j} \ln[x_+(\bar{j}) + \sqrt{x_+^2(\bar{j}) - 1}] + \sqrt{b + ax_+(\bar{j})} \quad (6.46)$$

and

$$A_{\tilde{c}} = \frac{1}{\sqrt{\tau}} \frac{1}{2\sqrt{2\pi}} \frac{1}{x_c \sqrt{d^2 \Phi^{(+)} / dz^2|_{z=x_c}}} \quad (6.47)$$

with $x_c = x_+(\bar{j}) + \sqrt{x_+^2(\bar{j}) - 1}$ and

$$d^2 \Phi^{(+)} / dz^2|_{z=x_c} = \frac{2}{a} \bar{j} \frac{\sqrt{\bar{j}^4 + b\bar{j}^2 + \frac{a^2}{4}}}{\left[x_+(\bar{j}) + \sqrt{x_+^2(\bar{j}) - 1}\right]^2 \sqrt{x_+^2(\bar{j}) - 1}}. \quad (6.48)$$

The same argument can be performed for $K_s^{(\Delta n)}(\bar{j}; \tau)$ defined in (6.27), with the result

$$\tilde{s}_{\tau \bar{j} + \Delta n}|_{\tau \bar{j} \text{ integer}}(\tau) \underset{\tau \rightarrow +\infty}{\sim} A_{\tilde{s}} \times e^{\tau \Phi^{(+)}(x_c)} \quad \text{with} \quad A_{\tilde{s}} = \left[x_c^{\Delta n} \sqrt{\tilde{\Delta}_+(x_c)}\right]^{-1} A_{\tilde{c}}. \quad (6.49)$$

As a consequence,

$$\lim_{\tau \rightarrow +\infty} \frac{1}{\tau} \ln [e^{-\tau} \rho^{\tau \bar{j}} \tilde{c}_{\tau \bar{j}}(\tau)] = \bar{f}(\bar{j}) \quad (6.50)$$

and

$$\lim_{\tau \rightarrow +\infty} \frac{1}{\tau} \ln [e^{-\tau} \rho^{\tau \bar{j}} \tilde{s}_{\tau \bar{j} + \Delta n}(\tau)] = \bar{f}(\bar{j}) \quad (6.51)$$

with $\bar{f}(\bar{j}) = -1 + \bar{j} \ln \rho + \Phi^{(+)}(x_c)$. By virtue of (4.53) $\ln \rho = \ln \sqrt{\frac{A+B}{A-B}}$ and, according to the definitions in (4.56), $\cosh^{-1}\left(\frac{1-b}{a}\right) = \cosh^{-1}\left(\frac{A}{\sqrt{A^2 - B^2}}\right) = \ln \sqrt{\frac{A+B}{A-B}}$. Therefore $\bar{f}(\bar{j})$ reads

$$\bar{f}(\bar{j}) = -1 + \bar{j} \cosh^{-1}\left(\frac{1-b}{a}\right) - |\bar{j}| \ln \left[x_+(\bar{j}) + \sqrt{x_+^2(\bar{j}) - 1}\right] + \sqrt{b + ax_+(\bar{j})} \quad (6.52)$$

where $x_+(\bar{j})$ is defined in (6.32).

Eventually, according to (4.45), $P\left(\frac{\mathcal{Q}_2}{\Delta e} = n; \tau\right)$ is a finite linear combination of functions of n plus a finite increment Δn , which can be rewritten as

$$P\left(\frac{\mathcal{Q}_2}{\Delta e} = n; \tau\right) = \sum_{n=\tau \bar{j}} \sum_{\Delta n=0,1,-1} b_{\Delta n} g_{\Delta n}(\tau \bar{j} + \Delta n; \tau), \quad (6.53)$$

and all functions $g_{\Delta n}$ prove to have the same “large deviation function” $\bar{f}(\bar{j})$ in the sense of definition (6.25),

$$g_{\Delta n}(\tau\bar{j} + \Delta n; \tau) \underset{\substack{\tau \rightarrow +\infty \\ \tau\bar{j} \text{ integer}}}{\sim} A_{\Delta n}(\bar{j}, \tau) e^{\tau\bar{f}(\bar{j})}. \quad (6.54)$$

Therefore, by comparison with (6.23) and (6.24) we get

$$\bar{f}_{\mathcal{Q}_2}(\bar{j}) = \bar{f}(\bar{j}) \quad \text{and} \quad A(\bar{j}, \tau) = \tau \sum_{\Delta n=0,1,-1} b_{\Delta n} A_{\Delta n}(\bar{j}, \tau) \quad (6.55)$$

where the expression (6.52) of $\bar{f}(\bar{j})$ indeed coincides with the result (6.10).

6.3 Derivation by Laplace’s method on a discrete sum

As the reader may have noticed, the computation of the large deviation function via contour integrals is a bit tricky and clumsy due to the cuts, and relies on some compensations which are not totally obvious to foresee.

In the case at hand it is possible to derive the large deviation function via Laplace’s method applied to a discrete sum of non-negative contributions. We illustrate this briefly in the case of $K_s^{(\Delta n)}$.

The key is an explicit formula for $\sin\left(\tau\sqrt{\tilde{\Delta}_+(z)}\right)/\sqrt{\tilde{\Delta}_+(z)}$ as a Laurent series in z . From the symmetry $z \leftrightarrow 1/z$ we can concentrate on positive powers of z . We start with

$$\frac{\sin\left(\tau\sqrt{\tilde{\Delta}_+(z)}\right)}{\sqrt{\tilde{\Delta}_+(z)}} = \sum_{k=0}^{+\infty} \frac{\tau^{2k+1}}{(2k+1)!} [\tilde{\Delta}_+(z)]^k \quad (6.56)$$

and expand $[\tilde{\Delta}_+(z)]^k$ as a Laurent polynomial in z ,

$$\left(b + \frac{a}{2} \left(z + \frac{1}{z}\right)\right)^k = \sum_{\substack{l,m \geq 0 \\ l+m \leq k}} z^{l-m} \left(\frac{a}{2}\right)^{l+m} b^{k-l-m} \frac{k!}{l!m!(k-l-m)!}. \quad (6.57)$$

So for $n \geq 0$ one gets (take $l = m + n$ above)

$$\oint_{|z|=1} \frac{dz}{2\pi i} \frac{1}{z^{n+1}} \frac{\sin\left(\tau\sqrt{\tilde{\Delta}_+(z)}\right)}{\sqrt{\tilde{\Delta}_+(z)}} = \sum_{m \geq 0} \sum_{k \geq 2m+n} \frac{\tau^{2k+1}}{(2k+1)!} \left(\frac{a}{2}\right)^{2m+n} b^{k-2m-n} \frac{k!}{m!(m+n)!(k-2m-n)!}. \quad (6.58)$$

As τ, a and b are > 0 , this is a (double) sum of positive terms, and we are interested in the limit

$$\tau \rightarrow +\infty \quad n = \tau\bar{j} + \Delta n \quad \text{with} \quad \Delta n \in \{0, -1, 1\}. \quad (6.59)$$

It is straightforward to check that in this limit the maximal term in the (double) sum is in the bulk (i.e. not for $m = 0$ or $k = 2m + n$) and such that k and m scale linearly with τ . One can use the Stirling approximation for all factorials and obtain the large deviation function straightforwardly, the most painful part of the computation being the location of the maximal term. We omit all details.

7 Dependence upon typical time scales of the thermostats

In this section we are interested in the influence of the typical time scales in the limit where the heat exchanges with one of the two thermostats become infinitely fast. We only explicitly consider the limit when ν_2/ν_1 goes to infinity, namely when heat bath 2 exchanges heat with the two-spin system far faster than heat bath 1 does. Indeed, the opposite limit when ν_2/ν_1 vanishes involves similar calculations (where the roles of $\mathcal{Q}_1^d \equiv -\mathcal{Q}_1$ and $\mathcal{Q}_2$ are interchanged), and the results in both limits are essentially the same when they are stated in terms of quantities pertaining either to the “fast” heat bath or to the “slow” heat bath. The main results are summarized with the latter terminology in subsection 1.2.

7.1 Stationary mean values in the infinite ν_2/ν_1 limit

The evolution of the probability distribution $P(\sigma_1, \sigma_2; t)$ is given in (4.39) where the time scales involve the parameter $\alpha = \sqrt{1 - 4A}$ with $A = \bar{\nu}_1 \bar{\nu}_2 (1 - \gamma_1 \gamma_2)$. When the ratio of inverse time scales ν_2/ν_1 goes to $+\infty$, $\lim_{\nu_2/\nu_1 \rightarrow +\infty} \frac{1}{2}(\nu_1 + \nu_2) [1 + \alpha] = \nu_2$ and $\lim_{\nu_2/\nu_1 \rightarrow +\infty} \frac{1}{2}(\nu_1 + \nu_2) [1 - \alpha] = \nu_1(1 - \gamma_1 \gamma_2)$, so that the probability distribution $P(\sigma_1, \sigma_2; t)$ goes exponentially fast to its stationary value over the time scale $1/[\nu_1(1 - \gamma_1 \gamma_2)]$. Moreover, since $\gamma_\star \equiv \bar{\nu}_1 \gamma_1 + \bar{\nu}_2 \gamma_2$, $\lim_{\nu_2/\nu_1 \rightarrow +\infty} \gamma_\star = \gamma_2$, and, according to (3.6), the stationary probability distribution $P_{\text{st}}(\sigma_1, \sigma_2)$ coincides with the canonical distribution at the inverse temperature β_2 of the fast thermostat, namely

$$\lim_{\nu_2/\nu_1 \rightarrow +\infty} P_{\text{st}}(\sigma_1, \sigma_2) = \frac{1}{4}(1 + \gamma_2 \sigma_1 \sigma_2). \quad (7.1)$$

As a consequence the heat capacities $C_{\text{st}}^{[a]}(T_1, T_2)$ ’s corresponding to a variation of the temperature T_a of bath a (for $a = 1, 2$) and given in (3.22), become $\lim_{\nu_2/\nu_1 \rightarrow +\infty} C_{\text{st}}^{[1]}(T_1, T_2) = 0$ and $\lim_{\nu_2/\nu_1 \rightarrow +\infty} C_{\text{st}}^{[2]}(T_1, T_2) = C_{\text{eq}}(T_2)$ respectively, while the heat capacity $C_{\text{st}}(T_1, T_2)$ corresponding to equal variations of both bath temperatures, and given at the end of (3.2.3), becomes $\lim_{\nu_2/\nu_1 \rightarrow +\infty} C_{\text{st}}(T_1, T_2) = C_{\text{eq}}(T_2)$.

On the other hand the time scale of the mean currents of exchanged quantities is that of the slow thermostat. Indeed, according to (3.27),

$$\lim_{\nu_2/\nu_1 \rightarrow +\infty} \langle j_2 \rangle_{\text{st}} = (\gamma_1 - \gamma_2) \frac{\nu_1 \Delta e}{2}, \quad (7.2)$$

and the thermal conductivity is determined by the typical time scale $1/\nu_1$ of the slow thermostat. Similarly the housekeeping entropy flow (3.47), which is equal to the opposite of the mean exchange entropy flow in the stationary state $d_{\text{exch}} S/dt|_{\text{st}}$, becomes

$$\lim_{\nu_2/\nu_1 \rightarrow +\infty} \sigma_{\text{hk}}[P_{\text{st}}] = \nu_1 (\gamma_1 - \gamma_2) (\beta_1 - \beta_2) \frac{\Delta e}{2}. \quad (7.3)$$

7.2 Various probabilities in the infinite ν_2/ν_1 limit

The stationary probability that the heat received from bath 1 is equal to $\mathcal{Q}_1 = -n_1 \Delta e$ is given by (4.44) where γ_0 is to be replaced by $\lim_{\nu_2/\nu_1 \rightarrow +\infty} \gamma_\star = \gamma_2$, with the result

$$\lim_{\nu_2/\nu_1 \rightarrow +\infty} P_{\text{st}}(n_1; t) = \lim_{\nu_2/\nu_1 \rightarrow +\infty} e^{-\tau} [c_{n_1}(\tau) + s_{n_1}(\tau)]. \quad (7.4)$$

According to the definitions (4.34) and (4.35), the integral representations of $c_n(\tau)$ and $s_n(\tau)$ in the complex plane involve the discriminant $\Delta_+(z)$. In order to discuss the dependence upon the inverse time scales ν_1 and ν_2 , it is convenient to rewrite the expression (4.22) of $\Delta_+(z)$ in terms of the parameters p_+ and p_- defined in (6.15). The correspondence with A and B reads $A = \bar{\nu}_1 \bar{\nu}_2 (p_+ + p_-)$ and $B = \bar{\nu}_1 \bar{\nu}_2 (p_+ - p_-)$, and

$$\Delta_+(z) = 1 + 2\bar{\nu}_1 \bar{\nu}_2 \left[-(p_+ + p_-) + p_+ z + p_- \frac{1}{z} \right]. \quad (7.5)$$

Therefore $\lim_{\nu_2/\nu_1 \rightarrow +\infty} \Delta_+(z) = 1$ and the leading order of $\frac{1}{2}(\nu_1 + \nu_2)[-1 + \sqrt{\Delta_+(z)}]$ is merely $\frac{1}{2}\nu_1[-(p_+ + p_-) + p_+z + p_-(1/z)]$. Consequently $e^{-\tau}c_{n_1}(\tau)$ and $e^{-\tau}s_{n_1}(\tau)$ have the same asymptotic behavior and

$$\lim_{\nu_2/\nu_1 \rightarrow +\infty} e^{-\tau}[c_n(\tau) + s_n(\tau)] = \oint_{|z|=1} \frac{dz}{2\pi i} \frac{1}{z^{n+1}} G_{RW}(z; \nu_1 t), \quad (7.6)$$

where

$$G_{RW}(z; \nu_1 t) = \exp \left\{ -\frac{1}{2}(p_+ + p_-)\nu_1 t + \frac{1}{2} \left[p_+ z + p_- \frac{1}{z} \right] \nu_1 t \right\}. \quad (7.7)$$

This expression can be interpreted as $G_{RW}(z; \nu_1 t) = \sum_{n=-\infty}^{+\infty} z^n P_{RW}(n; \nu_1 t)$, namely $G_{RW}(z; \nu_1 t)$ is the generating function of the probability $P_{RW}(n; \nu_1 t)$ for the continuous-time random walk, also referred to as the “randomized” random walk (see for instance page 59 of Ref.[28]), which is determined by the Markov evolution equation

$$\frac{dP_{RW}(n; \nu_1 t)}{dt} = \frac{\nu_1}{2} [-(p_+ + p_-)P_{RW}(n; \nu_1 t) + p_+ P_{RW}(n-1; \nu_1 t) + p_- P_{RW}(n+1; \nu_1 t)], \quad (7.8)$$

and the initial condition $P_{RW}(n; t=0) = \delta_{n,0}$. By virtue of the identity which defines the generating function of modified Bessel functions $I_n(x)$,

$$e^{\frac{1}{2}[p_+ z + p_- \frac{1}{z}]\nu_1 t} = \sum_{n=-\infty}^{+\infty} \left(z \sqrt{\frac{p_+}{p_-}} \right)^n I_n(\nu_1 t \sqrt{p_+ p_-}), \quad (7.9)$$

where

$$I_n(x) \equiv \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-in\theta} e^{x \cos \theta}. \quad (7.10)$$

As can be derived from the latter integral representation, the modified Bessel function $I_n(x)$ is an even function of n , $I_{-n} = I_n$. Therefore the series representation of the expression (7.7) for the generating function $G_{RW}(z; \nu_1 t)$ yields

$$P_{RW}(n; \nu_1 t) = \left(\sqrt{\frac{p_+}{p_-}} \right)^n I_{|n|}(\sqrt{p_+ p_-} \times \nu_1 t) e^{-\frac{1}{2}(p_+ + p_-)\nu_1 t}. \quad (7.11)$$

Eventually the limit in (7.4) reads

$$\lim_{\nu_2/\nu_1 \rightarrow +\infty} P_{\text{st}}(n_1; t) = P_{RW}(n_1; \nu_1 t), \quad (7.12)$$

where the thermodynamic parameters of the thermal baths appear through the combinations of p_+ and p_- explicitly given in (6.18). On the other hand the probability that at time t the system has received a heat amount $Q_2 = n_2 \Delta e$ from bath 2 is given by (4.45). Since $\lim_{\nu_2/\nu_1 \rightarrow +\infty} e^{-\tau}c_n(\tau) = \frac{1}{2}P_{RW}(n; \nu_1 t)$ and $\lim_{\nu_2/\nu_1 \rightarrow +\infty} e^{-\tau}s_n(\tau) = \frac{1}{2}P_{RW}(n; \nu_1 t)$,

$$\lim_{\nu_2/\nu_1 \rightarrow +\infty} P_{\text{st}}(n_2; t) = \frac{1 + \gamma_2^2}{2} P_{RW}(n_2; \nu_1 t) + \frac{1 - \gamma_2^2}{4} [P_{RW}(n_2 + 1; \nu_1 t) + P_{RW}(n_2 - 1; \nu_1 t)]. \quad (7.13)$$

We notice that at equilibrium, namely in the stationary state where $\beta_1 = \beta_2 = \beta$, $p_+ = p_-$ according to (6.18), and the limit (7.12) reads

$$\lim_{\nu_2/\nu_1 \rightarrow +\infty} P_{\text{eq}}(n_1; t) = I_{|n_1|} \left(\frac{1}{2}(1 - \gamma^2)\nu_1 t \right) e^{-\frac{1}{2}(1 - \gamma^2)\nu_1 t}. \quad (7.14)$$

Similar formulæ hold for the three contributions in $\lim_{\nu_2/\nu_1 \rightarrow +\infty} P_{\text{eq}}(n_2; t)$, which is derived from (7.13).

7.3 Interpretation: mean-field regime

When the initial state is distributed according to the stationary measure, the probability that at time t the system is in a configuration where $\sigma_1\sigma_2$ is equal to ± 1 and that the system has received a heat amount $\mathcal{Q}_1 = -n_1\Delta e$ from bath 1 is given by (4.42) and (4.43) where $\gamma_\star$ is to be replaced by $\lim_{\nu_2/\nu_1 \rightarrow +\infty} \gamma_\star = \gamma_2$, namely

$$\lim_{\nu_2/\nu_1 \rightarrow +\infty} P_{\text{st}}(\sigma_1\sigma_2 = \pm 1, n_1; t) = \frac{1}{2} [1 \pm \gamma_2] \lim_{\nu_2/\nu_1 \rightarrow +\infty} e^{-\tau} [c_{n_1}(\tau) + s_{n_1}(\tau)]. \quad (7.15)$$

Comparison with (7.1) and (7.4) shows that the latter equation can be interpreted as

$$P_{\text{st}}(\sigma_1\sigma_2 = \pm 1, n_1; t) \underset{\nu_2/\nu_1 \rightarrow +\infty}{\sim} P_{\text{st}}(\sigma_1\sigma_2 = \pm 1) \times P_{\text{st}}(n_1; t). \quad (7.16)$$

This is a mean-field property: between two flips of spin σ_1 , spin σ_2 is flipped so many times by thermostat 2 that, when spin σ_1 is flipped again, the sign of $\sigma_1\sigma_2$ is no longer correlated to its value when the previous flip of σ_1 occurred. Therefore the variation of n_1 , which is generated by the flip of σ_1 and the value of which is determined by the sign of $\sigma_1\sigma_2$, is no longer correlated to the sign which $\sigma_1\sigma_2$ had when the previous variation of n_1 occurred: the probability distributions of $\sigma_1\sigma_2$ and n_1 are independent from each other.

On the other hand, the probability that at time t the system is in a configuration where $\sigma_1\sigma_2$ is equal to 1 and that the system has received a heat amount $\mathcal{Q}_2 = n_2\Delta e$ from bath 2 has an expression given by the remark after (4.42) and (4.43). We get

$$\lim_{\nu_2/\nu_1 \rightarrow +\infty} P_{\text{st}}(\sigma_1\sigma_2 = +1, n_2; t) = \frac{(1 + \gamma_2)^2}{4} P_{\text{RW}}(n_2; \nu_1 t) + \frac{1 - \gamma_2^2}{4} P_{\text{RW}}(n_2 + 1; \nu_1 t), \quad (7.17)$$

while

$$\lim_{\nu_2/\nu_1 \rightarrow +\infty} P_{\text{st}}(\sigma_1\sigma_2 = -1, n_2; t) = \frac{(1 - \gamma_2)^2}{4} P_{\text{RW}}(n_2; \nu_1 t) + \frac{1 - \gamma_2^2}{4} P_{\text{RW}}(n_2 - 1; \nu_1 t). \quad (7.18)$$

Comparison with (7.13) shows that there is no factorization similar to (7.16). In other words the variables n_2 and $\sigma_1\sigma_2$ are still correlated.

7.4 Symmetry property specific to the probability of $\mathcal{Q}_1$

The probability $P_{\text{RW}}(n; \nu_1 t)$ for the continuous-time random walk, recalled in (7.11), obeys the symmetry

$$\ln \frac{P_{\text{RW}}(n; \nu_1 t)}{P_{\text{RW}}(-n; \nu_1 t)} = n \ln \frac{p_+}{p_-} \quad \text{at any time } t. \quad (7.19)$$

As a consequence, according to (7.12) and the relations (6.18), the probability distribution for the heat amount dissipated towards the slow bath, $\mathcal{Q}_1^{\text{d}} = -n_1\Delta e$, obeys the finite-time symmetry

$$\ln \frac{P_{\text{st}}(\mathcal{Q}_1^{\text{d}}; t)}{P_{\text{st}}(-\mathcal{Q}_1^{\text{d}}; t)} \underset{\nu_2/\nu_1 \rightarrow +\infty}{=} (\beta_1 - \beta_2) \mathcal{Q}_1^{\text{d}}. \quad (7.20)$$

However, by virtue of (7.13), there is no similar finite-time symmetry property for $P_{\text{st}}(\mathcal{Q}_2; t)$.

7.5 Cumulants per unit time for $\mathcal{Q}_1^{\text{d}}$ and $\mathcal{Q}_2$

At any time, according to (7.7), the characteristic function $G_{\text{RW}}(e^{\bar{\lambda}}; \nu_1 t)$ for the continuous-time random walk takes the very simple form

$$G_{\text{RW}}(e^{\bar{\lambda}}; \nu_1 t) = \exp[t\alpha_{\text{RW}}(\bar{\lambda}; \nu_1)] \quad (7.21)$$

where

$$\alpha_{RW}(\bar{\lambda}; \nu_1) = \frac{\nu_1}{2} \left[-(p_+ + p_-) + p_+ e^{\bar{\lambda}} + p_- e^{-\bar{\lambda}} \right]. \quad (7.22)$$

On the other hand, according to (7.12), $\sum_{n_1=-\infty}^{+\infty} e^{\bar{\lambda} n_1} \lim_{\nu_2/\nu_1 \rightarrow +\infty} P_{\text{st}}(n_1; t) = G_{RW}(e^{\bar{\lambda}}; \nu_1 t)$.

Therefore, the cumulants for $\mathcal{Q}_1^d$ are given at any finite time by the formulæ $\kappa_{n_1}^{[q]} = \partial^q \ln G_{RW}(e^{\bar{\lambda}}; \nu_1 t) / \partial \lambda^q|_{\lambda=0}$ and the cumulants per unit time read for $p \geq 0$

$$\frac{1}{t} \kappa_{n_1}^{[2p+1]} \Big|_{\nu_2/\nu_1 \rightarrow +\infty} = \frac{\nu_1}{2} (p_+ - p_-) = \frac{\nu_1}{2} (\gamma_1 - \gamma_2) \quad (7.23)$$

and for $p \geq 1$

$$\frac{1}{t} \kappa_{n_1}^{[2p]} \Big|_{\nu_2/\nu_1 \rightarrow +\infty} = \frac{\nu_1}{2} (p_+ + p_-) = \frac{\nu_1}{2} (1 - \gamma_1 \gamma_2). \quad (7.24)$$

In the case of $\mathcal{Q}_2$, by virtue of (7.13)

$$\sum_{n_2=-\infty}^{+\infty} e^{\bar{\lambda} n_2} P_{\text{st}}(n_2; t) \Big|_{\nu_2/\nu_1 \rightarrow +\infty} = \frac{1}{2} \left[1 + \gamma_2^2 + (1 - \gamma_2^2) \cosh \bar{\lambda} \right] G_{RW}(e^{\bar{\lambda}}; \nu_1 t). \quad (7.25)$$

The cumulants per unit time of $\mathcal{Q}_2$ coincide with the cumulants per unit time of $\mathcal{Q}_1^d$ only in the long-time limit, in agreement with (5.10) and because $\mathcal{Q}_1^d = -\mathcal{Q}_1 = n_1 \Delta e$,

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \kappa_{n_2}^{[p]} = \lim_{t \rightarrow +\infty} \frac{1}{t} \kappa_{n_1}^{[p]}. \quad (7.26)$$

When the system is at equilibrium $\gamma_1 = \gamma_2$ and the long-time behavior of all odd cumulants per unit time vanish, as already noticed in subsection 5.2.

We notice that the previous results can also be retrieved directly from the expression for the generating function $\alpha_2(\bar{\lambda})$ of the long-time cumulants per unit time. (The limits $\nu_2/\nu_1 \rightarrow +\infty$ and $t \rightarrow +\infty$ do commute with each other.) The expression of $\alpha_2(\bar{\lambda})$ is given in (5.14), and, according to the relations after (6.15), it reads

$$\alpha_2(\bar{\lambda}) = \frac{1}{2} \left\{ -(\nu_1 + \nu_2) + \sqrt{(\nu_1 + \nu_2)^2 + 2\nu_1 \nu_2 \left[-(p_+ + p_-) + p_+ e^{\bar{\lambda}} + p_- e^{-\bar{\lambda}} \right]} \right\}. \quad (7.27)$$

$\alpha_2(\bar{\lambda})$ is a symmetric function of ν_1 and ν_2 . In the limit $\nu_2/\nu_1 \rightarrow +\infty$ the generating function of the long-time cumulants per unit time becomes

$$\lim_{\nu_2/\nu_1 \rightarrow +\infty} \alpha_2(\bar{\lambda}) = \alpha_{RW}(\bar{\lambda}; \nu_1), \quad (7.28)$$

where $\alpha_{RW}(\bar{\lambda}; \nu_1)$ is given in (7.22).

7.6 Long-time current distribution in the infinite ν_2/ν_1 limit

According to the definition (7.10) of the modified Bessel function

$$I_{n=t\tilde{j}}(\alpha t) = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{tg(\theta; \tilde{j})} \quad \text{with} \quad g(\theta; \tilde{j}) = -i\tilde{j}\theta + \alpha \cos \theta, \quad (7.29)$$

where α denotes some parameter. In the complex plane where the affix reads $z = \theta + i\theta'$, $g(z; \tilde{j})$, the analytic continuation of $g(\theta; \tilde{j})$, is a periodic function of z with period 2π when $t\tilde{j}$ is equal to an integer. Therefore $I_{n=t\tilde{j}}$ can be rewritten as $I_{n=t\tilde{j}} = \int_{[-\pi, \pi]} (dz/2\pi) \exp[tg(z; \tilde{j})]$ and, by applying a saddle-point method to the latter integral, with a deformation of the initial contour in order to exhibit the constant phase path which goes through the saddle-point in the direction

where it is indeed a maximum, as done in subsection 6.2, one obtains (with the relevant saddle point $z_c = -i \ln[(|\tilde{j}|/\alpha) + \sqrt{(\tilde{j}/\alpha)^2 + 1}]$) that

$$I_{n=t\tilde{j}}(\alpha t) \underset{t \rightarrow +\infty}{\sim} \frac{1}{\sqrt{2\pi\alpha t \sqrt{(\tilde{j}/\alpha)^2 + 1}}} \exp \left[t \left(\alpha \sqrt{\left(\frac{\tilde{j}}{\alpha}\right)^2 + 1} - |\tilde{j}| \ln \left[\frac{|\tilde{j}|}{\alpha} + \sqrt{\left(\frac{\tilde{j}}{\alpha}\right)^2 + 1} \right] \right) \right]. \quad (7.30)$$

(The latter asymptotic behavior can also be directly read at page 378 of Ref.[29]). The long-time behavior of a current density $\Pi(\tilde{j}; t)$ is given in terms of $P(n = t\tilde{j}; t)$ by (6.22)-(6.23). From the expression (7.11) for $P_{RW}(n = t\tilde{j}; \nu_1 t)$ with $\alpha = \nu_1 \sqrt{p_+ p_-}$, we get

$$P_{RW}^{\text{as}}(\tilde{j}; t; \nu_1) = A_{RW}(\tilde{j}; t; \nu_1) e^{t f_{RW}(\tilde{j}; \nu_1)} \quad (7.31)$$

with $A_{RW}(\tilde{j}; t; \nu_1) = 1/\sqrt{2\pi\nu_1 t \sqrt{p_+ p_- + (\tilde{j}/\nu_1)^2}}$ and

$$f_{RW}(\tilde{j}; \nu_1) = -\frac{\nu_1}{2}(p_+ + p_-) + \tilde{j} \ln \sqrt{\frac{p_+}{p_-}} + |\tilde{j}| \ln \sqrt{p_+ p_-} + \nu_1 \sqrt{\frac{\tilde{j}^2}{\nu_1^2} + p_+ p_-} - |\tilde{j}| \ln \left[\frac{|\tilde{j}|}{\nu_1} + \sqrt{\frac{\tilde{j}^2}{\nu_1^2} + p_+ p_-} \right] \quad (7.32)$$

$f_{RW}(\tilde{j}; \nu_1)$ is the large deviation function for the randomized random walk described by the Markov equation (7.8). We notice that, as predicted by large deviation theory and in particular the Gärtner-Ellis theorem, $f_{RW}(\tilde{j}; \nu_1)$ can also be retrieved as the inverse Legendre transform of the generating function of the long-time cumulants per unit time $\alpha_{RW}(\bar{\lambda}; \nu_1)$ written in (7.22).

The probability density of the cumulative heat current $\mathcal{Q}_1^d/t$ in the long-time limit is given by (6.22), (7.12) and (7.31), with the result

$$\Pi^{\text{as}} \left(\frac{\mathcal{Q}_1^d}{t\Delta e} = \tilde{j}; t \right) \underset{\nu_2/\nu_1 \rightarrow +\infty}{=} \sum_{n=-\infty}^{+\infty} \delta \left(\tilde{j} - \frac{n}{t} \right) t A_{RW}(\tilde{j}; t; \nu_1) e^{t f_{RW}(\tilde{j}; \nu_1)} \quad (7.33)$$

Similarly, according to (7.13), by an argument similar to that leading to (6.55), the probability density of the cumulative heat current $\mathcal{Q}_2/t$ in the long-time limit is shown to read

$$\Pi^{\text{as}} \left(\frac{\mathcal{Q}_2}{t\Delta e} = \tilde{j}; t \right) \underset{\nu_2/\nu_1 \rightarrow +\infty}{=} \sum_{n=-\infty}^{+\infty} \delta \left(\tilde{j} - \frac{n}{t} \right) t A_2(\tilde{j}; t; \nu_1) e^{t f_{RW}(\tilde{j}; \nu_1)}, \quad (7.34)$$

with $A_2(\tilde{j}; t; \nu_1) \neq A_{RW}(\tilde{j}; t; \nu_1)$. We retrieve that the cumulative heat $\mathcal{Q}_1^d = n_1 \Delta e$ and $\mathcal{Q}_2 = n_2 \Delta e$ have the same large deviation function, and, more precisely,

$$\lim_{\nu_2/\nu_1 \rightarrow +\infty} f_{\mathcal{Q}_2}(\mathcal{J}) = f_{RW} \left(\frac{\mathcal{J}}{\Delta e}; \nu_1 \right). \quad (7.35)$$

We notice that the expression (7.32) for the large deviation function $f_{RW}(\tilde{j}; \nu_1)$ agrees with the limit of the expression (6.16) for $\tilde{f}_{\mathcal{Q}_2}(\tilde{j})$, when ν_2/ν_1 goes to infinity and $\tilde{j}/\nu_1$ is fixed. Indeed, in the expression (6.16), which is valid when $T_1 \neq 0$, the function $Z(\tilde{j})$ given in (6.17) is such that

$$\lim_{\substack{\nu_2/\nu_1 \rightarrow +\infty \\ \tilde{j}/\nu_1 \text{ fixed}}} Z(\tilde{j}) = \sqrt{\frac{\tilde{j}^2}{\nu_1^2} + p_+ p_-}, \quad (7.36)$$

while $(\nu_1 + \nu_2) \left(-1 + \sqrt{1 - 2\bar{\nu}_1 \bar{\nu}_2 [p_+ + p_- - 2Z(\tilde{j})]} \right) \sim -\nu_1 [p_+ + p_- - 2Z(\tilde{j})]$.

8 Case where $T_1 = 0$: pure energy dissipation towards thermal bath 1

8.1 Microscopic irreversibility

When the temperature T_1 of the colder bath vanishes, in the sense that $\beta_1 \Delta e$ goes to infinity, the microscopic reversibility (2.2) is broken,

$$(-\sigma, \sigma | \mathbb{W} | \sigma, \sigma) = 0 \quad \text{whereas} \quad (\sigma, \sigma | \mathbb{W} | -\sigma, \sigma) \neq 0 \quad (8.1)$$

by virtue of the expression (2.9) for the transition rates when $\gamma_1 = 1$. In other words the thermal bath at zero temperature cannot provide energy to the system, i.e., it cannot flip spin σ_1 if the flip corresponds to an increase of the two-spin system energy. There is only energy dissipation towards the zero-temperature bath.

When $\beta_1 \Delta e = +\infty$, the ratio in the modified detailed balance (2.3) vanishes or is infinite when two configurations differs from each other by the sign of σ_1 , and when spin σ_1 is flipped by thermostat 1 the corresponding variation of the thermostat entropy, $\delta S_1^{T^H}(\mathcal{C}' \leftarrow \mathcal{C}) \equiv -\beta_1 \delta q_1(\mathcal{C}' \leftarrow \mathcal{C})$ with definition (3.26), is infinite. All direct consequences of the modified detailed balance (2.3) are no longer valid.

However the Markov matrix (2.10) of the configurations evolution is still irreducible (see the definition after (2.10)) because histories such as

$$(\sigma, \sigma) \rightarrow (\sigma, -\sigma) \rightarrow (-\sigma, -\sigma) \rightarrow (-\sigma, \sigma) \quad (8.2)$$

correspond to a succession of flips with non-vanishing transition rates. Therefore, according to the Perron-Frobenius theorem, there still exists a single stationary distribution and in the latter distribution every configuration has a non-vanishing weight. The stationary probability given by (3.6) is still a canonical distribution with an effective inverse temperature $\beta_*^0 = (2/\Delta e) \tanh^{-1} \gamma_*^0$, with $\gamma_*^0 = \bar{\nu}_1 + \bar{\nu}_2 \gamma_2$,

$$\lim_{\beta_1 \Delta e \rightarrow +\infty} P_{\text{st}}(\sigma_1, \sigma_2) = \frac{1}{4} [1 + \gamma_*^0 \sigma_1 \sigma_2]. \quad (8.3)$$

According to (3.27), the mean current is finite,

$$\lim_{\beta_1 \Delta e \rightarrow +\infty} \langle j_2 \rangle_{\text{st}} = \frac{\nu_1 \nu_2}{\nu_1 + \nu_2} (1 - \gamma_2) \frac{\Delta e}{2}. \quad (8.4)$$

Since $\beta_1 \Delta e = +\infty$, the stationary exchange entropy flow (3.34) is infinitely negative in the stationary state

$$\lim_{\beta_1 \Delta e \rightarrow +\infty} \left. \frac{d_{\text{exch}} S}{dt} \right|_{\text{st}} = -\infty, \quad (8.5)$$

while the rate of entropy production, which has the opposite value in the stationary state, is infinitely positive, $\lim_{\beta_1 \Delta e \rightarrow +\infty} d_{\text{int}} S^{SG}/dt|_{\text{st}} = +\infty$. We also notice that the heat capacity with respect to a variation of the temperature T_1 from the zero value, $C_{\text{st}}^{[1]}(T_1 = 0, T_2)$, defined in (3.19) vanishes according to (3.22).

As a consequence of the fact that the thermal bath at zero temperature cannot give energy to the system, $\mathcal{Q}_1 = -n_1 \Delta e$ is necessarily negative and

$$P(\mathcal{Q}_1, \mathcal{Q}_2; t) = 0 \quad \text{if } \mathcal{Q}_1 > 0. \quad (8.6)$$

This can be checked on the explicit expressions of subsection 4.2 as follows. The probability that the system is in configuration (σ_1, σ_2) at time $t_0 = 0$, in configuration (σ'_1, σ'_2) at time t and receives $\mathcal{Q}_1 = -n \Delta e$ and $\mathcal{Q}_2 = (n + \Delta n) \Delta e$ during the time interval $[0, t]$ is $(\sigma'_1, \sigma'_2 | \mathbb{U}(n, n + \Delta n; t) | \sigma_1, \sigma_2)$. According to (4.36)-(4.37) where γ_1 is to be set equal to 1, the latter matrix elements involve the functions $c_n(\tau)$ and $s_n(\tau)$ defined in (4.34) and (4.35). When T_1 vanishes, γ_1 tends to 1, $A - B = \bar{\nu}_1 \bar{\nu}_2 (1 - \gamma_1)(1 + \gamma_2)$ goes to zero and, by virtue of (4.22), $\lim_{\beta_1 \Delta e \rightarrow +\infty} \Delta_+(z) = 1 - 2A + 2Az$. As

a consequence, $\cosh(\tau\sqrt{\Delta_+(z)}) = \sum_{p=0} [1/(2p)!] \tau^{2p} (1 - 2A + 2Az)^p$ contains only positive powers of z , and so does $\sinh(\tau\sqrt{\Delta_+(z)})/\sqrt{\Delta_+(z)}$. Consequently $c_n(\tau)$ and $s_n(\tau)$ vanish for $n < 0$ and $\lim_{\beta_1 \Delta e \rightarrow +\infty} (\sigma'_1, \sigma'_2 | \mathcal{U}(n, n + \Delta n; t) | \sigma_1, \sigma_2) = 0$ for any $n < 0$.

8.2 Long-time behavior

The explicit values of the infinite-time limit for the cumulants per unit time of the heats $\mathcal{Q}_1^d$ and $\mathcal{Q}_2$ can be calculated as in subsection 5.2. When T_1 vanishes, $\gamma_1 = 1$, A tends to $A^0 = \bar{\nu}_1 \bar{\nu}_2 (1 - \gamma_2)$, and $A - B$ vanishes. Then the expression (5.14) of $\alpha_2(\bar{\lambda})$ is reduced to

$$\lim_{\beta_1 \Delta e \rightarrow +\infty} \alpha_2(\bar{\lambda}) = \frac{1}{2} \left\{ -(\nu_1 + \nu_2) + \sqrt{(\nu_1 + \nu_2)^2 + 2\nu_1 \nu_2 (1 - \gamma_2)} [-1 + e^{\bar{\lambda}}] \right\}. \quad (8.7)$$

The expressions of the first three cumulants can be retrieved by setting $\gamma_1 = 1$ in the expressions (5.16).

As in section 6, the large deviation function can be derived either as the Legendre transform of $\alpha_2(\bar{\lambda})$ or by a saddle-point method similar to that performed in subsection 6.2 for $P_{P_0}(n_2; t)$ given by (4.45), which also provides the amplitude of the probability, or it can be retrieved directly by taking the limit $\gamma_1 \rightarrow 1$ in the expressions (6.13)-(6.14) for the large deviation function $\bar{f}_{\mathcal{Q}_2}(\bar{j})$, as follows. In the limit where T_1 vanishes, so does $A - B$, and $Y(\bar{j})$ tends to $Y^0(\bar{j}) = \bar{j} \left[\bar{j} + \sqrt{\bar{j}^2 + 1 - 2A^0} \right]$ where $A^0 = \bar{\nu}_1 \bar{\nu}_2 (1 - \gamma_2)$, while, according to (3.27) and (6.3),

$$\langle \bar{j} \rangle_{\text{st}}^0 = \bar{\nu}_1 \bar{\nu}_2 (1 - \gamma_2). \quad (8.8)$$

As a result,

$$\begin{aligned} \lim_{\beta_1 \Delta e \rightarrow +\infty} f_{\mathcal{Q}_2}(\mathcal{J}) &\stackrel{\mathcal{J} < 0}{=} -\infty \\ \lim_{\beta_1 \Delta e \rightarrow +\infty} f_{\mathcal{Q}_2}(\mathcal{J}) &\stackrel{\mathcal{J} > 0}{=} \frac{\nu_1 + \nu_2}{2} \left(-1 + [1 + \ln \langle \bar{j} \rangle_{\text{st}}^0] \bar{j} + \sqrt{\bar{j}^2 + 1 - 2\langle \bar{j} \rangle_{\text{st}}^0} - \bar{j} \ln \bar{j} \left[\bar{j} + \sqrt{\bar{j}^2 + 1 - 2\langle \bar{j} \rangle_{\text{st}}^0} \right] \right) \end{aligned} \quad (8.9)$$

where $\bar{j} = [2/(\nu_1 + \nu_2)] \mathcal{J} / \Delta e$.

8.3 Limit where ν_2/ν_1 becomes infinite

8.3.1 Finite-time behaviors

The discussion can be performed along the same lines as in section 7. We only point out the features which are qualitatively different when $\gamma_1 = 1$. The discrepancies are due to the fact that, according to (6.15), when $\gamma_1 = 1$, though p_+ remains finite,

$$p_- = 0, \quad (8.10)$$

and the random walk process associated to the variation of the heat amounts $\mathcal{Q}_1^d$ or $\mathcal{Q}_2$ can have only positive increments. As shown below, a Poisson process shows off as a randomized random walk which can proceed only in the sense of increasing positive n_2 .

Since p_- vanishes, while $p_+ = 1 - \gamma_2$, the expression (7.5) for $\Delta_+(z)$ becomes $\lim_{\beta_1 \Delta e \rightarrow +\infty} \Delta_+(z) = 1 + 2\bar{\nu}_1 \bar{\nu}_2 (1 - \gamma_2) [-1 + z]$ and the generating function $G_{RW}(z; \nu_1 t)$ which appears in subsection 7.2 (see (7.7)) is to be replaced by

$$G_{\text{Pois}}(z; \nu_1 t) = e^{[-1+z]p_+ \nu_1 t} \quad (8.11)$$

with

$$p_+ \nu_1 = \frac{1}{2} (1 - \gamma_2) \nu_1 = \lim_{\nu_2/\nu_1 \rightarrow +\infty} \langle \bar{j} \rangle_{\text{st}}^0 \equiv \langle \bar{j} \rangle_{\text{st}}^{0*}. \quad (8.12)$$

$G_{\text{Pois}}(z; \nu_1 t)$ is the generating function $G_{\text{Pois}}(z; \nu_1 t) = \sum_{n=0}^{+\infty} z^n P_{\text{Pois}}(n; \nu_1 t)$ for the Poisson process ruled by the Markov evolution equation

$$\frac{dP_{\text{Pois}}(n; \nu_1 t)}{dt} = p_+ \nu_1 [-P_{\text{Pois}}(n; \nu_1 t) + P_{\text{Pois}}(n-1; \nu_1 t)], \quad (8.13)$$

and the initial condition $P_{\text{Pois}}(n; t=0) = \delta_{n,0}$. The solution reads

$$P_{\text{Pois}}(n; \nu_1 t) = \frac{(p_+ \nu_1 t)^n}{n!} e^{-p_+ \nu_1 t}. \quad (8.14)$$

Eventually the probability that at time t the system has dissipated a heat amount $\mathcal{Q}_1^d = n_1 \Delta e$ towards bath 1 at zero temperature reads

$$\lim_{\nu_2/\nu_1 \rightarrow +\infty} \lim_{\beta_1 \Delta e \rightarrow +\infty} P_{\text{st}}(n_1; t) = P_{\text{Pois}}(n_1; \nu_1 t). \quad (8.15)$$

A formula similar to (7.13), where $P_{RW}(n_2; \nu_1 t)$ is to be replaced by $P_{\text{Pois}}(n_2; \nu_1 t)$, holds for the probability that at time t the system has received a heat amount $\mathcal{Q}_2 = n_2 \Delta e$ from bath 2.

8.3.2 Long-time behavior

The infinite-time cumulants per unit time for the heat $\mathcal{Q}_2$ can be obtained by noticing that, when ν_1/ν_2 vanishes, the expression (8.7) for $\lim_{\beta_1 \Delta e \rightarrow +\infty} \alpha_2(\bar{\lambda})$ becomes

$$\lim_{\nu_2/\nu_1 \rightarrow +\infty} \lim_{\beta_1 \Delta e \rightarrow +\infty} \alpha_2(\bar{\lambda}) = \alpha_{\text{Pois}}(\bar{\lambda}; \nu_1), \quad (8.16)$$

where

$$\alpha_{\text{Pois}}(\bar{\lambda}; \nu_1) = \frac{\nu_1}{2} (1 - \gamma_2) \left[-1 + e^{\bar{\lambda}} \right] \quad (8.17)$$

is the cumulant generating function for a Poisson process with average $\nu_1 p_+ = \frac{\nu_1}{2} (1 - \gamma_2)$. We notice that the two limits can be taken in the reverse order: by virtue of (7.28) $\lim_{\beta_1 \Delta e \rightarrow +\infty} \lim_{\nu_2/\nu_1 \rightarrow +\infty} \alpha_2(\bar{\lambda}) = \lim_{\beta_1 \Delta e \rightarrow +\infty} \alpha_{RW}(\bar{\lambda}; \nu_1) = \alpha_{\text{Pois}}(\bar{\lambda}; \nu_1)$. The long-time cumulants per unit time are all equal

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \left(\lim_{\nu_2/\nu_1 \rightarrow +\infty} \lim_{\beta_1 \Delta e \rightarrow +\infty} \kappa_{n_2}^{[p]} \right) = \frac{\nu_1}{2} (1 - \gamma_2). \quad (8.18)$$

The large deviation function can be retrieved

- either as the Legendre transform of $\alpha_2(\bar{\lambda}) = \alpha_{\text{Pois}}(\bar{\lambda}; \nu_1)$,
- or by a saddle-point method applied to the expression (8.14) of $P_{\text{Pois}}(n; \nu_1 t)$ and similar to that performed for $P_{RW}(n; \nu_1 t)$, with the result,

$$P_{\text{Pois}}(t\tilde{\mathcal{J}}; \nu_1 t) |_{t\tilde{\mathcal{J}} \text{ integer}} \underset{t \rightarrow +\infty}{\sim} \frac{1}{\sqrt{2\pi t\tilde{\mathcal{J}}}} e^{t f_{\text{Pois}}(\tilde{\mathcal{J}}; \nu_1)} \quad (8.19)$$

with

$$f_{\text{Pois}}(\tilde{\mathcal{J}}; \nu_1) = -\langle \tilde{\mathcal{J}} \rangle_{\text{st}}^{0*} + \tilde{\mathcal{J}} - \tilde{\mathcal{J}} \ln \frac{\tilde{\mathcal{J}}}{\langle \tilde{\mathcal{J}} \rangle_{\text{st}}^{0*}}, \quad (8.20)$$

where $\langle \tilde{\mathcal{J}} \rangle_{\text{st}}^{0*}$ is defined in (8.12),

- or directly by taking the limit $\nu_2/\nu_1 \rightarrow +\infty$ with $\tilde{\mathcal{J}}/\nu_1$ fixed in the expression for the large deviation function $\lim_{\beta_1 \Delta e \rightarrow +\infty} f_{\mathcal{Q}_2}(\mathcal{J})$ given in (8.9) (and by noticing, that $\langle \tilde{\mathcal{J}} \rangle_{\text{st}}^0$ is of order $\frac{\nu_1}{\nu_2}$ while $\tilde{\mathcal{J}}^2$ is of order $(\frac{\tilde{\mathcal{J}}}{\nu_1})^2 \times \left(\frac{\nu_1}{\nu_2}\right)^2$),

- or by taking first the expression for $\mathcal{J} > 0$ of $\lim_{\nu_2/\nu_1 \rightarrow +\infty} f_{\mathcal{Q}_2}(\mathcal{J})$ given by (7.35) and (7.32) and then taking the limit $\beta_1 \Delta e \rightarrow +\infty$, namely $p_- \rightarrow 0$.

Eventually, the large deviation function takes the simple form

$$\lim_{\nu_2/\nu_1 \rightarrow +\infty} \lim_{\beta_1 \Delta e \rightarrow +\infty} f_{\mathcal{Q}_2}(\mathcal{J}) \underset{\mathcal{J} < 0}{=} -\infty \quad (8.21)$$

$$\lim_{\nu_2/\nu_1 \rightarrow +\infty} \lim_{\beta_1 \Delta e \rightarrow +\infty} f_{\mathcal{Q}_2}(\mathcal{J}) \underset{\mathcal{J} > 0}{=} f_{\text{Pois}}\left(\frac{\mathcal{J}}{\Delta e}; \nu_1\right) = \frac{1}{\Delta e} \left[-\langle \mathcal{J}_2 \rangle_{\text{st}}^{0\star} + \mathcal{J} - \mathcal{J} \ln \frac{\mathcal{J}}{\langle \mathcal{J}_2 \rangle_{\text{st}}^{0\star}} \right]$$

where $\langle \mathcal{J}_2 \rangle_{\text{st}}^{0\star} = (\nu_1/2)(1 - \gamma_2)\Delta e$.

The expression of $\langle \mathcal{J}_2 \rangle_{\text{st}}^{0\star}$ can be interpreted as follows. Since bath 1 is at zero temperature, spin σ_1 may be flipped only when it is opposite to spin σ_2 . Moreover, since $\nu_2 \gg \nu_1$, once spin 1 has been flipped so that $\sigma_1\sigma_2 = 1$, on average spin 2 is flipped a great odd number of times with a net energy transfer Δe from heat bath 2 until spin σ_1 is again flipped with an energy transfer Δe to heat bath 1 so that $\sigma_1\sigma_2 = 1$ again. As a consequence the mean energy current through the spins system is equal to Δe times the typical inverse time ν_1 between two possible flips induced by thermal bath 1 times the probability that σ_2 is opposite to σ_1 , namely $(1 - \gamma_2)/2$.

8.4 Limit where ν_2/ν_1 tends to zero

In the reverse limit where $\nu_2 \ll \nu_1$, the roles of the two heat baths in the discussion of section 7 are interchanged (see comment after (4.42) and (4.43), as well as comparison of (4.44) and (4.45)). The slow thermostat is heat bath 2 and the evolution of $\mathcal{Q}_2$ is a Poisson process with the kinetic parameter ν_2 .

The stationary mean heat current received by the system is now $\langle \mathcal{J}_2 \rangle_{\text{st}}^{0\star} = (\nu_2/2)(1 - \gamma_2)\Delta e$. The interpretation of the latter expression is the following. Since $\nu_2 \ll \nu_1$, as soon as spin σ_2 is flipped to a value opposite to spin σ_1 with an energy transfer Δe from heat bath 2, bath 1 flips spin σ_1 so that $\sigma_1\sigma_2 = 1$ and an energy Δe is transferred to heat bath 1. The next flip can be only a flip of spin σ_2 and its probability per time unit is the value of the transition rate of spin 2 when $\sigma_1\sigma_2 = 1$, namely $(1/2)\nu_2(1 - \gamma_2)$. As a consequence the mean energy current through the spins system is equal to Δe times $(\nu_2/2)(1 - \gamma_2)$.

9 Thermal cycles

As recalled in the introduction, part of the physical relevance of the two-spin system is as an idealized mesoscopic thermal machine, with heat flowing from the high temperature reservoir to the low temperature reservoir in average. This flow of heat results from thermal cycles made by the system. After each thermal cycle, the two spins have returned to their original state, but an amount of heat $2\Delta e$ has been transferred from heat bath 2 (the hot bath) to heat bath 1 (the cold bath). Thermal fluctuations do occur however, and with this interpretation two questions come naturally. What is the distribution of the time needed to make a thermal cycle? What is the probability that the machine will perform a thermal cycle in the wrong direction?

The graph showing the possible transitions in the two-spin system looks as follows

$$\begin{array}{ccc} (+, +) & \leftrightarrow & (+, -) \\ \updownarrow & & \updownarrow \\ (-, +) & \leftrightarrow & (-, -) \end{array} .$$

This graph looks like a square, i.e. a cycle with 4 edges. Most of what we shall have to say applies equally well to a Markov process with a finite number of states and whose associated transition graph is a general cycle. As the probabilistic reasoning is more transparent in this more general framework, we shall devote a separate section to it for completeness. It is likely that the forthcoming analysis has already been performed (more than once) in the literature, but we have not found it.

9.1 Markov processes with a cyclic transition graph

We label the $n \geq 3$ configurations as $1, \dots, n$, and identify configuration m with configuration $m + n$ in all subsequent formulæ. We view the process as the motion of a particle along the cycle, jumping from time to time from a site to one of its two neighbors. We choose arbitrarily an orientation of the cycle. A jump from m to $m + 1$ (resp. $m - 1$) is said to be clockwise (resp. anti clockwise). We let c_m be the transition rate from configuration m to configuration $m + 1$ and a_m be the transition rate from configuration m to configuration $m - 1$. All other transition rates vanish. We set $b_m \equiv a_m + c_m$. If the particle sits at m at time t , the probability that the next jump will be clockwise (resp. anti clockwise) is c_m/b_m (resp. a_m/b_m). By saying that the graph associated to the Markov process is a cycle we mean that all a_m 's and c_m 's are > 0 . To be totally explicit, with the conventions of this article, the generator of the Markov process looks like

$$\begin{pmatrix} -b_1 & a_2 & 0 & \cdots & \cdots & 0 & c_n \\ c_1 & -b_2 & a_3 & 0 & \cdots & \cdots & 0 \\ 0 & c_2 & -b_3 & a_4 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & c_{n-3} & -b_{n-2} & a_{n-1} & 0 \\ 0 & \cdots & \cdots & 0 & c_{n-2} & -b_{n-1} & a_n \\ a_1 & 0 & \cdots & \cdots & 0 & c_{n-1} & -b_n \end{pmatrix} \quad (9.1)$$

Starting at some arbitrary reference configuration, each later visit to this configuration defines an integer, namely the algebraic number of times the cycle has been traversed, i.e. the homotopy class of the trajectory between the two passages at the reference configuration, or the winding number. The variation of winding number between two successive visits to the reference configuration belongs to $\{-1, 0, 1\}$. We can refine the definition of winding to have it defined at all times, i.e. as a process W_t , via the following trick: nW_t is just the number of clockwise jumps minus the number of anti clockwise jumps that have occurred up to time t (included). Note that W_t is an integer if and only if the positions of the particle on the cycle are the same at time 0 and t , and then W_t is simply the previously defined winding number.

9.1.1 Forward and backward thermal cycles

The easiest question to answer is whether or not the winding number will ever reach ± 1 ? The (strong) Markov property is the crucial ingredient.

Let π_m^- be the probability that the winding number of a trajectory started at m ever reaches the value $-1/n$. Then by the Markov property

$$\pi_m^- = \frac{a_m}{b_m} + \frac{c_m}{b_m} \pi_{m+1}^- \pi_m^-. \quad (9.2)$$

The meaning of this equation is clear : either the first jump is anti clockwise (probability a_m/b_m) and the winding number reaches its target $-1/n$ or the first jump is clockwise (probability c_m/b_m), and then the particle has “lost” a winding $1/n$ so it has to go from $m + 1$ to m with winding number $-1/n$ to compensate (probability π_{m+1}^-), and take a new chance.

Luckily, we do not need to solve the full system: the probability that starting from m the winding number reaches -1 is, by the Markov property again, $\prod_{l=0}^{n-1} \pi_{m-l}^-$, which is independent of m . So we denote this probability simply by $\Pi^- \equiv \prod_{l=0}^{n-1} \pi_{m-l}^-$. We rewrite (9.2) as

$$a_m(1 - \pi_m^-) = c_m \pi_m^- (1 - \pi_{m+1}^-). \quad (9.3)$$

A first consequence is that if $\pi_m^- = 1$ for some m then also $\pi_{m+1}^- = 1$ and so on, so that all π_m^- 's are equal to one, and the probability to reach winding number -1 is unity. On the other hand, in the case when no π_m^- equals one,

$$\Pi^- = \prod_{m=1}^n \frac{a_m}{c_m}. \quad (9.4)$$

Indeed, in (9.3) take the product over all m 's in the cycle and simplify both sides by $\prod_{m=1}^n (1 - \pi_m^-) \neq 0$ to get $\prod_{m=1}^n a_m = \prod_{m=1}^n c_m \prod_{m=1}^n \pi_m^-$, i.e. $\Pi^- = \prod_{m=1}^n a_m / c_m$. As Π^- is a probability, this is possible only if $\prod_{m=1}^n a_m / c_m \leq 1$.

We have already proved the following : if $\prod_{m=1}^n a_m / c_m \geq 1$ then $\Pi^- = 1$, i.e. winding number -1 is reached with probability one. We could reproduce the above argument with π_m^+ , the probability that the winding of a trajectory started at m ever reaches the value $1/n$, and Π^+ . We would get : if $\prod_{m=1}^n a_m / c_m \leq 1$ then $\Pi^+ = 1$, i.e. winding number 1 is reached with probability one. In particular, if $\prod_{m=1}^n a_m = \prod_{m=1}^n c_m$ then the probability to reach winding numbers -1 and 1 is unity, and by the Markov property, the probability to reach any winding number an infinite number of times is also unity: the winding number W_t will oscillate and take arbitrarily large positive and negative values as $t \rightarrow +\infty$.

To deal with the case $\prod_{m=1}^n a_m \neq \prod_{m=1}^n c_m$, we need a deeper result which we shall not prove here: the ergodic theorem for a finite state Markov process, which, stated informally and adapted to the case at hand, says that W_t/t will be close to $\langle W_t \rangle_{\text{st}}/t$ with probability close to 1 at large times. We shall “prove” below a formula for $\langle W_t \rangle_{\text{st}}$:

$$\langle W_t \rangle_{\text{st}} = t \frac{\prod_{m=1}^n c_m - \prod_{m=1}^n a_m}{P_{n-1}(a, c)}, \quad (9.5)$$

where $P_{n-1}(a, c)$ is the principal minor of the generator (9.1) of the Markov process (note that by construction the determinant of the generator is 0). It is a homogeneous polynomial of degree $n-1$ in the a_m 's and c_m 's given explicitly by

$$P_{n-1}(a, c) \equiv \sum_{l=1}^n \sum_{m=1}^n \prod_{1 \leq j < l} a_{m+j} \prod_{l \leq k < n} c_{m+k}, \quad (9.6)$$

where as usual an empty product stands for 1. We observe simply that the denominator in (9.5) is a sum of positive terms so that the sign of $\langle W_t \rangle_{\text{st}}/t$ is that of the numerator. So if $\prod_{m=1}^n c_m > \prod_{m=1}^n a_m$, $W_t \rightarrow +\infty$ with probability 1, implying that Π^+ , the probability to make one clockwise cycle, is 1 as well. The above discussion implies that

$$\Pi^- = \min\{1, \prod_{m=1}^n \frac{a_m}{c_m}\}, \quad \Pi^+ = \min\{1, \prod_{m=1}^n \frac{c_m}{a_m}\}. \quad (9.7)$$

We observe that the quantity $A \equiv -\ln \min\{\Pi^-, \Pi^+\}$ plays an important role in the context of chemical reactions, where, after multiplication by $k_B T$, it is called the affinity of the cycle properly oriented. If $A = -\log \Pi^-$ (resp. $A = -\log \Pi^+$) the reaction has a tendency to evolve clockwise (resp. anticlockwise).

Coming back to the physical interpretation, we may view the cycle as a mesoscopic thermal machine, and imagine that when the winding number changes by one, some reference heat bath has collected one unit of energy. This is what happens in the simple two-spin system, the “unit” being $2\Delta e$. For arbitrary n , a microscopic implementation of this behavior is not so obvious, but is not needed for the discussion either. By the ergodic theorem for a finite state Markov process as recalled above, W_t/t will be close to $\langle W_t \rangle_{\text{st}}/t$ with probability close to 1 at large times. So the behavior of the thermal machine is deterministic at large times. But fluctuations may occur, and e^{-A} is the probability that a time exists at which the net heat transfer is opposite to that of an expected working cycle, i.e. the thermal machine has performed the equivalent of a working cycle in the wrong direction. By the Markov property, e^{-kA} , $k = 1, 2, \dots$ is the probability that a time exists at which the net heat transfer is $-k$ times that of an expected working cycle.

9.1.2 Fluctuations in the time it takes to make a thermal cycle

We can now come to the second question, namely what are the fluctuations of the time it takes to make one cycle in the direction the machine is built for?

As usual, the object that satisfies simple equations is a Laplace transform. We assume that $\prod_{m=1}^n c_m > \prod_{m=1}^n a_m$, i.e. that winding number grows in average. We denote by t_m the random time it takes, starting from m to reach winding number $1/n$. Our real interest is the random time T it takes to reach winding number 1, and by the (strong) Markov property, T is distributed as a sum of n independent random variables each of which is distributed like a t_m . We write $f_m(\lambda) \equiv \langle e^{-\lambda t_m} \rangle_m$ where $\langle \dots \rangle_m$ is expectation with respect to trajectories starting at m . This is a bit redundant here, but we want to stress that T is a cyclic invariant so that we may write $F(\lambda) \equiv \langle e^{-\lambda T} \rangle$ because the expectation with respect to any initial probability distribution gives the same result.

The Markov property says that $F = \prod_{m=1}^n f_m$. The f_m 's satisfy

$$f_m = \frac{c_m}{b_m + \lambda} + \frac{a_m}{b_m + \lambda} f_{m-1} f_m. \quad (9.8)$$

The interpretation of this equation is analogous to that for $\pi_m^\pm$. If the particle is at m at some time, it waits an exponential time with parameter b_m and then jumps to $m+1$ (resp. $m-1$) with probability c_m/b_m (resp. a_m/b_m). The computation of the Laplace transform of the waiting time gives

$$\int_0^{+\infty} dt b_m e^{-b_m t} e^{-\lambda t} = \frac{b_m}{b_m + \lambda}, \quad (9.9)$$

which multiplied by the jump probability c_m/b_m (resp. a_m/b_m) yields the prefactors above. Then writing t_m as a sum of the exponential waiting time to leave m and some other (independent of the waiting time at m by the Markov property) random time, we note that if the jump is to $m+1$ this random time is 0, whereas if the jump is to $m-1$ this random time is, by the (strong) Markov property, distributed like the sum of two independent random times, one distributed like t_{m-1} and the other like t_m .

A standard trick to deal with the quadratic equations for the f_m 's is by a Riccati transform, to linearize them. Then $F(\lambda)$ acquires an interpretation as an holonomy. So we write $f_m \equiv g_{m-1}/g_m$ for $m = 1, \dots, n$ and observe that $F = g_0/g_n$: whereas the sequence f_m is periodic by construction, the sequence g_m is not, and F is the holonomy along the cycle. With this in mind, we set $g_{-1} \equiv F g_{n-1}$. Then we define

$$\mathbb{F}_m \equiv \begin{pmatrix} (b_m + \lambda)/a_m & -c_m/a_m \\ 1 & 0 \end{pmatrix} \quad (9.10)$$

and check that (9.8) turns into the linear equation

$$\begin{pmatrix} g_{m-2} \\ g_{m-1} \end{pmatrix} = \mathbb{F}_m \begin{pmatrix} g_{m-1} \\ g_m \end{pmatrix} \quad (9.11)$$

valid for $m = 1, \dots, n$. Setting $\mathbb{F} \equiv \mathbb{F}_1 \cdots \mathbb{F}_n$, one finds by iterating the above formula that

$$F \begin{pmatrix} g_{n-1} \\ g_n \end{pmatrix} = \begin{pmatrix} g_{-1} \\ g_0 \end{pmatrix} = \mathbb{F} \begin{pmatrix} g_{n-1} \\ g_n \end{pmatrix} \quad (9.12)$$

so that F is an eigenvalue of the transfer (or Bloch-Floquet or $\dots$ depending on the community) matrix $\mathbb{F}$, i.e. a solution of

$$F^2 - F \text{Tr } \mathbb{F} + \text{Det } \mathbb{F} = 0. \quad (9.13)$$

This formula shows clearly that F is a cyclic invariant, because $\text{Tr } \mathbb{F}$ and $\text{Det } \mathbb{F}$ are. By continuity, $F(0) = 1$, and we shall see shortly that this allows to choose the right branch.

Equation (9.12) allows to express $f_n = g_{n-1}/g_n$ in terms of F and the matrix elements of $\mathbb{F}$. A moment thinking shows that the same formula expresses any other f_m in terms of the same F and the matrix elements of the matrix obtained by applying a cyclic permutation of order m to the factors defining $\mathbb{F}$.

The determinant of $\mathbb{F}$ is easily seen by multiplicativity to be $e^A = \prod_{m=1}^n c_m/a_m$ (which by the way does not depend on λ), so the affinity has also something to say on F .

The trace of $\mathbb{F}$ can be computed in any specific case, but it is complicated even for our simple two-spin system. We conclude this short digression in the world of general cycles by computing $\mathbb{F}$ to first order in λ . This will allow us to give a “proof” of formula (9.5) for $\langle W_t \rangle_{\text{st}}$. We observe that $\mathbb{F}_m(\lambda) = \mathbb{F}_m(0) + \lambda/a_m \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, so that

$$\text{Tr } \mathbb{F}'(0) = \text{Tr} \left(\sum_{m=1}^n \mathbb{F}_1(0) \cdots \mathbb{F}_{m-1}(0) \frac{1}{a_m} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbb{F}_{m+1}(0) \cdots \mathbb{F}_n(0) \right) \quad (9.14)$$

$$= \sum_{m=1}^n \frac{1}{a_m} [\mathbb{F}_{m+1}(0) \cdots \mathbb{F}_n(0) \mathbb{F}_1(0) \cdots \mathbb{F}_{m-1}(0)]_{11} \quad (9.15)$$

where $[\cdots]_{11}$ stands for the top left corner matrix element of the matrix product. Expanding (9.13) to first order in λ we get $1 - \text{Tr } \mathbb{F}(0) + \text{Det } \mathbb{F} = 0$ and $[1 - \text{Det } \mathbb{F}] F'(0) = \text{Tr } \mathbb{F}'(0)$. The first equation implies that the right branch for (9.13) is (under our assumption that $A > 0$)

$$F = \frac{\text{Tr } \mathbb{F} - \sqrt{(\text{Tr } \mathbb{F})^2 - 4 \text{Det } \mathbb{F}}}{2}, \quad (9.16)$$

and the second then gives

$$\langle T \rangle = \frac{\text{Tr } \mathbb{F}'(0)}{\text{Det } \mathbb{F} - 1}. \quad (9.17)$$

By computing $\text{Tr } \mathbb{F}'(0)$ we shall now show that

$$\langle T \rangle = \frac{P_{n-1}(a, c.)}{\prod_{m=1}^n c_m - \prod_{m=1}^n a_m}. \quad (9.18)$$

This is nothing but the inverse of the value announced in (9.5) for $\langle W_t \rangle_{\text{st}}/t$. That the relation $\langle T \rangle \langle W_t \rangle_{\text{st}}/t = 1$ should hold is intuitively clear: $\langle W_t \rangle_{\text{st}}/t$ is the average of the growth of the winding number per unit time, and $\langle T \rangle$ is the average time it takes to increase the winding number by one unit. Intuition is not proof, but modulo that, we shall have given a proof of the formula for $\langle W_t \rangle_{\text{st}}/t$. Just note that in $\langle T \rangle$, the expectation is with respect to any initial distribution, while the average of W_t is exactly proportional to t only if expectation is taken with respect to the stationary measure.

We sketch the derivation that $(\prod_{l=1}^n a_l) \text{Tr } \mathbb{F}'(0) = P_{n-1}(a, c.)$, which gives immediately the announced formula for $\langle T \rangle$. We first note that $\mathbb{F}_m(0)$ has $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ as right eigenvector with eigenvalue 1 whatever the value of m , so that the same is true for any product of $\mathbb{F}_m(0)$'s. The generic matrix with this property can be parameterised by $\mathbb{K}(x, y) \equiv \begin{pmatrix} 1+x & -x \\ 1+y & -y \end{pmatrix}$. We observe that $\mathbb{F}_m(0) = \mathbb{K}(c_m/a_m, 0)$ and a simple recursive computation shows that $\mathbb{K}(x_1, 0) \cdots \mathbb{K}(x_k, 0) = \mathbb{K}(x_k + x_k x_{k-1} + \cdots + x_k x_{k-1} \cdots x_2)$. In particular, the top left matrix element of $\mathbb{F}_{m+1}(0) \cdots \mathbb{F}_n(0) \mathbb{F}_1(0) \cdots \mathbb{F}_{m-1}(0)$ is

$$1 + \frac{c_{m-1}}{a_{m-1}} + \cdots + \frac{c_{m-1} \cdots c_1}{a_{m-1} \cdots a_1} + \frac{c_{m-1} \cdots c_1 c_{m+1}}{a_{m-1} \cdots a_1 a_{m+1}} + \cdots + \frac{c_{m-1} \cdots c_1 c_{m+1} \cdots c_n}{a_{m-1} \cdots a_1 a_{m+1} \cdots a_n}. \quad (9.19)$$

Multiplying by $(\prod_{l=1}^n a_l)/a_m$ and summing over m one recovers the formula (9.6) for $P_{n-1}(a, c.)$.

9.1.3 Degeneration to random walks

There is a degeneration of the above family of Markov processes for which computations are totally explicit. It corresponds to the case $n = 1$ in which each single step is interpreted as a cycle. So, suppressing indices, c is the rate for making a cycle in the clockwise direction, and a is the rate for making a cycle in the anticlockwise direction. We assume $c \geq a$. The evolution of the winding number in that case is that of a simple continuous-time random walk. Equation (9.8) for the distribution of first passage times degenerates to a simple quadratic equation

$$F = \frac{c}{b + \lambda} + \frac{a}{b + \lambda} F^2, \quad (9.20)$$

which, as was to be expected, is the same as (9.13) when $n = 1$. As $a \leq c$, the solution taking value 1 at $\lambda = 0$ is $F = \frac{b+\lambda-\sqrt{(b+\lambda)^2-4ac}}{2a}$. When $a = c$, F is non-analytic in λ , a behavior consistent with the well-known divergence of the first passage time in that case. Setting $\bar{T} \equiv (c-a)^2 T$ it is easily checked that

$$\lim_{a,c \rightarrow g} \frac{1}{c-a} \ln \langle e^{-\lambda \bar{T}} \rangle = \frac{1 - \sqrt{1+4g\lambda}}{2g}, \quad (9.21)$$

which is essentially the generating function for Catalan numbers.

9.2 The case of the two-spin system

We apply the formulæ of the previous subsection to our concrete model. We choose the cyclic order

$$\begin{array}{ccc} (+, +) & \rightarrow & (+, -) \\ \uparrow & & \downarrow \\ (-, +) & \leftarrow & (-, -) \end{array}, \quad (9.22)$$

with corresponding $a; c$ transition coefficients (remember c is for the clockwise transition and a for the anticlockwise one)

$$\begin{array}{ccc} \nu_1(1-\gamma_1)/2; \nu_2(1-\gamma_2)/2 & \leftrightarrow & \nu_2(1+\gamma_2)/2; \nu_1(1+\gamma_1)/2 \\ \updownarrow & & \updownarrow \\ \nu_2(1+\gamma_2)/2; \nu_1(1+\gamma_1)/2 & \leftrightarrow & \nu_1(1-\gamma_1)/2; \nu_2(1-\gamma_2)/2 \end{array}. \quad (9.23)$$

The meaning of this diagram is that, starting at the upper-left corner $(+, +)$ for instance, the rate from $(+, +)$ to $(+, -)$ is the clockwise coefficient $\nu_2(1-\gamma_2)/2$, while the rate from $(+, +)$ to $(-, +)$ is the anticlockwise coefficient $\nu_1(1-\gamma_1)/2$.

The product $\prod_{m=1}^n a_m/c_m$ is readily evaluated to be $e^{-A} = e^{-2(\beta_1-\beta_2)\Delta e}$. Note that this quantity has a purely thermodynamic interpretation : this is due to the modified detailed balance. Under our standard assumption $\beta_1 \geq \beta_2$, we see that in average the thermal machine works in the clockwise direction, and the probability that at some time the net heat transfer is $-2k\Delta e$ ($k = 0, 1, \dots$) is $e^{-2k(\beta_1-\beta_2)\Delta e}$.

Using the formulæ of the previous subsection, a closed formula for $\langle e^{-\lambda T} \rangle$ is not difficult to write down, but it is very complicated and not particularly illuminating. We content to give the first two cumulants:

$$\langle T \rangle = \left(\frac{1}{\nu_1} + \frac{1}{\nu_2} \right) \frac{4}{\gamma_1 - \gamma_2}, \quad (9.24)$$

$$\langle T^2 \rangle - \langle T \rangle^2 = \left[(1 - \gamma_1 \gamma_2) \left(\frac{1}{\nu_1^2} + \frac{1}{\nu_2^2} \right) + (2 - \gamma_1^2 - \gamma_2^2) \frac{1}{\nu_1 \nu_2} \right] \frac{8}{(\gamma_1 - \gamma_2)^3}. \quad (9.25)$$

As expected, the cumulants of T diverge when $\gamma_1 - \gamma_2 \searrow 0$: for $\gamma_1 = \gamma_2$ there is no net heat current and the winding number is just a simple symmetric random walk.

We conclude by observing that the formulæ for the normalized random variable

$$\bar{T} \equiv T(\gamma_1 - \gamma_2)^2 \nu_1 \nu_2 / 2 \quad (9.26)$$

are slightly simpler: writing

$$\frac{1}{2(\gamma_1 - \gamma_2)} \log \langle e^{-\lambda \bar{T}} \rangle \equiv \sum_{k=1}^{+\infty} \frac{p_k}{k!} \lambda^k, \quad (9.27)$$

the p_k 's are polynomials in $\gamma_1, \gamma_2, \nu_1, \nu_2$, with integral coefficients, homogeneous of degree k in ν_1, ν_2 by dimensional analysis, non-homogeneous but of degree $2k-2$ in γ_1, γ_2 and symmetric both under the exchange of ν_1, ν_2 and under the exchange of γ_1, γ_2 . It is easy to check that in the limit when $\gamma_1 = \gamma_2 \equiv \gamma$ the formulæ simplify dramatically. In fact, setting $\nu \equiv \nu_1 + \nu_2$, one has

$$\lim_{\gamma_1, \gamma_2 \rightarrow \gamma} \frac{1}{2(\gamma_1 - \gamma_2)} \log \langle e^{-\lambda \bar{T}} \rangle = \frac{1 - \sqrt{1 + 2\lambda\nu(1 - \gamma^2)}}{(1 - \gamma^2)}, \quad (9.28)$$

which is again essentially the generating function for Catalan numbers, a result reminiscent of the case of the simple random walk recalled in subsection 9.1.3, see in particular (9.21).

ACKNOWLEDGMENTS

F. Cornu is indebted to C. Van den Broeck for bringing her interest to thermal contact modelization a few years ago.

References

- [1] F. Cornu and M. Bauer. Thermal Contact I. Symmetries ruled by Exchange Entropy Variations. Referred to as Paper I. *arXiv:1302.4538 [cond-mat.stat-mech]*, 2013.
- [2] E. Fermi, J. Pasta, and S. Ulam. Studies of Nonlinear Problems: I. *Los Alamos Document No. LA-1940*; also at <http://www.physics.utah.edu/~detar/phys6720/handouts/fpu/FermiCollectedPapers1965.pdf>, 1955.
- [3] D. K. Campbell, P. Rosenau, and G. M. Zaslavsky. Introduction: The Fermi-Pasta-Ulam problem - The first fifty years. *Chaos*, 15:015101, 2005.
- [4] T. Prosen and D. K. Campbell. Normal and anomalous heat transport in one-dimensional classical lattices. *Chaos*, 15:015117, 2005.
- [5] Lena Nicolin and Dvira Segal. Thermal conductance of the Fermi-Pasta-Ulam chains: Atomic to mesoscopic transition. *Phys. Rev. E*, 81:040102, Apr 2010.
- [6] C. Bernardin. Thermal conductivity for a chain of anharmonic oscillators perturbed by a conservative noise. *Journal of Difference Equations and Applications*, page 1563, 2011.
- [7] É. Brunet, B. Derrida, and A. Gerschenfeld. Fluctuations of the heat flux of a one-dimensional hard particle gas. *EPL*, 90:20004, 2010.
- [8] A. Gerschenfeld, B. Derrida, and J. L. Lebowitz. Anomalous Fourier’s law and long range correlations in a 1 d non-momentum conserving mechanical model. *J. Stat. Phys.*, 141:757, 2010.
- [9] V. Lecomte, Z. Rácz, and F. van Wijland. Energy flux distribution in a two-temperature Ising model. *J. Stat. Mech.*, page P02008, 2005.
- [10] R. K. P. Zia and B. Schmittmann. Probability currents as principal characteristics in the statistical mechanics of non-equilibrium steady states. *J. Stat. Mech.*, page P07012, 2007.
- [11] M. O. Lavrentovich and R.K.P. Zia. Energy flux near the junction of two Ising chains at different temperatures. *Europhys. Lett.*, 91:50003, 2010.
- [12] F. Corberi, G. Gonnella, and A. Piscitelli. Heat exchanges in coarsening systems. *J. Stat. Mech.*, page P10022, 2011.
- [13] B. Derrida and É. Brunet. *Einstein aujourd’hui*, chapter Le mouvement brownien et le théorème de fluctuation-dissipation, page 203. EDP Sciences, Les Ulis, 2005.
- [14] P. Visco. Work fluctuations for a Brownian particle between two thermostats. *J. Stat. Mech.*, page P06006, 2006.
- [15] K. Sekimoto. Kinetic characterization of heat bath and the energetics of thermal ratchet models. *J. Phys. Soc. Japan*, 66:1234, 1997.

- [16] T. Hatano and S. Sasa. Steady-State Thermodynamics of Langevin Systems. *Phys. Rev. Lett.*, 86:3463, 2001.
- [17] A. Gomez-Marín and J.M. Sancho. Heat fluctuations in Brownian transducers. *Phys. Rev. E*, 73:045101(R), 2006.
- [18] R.J. Glauber. Time-Dependent Statistics of the Ising Model. *Journal of Mathematical Physics.*, 4:294, 1963.
- [19] T. S. Komatsu, N. Nakagawa, S. Sasa, and H. Tasaki. Entropy and Nonlinear Nonequilibrium Thermodynamic Relation for Heat Conducting Steady States. *J. Stat. Phys.*, 142:127, 2011.
- [20] Y. Oono and M. Paniconi. Steady State Thermodynamics. *Prog. Theor. Phys. Suppl.*, 130:29, 1998.
- [21] K. Sekimoto. Langevin Equation and Thermodynamics. *Prog. Theor. Phys. Suppl.*, 130:17, 1998.
- [22] T. Speck and U. Seifert. Integral fluctuation theorem for the housekeeping heat. *J. Phys. A: Math. Gen.*, 38:581, 2005.
- [23] E. Boksenbojm, C. Maes, K. Netočný, and J. Pešek. Heat capacity in nonequilibrium steady states. *EPL*, 96:40001, 2011.
- [24] H. Touchette. The large deviation approach to statistical mechanics. *Phys. Rep.*, 478:1, 2009.
- [25] A. Dembo and O. Zeitouni. *Large Deviations Techniques and Applications*. Springer, New York, 2nd edition, 1998.
- [26] J. Farago. Injected power fluctuations in Langevin equation. *J. Stat. Phys.*, 107:781, 2002.
- [27] R. van Zon and E.G.D. Cohen. Extended heat-fluctuation theorems for a system with deterministic and stochastic forces. *Phys. Rev. E*, 69:056121, 2004.
- [28] W. Feller. *An introduction to probability theory and its applications*, volume II. Wiley, New York, 2nd edition, 1971.
- [29] M. Abramowitz and I. A. Stegun, editors. *Handbook of Mathematical Functions*. Dover, 1972.