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Collisionless propagation in the Lorentz model: solution of the virtual BBGKY hierarchy

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Abstract

The kinetic theory description of collisionless propagation of a particle through a homogeneous static medium composed of randomly distributed hard spheres (Lorentz's model with overlapping) is derived. Both free motion and the motion accelerated by an external field are considered. The relevant *virtual collision operator* is given a clear physical interpretation. A rigorous solution to the corresponding *virtual BBGKY hierarchy* is found based on an exact reduction of the infinite hierarchy to a system of two coupled equations. In the case of two-dimensional cyclotron motion induced by a uniform magnetic field the finite, non-zero probability of unperturbed everlasting circling is derived from the hierarchy.

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1. Introduction

Our object here is to develop an efficient kinetic theory approach for studying the probability of collisionless motion for a point particle, called hereafter particle e, propagating among hard sphere scatterers of radius a, in d-dimensional space R^d, with $d=2$ (the spheres are disks) and $d=3$. The spheres are at rest, randomly distributed (including overlapping configurations) with a constant number density n. Such a system with scatterers of infinite mass is called the d-dimensional Lorentz model.

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Notice that evaluating the probability of collisionless motion is equivalent to solving the problem of annihilation dynamics. Indeed, the annihilation problem consists in evaluating the survival probability for a particle supposed to propagate only up to its first encounter with a scatterer [1]. At the moment of impact it disappears instantaneously from the system. The survival probability is thus equal to the probability of collisionless propagation.

Suppose that, starting at the initial moment $t=0$ from the point $\mathbf{r}_0$ with velocity $\mathbf{v}_0$, particle e follows the unperturbed trajectory

$$\tau \rightarrow \mathbf{r}(\tau; \mathbf{r}_0, \mathbf{v}_0), \quad 0 \leqslant \tau \leqslant t , \tag{1}$$

up to time $t > 0$. This event can occur if and only if for any $\tau \in [0, t]$ the sphere $S_d(\tau)$ of radius a defined as the set of points $\mathbf{R}$ satisfying the condition

$$|\mathbf{r}(\tau; \mathbf{r}_0, \mathbf{v}_0) - \mathbf{R}| < a \tag{2}$$

is void of scatterer centers. The probability of collisionless motion $P_d(t)$ is thus equal to the probability of finding unoccupied the volume $A_d(t)$ covered owing to the motion of the sphere $S_d(\tau)$. Initially we have $P_d(0) = 1$, as the motion is known to start from a point lying outside all hard spheres, while by definition $A_d(0) = 0$.

The assumed complete randomness of the scatterer distribution around particle e with some constant number density n implies the applicability of the Poisson formula

$$P_d(t) = \exp[-nA_d(t)] . \tag{3}$$

Eq. (3) will serve as a basis for determining $P_d(t)$ in the first part of the paper (Section 2) where we start with a geometric derivation of the time evolution followed by $A_d(t)$.

In the second part of the paper, the problem of determining $P_d(t)$ is translated into the language of the kinetic theory. To begin with, we present the detailed derivation of the appropriate two-body collision operator by considering the annihilation problem for a simple system composed of particle e, submitted in general to an external field, and a single hard sphere (Section 3). Written mostly for pedagogical reasons, this section should provide a clear physical interpretation of the so-called *virtual collision operator*, used in the kinetic theory of hard sphere fluids. The results of Section 3 permit one to write down the corresponding *virtual BBGKY hierarchy* describing the annihilation dynamics in the Lorentz model (Section 4). The main results of Section 4 are the proof that the derived hierarchy can be rigorously reduced to a system of two coupled equations (for any value of the volume fraction na^d), and the resolution of these equations. The solution for $P_d(t)$ is consistent with the result of the geometric analysis of Section 2.

Our general considerations are applied throughout the paper to the case of free collisionless propagation and to the two-dimensional Lorentz model in which the point electric charge e performs the cyclotron motion induced by a constant and homogeneous external magnetic field perpendicular to the plane R^2. We point out that the cyclotron case is of particular interest because, in contradistinction to the free propagation dynamics, one finds here a non-zero long-time limit for the probability $P_d(t)$ of collisionless motion. This fact, occuring exclusively for $d = 2$, has been noticed and taken into account in recent studies of the kinetic equation for two-dimensional magnetotransport leading to the occurence of the so-called "circling electrons" [2]. In Section 4 we provide an original derivation of this phenomenon based on the proven reduction of the virtual BBGKY hierarchy. Our results for the cyclotron case, considered in the Grad limit [3] (definition recalled in Section 2), can be looked upon as the first step toward the systematic derivation of the so-called "generalized Boltzmann equation" (see Ref. [2]) from the BBGKY hierarchy, the problem left open in Ref. [4]. The paper ends with concluding comments (Section 5).

2. Geometric derivation of $P_d(t)$

2.1. Free motion

Consider first the free motion trajectory

$$\tau \to \mathbf{r}(\tau;\mathbf{r}_0,\mathbf{v}_0) = \mathbf{r}_0 + \mathbf{v}_0\tau, \quad 0 \leqslant \tau \leqslant t\,. \tag{4}$$

The volume of the region covered by the moving sphere (2) (without $S_d(0)$) equals

$$A_d^0(t) = \begin{cases} 2avt & \text{for } d=2\,, \\ \pi a^2 vt & \text{for } d=3\,, \end{cases} \tag{5}$$

where $v=|\mathbf{v}_0|$. So, the probability of free motion during time t through the homogeneous hard sphere medium with number density n is exponentially decaying according to the formula:

$$P_d(t) = \exp(-nA_d^0(t)) = \exp(-vt/\lambda_d)\,, \tag{6}$$

where

$$\lambda_d = \begin{cases} 1/2an & \text{for } d=2\,, \\ 1/\pi a^2 n & \text{for } d=3 \end{cases} \tag{7}$$

is the mean free path.

2.2. Cyclotron motion

Consider now the qualitatively different case of the two-dimensional cyclotron motion. Under the action of magnetic field $\mathbf{B}$ particle with charge e moves with a constant angular velocity $\omega = e|\mathbf{B}|/m$, where m denotes its mass. The center of the cyclotron orbit is situated at the point

$$\mathbf{r}_c = \mathbf{r}_0 + \frac{\mathscr{R}(\pi/2)\cdot\mathbf{v}_0}{\omega}\,, \tag{8}$$

where the tensor $\mathscr{R}(\phi)$ denotes the rotation of angle ϕ. The radius a_c of the cyclotron circle equals

$$a_c = \frac{v}{\omega}\,. \tag{9}$$

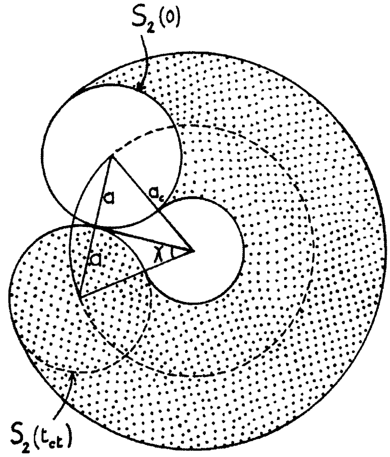

Fig. 1. $a < a_c$: contact configuration at $t = t_{ct}$.

In order to evaluate the probability of collisionless motion we have thus to calculate the area $A_2^c(t)$ of the region covered by the moving disc (two-dimensional sphere) $S_2(\tau)$ of radius a whose center follows the circular cyclotron orbit

$$\tau \to \mathbf{r}(\tau; \mathbf{r}_0, \mathbf{v}_0) = \mathbf{r}_c - \frac{\mathscr{R}(\omega\tau + \pi/2) \cdot \mathbf{v}_0}{\omega}, \quad 0 \leqslant \tau \leqslant t \, . \tag{10}$$

It is important to remember that the initially occupied area πa^2 should not be counted.

Denote by $T_c = 2\pi/\omega$ the period of the cyclotron motion. Clearly, for times $t > T_c$ particle e does not explore new regions on the plane any more, so $A_2^c(t)$ stays constant, equal to $A_2^c(T_c)$. Thus the probability of collisionless motion takes the form:

$$P_2^c(t) = \theta(T_c - t) \exp[-nA_2^c(t)] + \theta(t - T_c) \exp[-nA_2^c(T_c)] \, . \tag{11}$$

The explicit calculation of $A_2^c(t)$ for $t < T_c$ requires the distinction between two cases, depending on whether the radius of scatterers a is smaller or greater than that of the cyclotron orbit $a_c = v/\omega$.

$a < a_c$: When $a < a_c$ first the disc $S_2(\tau)$ moves away from $S_2(0)$, then it comes back in its circular motion and there exists a time of contact t_{ct} at which it reaches again the boundary of the initially occupied region $S_2(0)$ (see Fig. 1). At this moment the distance between the centers of $S_2(0)$ and $S_2(t_{ct})$ equals $2a$, preceding the occurence of overlapping between both discs. The angle 2χ of the arc of the cyclotron circle separating the centers of $S_2(0)$ and $S_2(t_{ct})$ satisfies the equation:

$$\sin \chi = \frac{a}{a_c} \, . \tag{12}$$

The renewal of the contact with the initial disc $S_2(0)$ occurs thus at time

$$t_{ct} = \frac{2(\pi - \chi)}{\omega} = T_c \left(1 - \frac{\mathrm{Arcsin}(a/a_c)}{\pi} \right) \, . \tag{13}$$

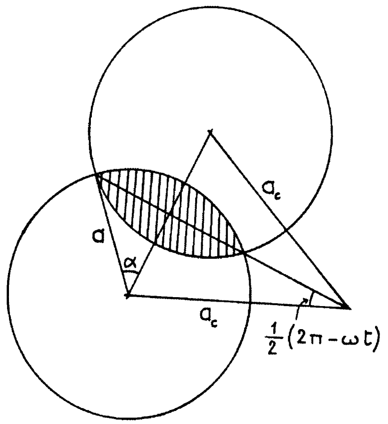

Fig. 2. $a < a_c$: overlapping configuration for $t_{ct} < t < T_c$.

When $t < t_{ct}$, no overlapping occurs between the visited region and the initial disc $S_2(0)$. The relevant area $A_2^c(t)$ reduces then to that of a region in a sector of angle ωt, contained between the arcs of two circles with radii $(a_c + a)$ and $(a_c - a)$ and common center point $\mathbf{r}_c$. The calculation yields the result

$$A_2^c(t) = \frac{\omega t}{2}[(a_c + a)^2 - (a_c - a)^2] = 2avt.$$

Hence, for $0 < t < t_{ct}$ the probability of collisionless motion equals

$$P_2^c(t) = \exp(-vt/\lambda_2)\,, \tag{14}$$

which coincides with Eq. (5) describing the free case for $d = 2$.

When $t_{ct} < \tau < T_c$, the disc $S_2(\tau)$ overlaps the initially occupied region $S_2(0)$. In order to calculate the area relevant for the probability of collisionless motion we have thus to subtract the area $I(t)$ of the intersection of discs $S_2(0)$ and $S_2(\tau)$ from the value $2avt$, obtained when no overlapping occured. The area $A_2^c(t)$ which must be free of scatterer centers to permit collisionless motion up to times $t_{ct} < t < T_c$ equals

$$A_2^c(t) = 2avt - I(t)\,. \tag{15}$$

Geometric considerations (see Fig. 2) yield the formula

$$I(t) = [2\alpha_t - \sin 2\alpha_t]a^2\,, \tag{16}$$

where

$$\cos\alpha_t = \frac{a_c}{a}\sin\frac{\omega t}{2}\,. \tag{17}$$

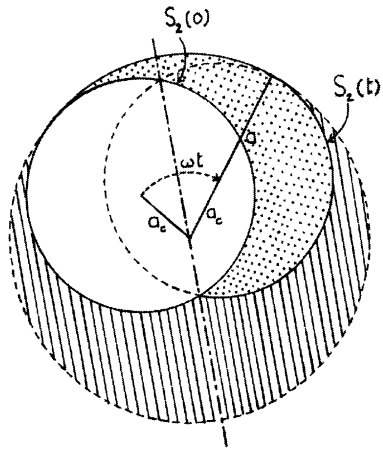

Fig. 3. Permanent overlapping during the cyclotron motion for $a > a_c$.

Combining Eqs. (16) and (17) we get

$$I(t) = 2\left[\mathrm{Arccos}\left(\frac{a_c}{a}\sin\frac{\omega t}{2}\right) - \frac{a_c}{a}\sin\frac{\omega t}{2}\sqrt{1-\left(\frac{a_c}{a}\sin\frac{\omega t}{2}\right)^2}\,\right]a^2 . \tag{18}$$

Eqs. (16) and (17) are consistent with the fact that at the moment of contact $t = t_{ct}$ the area of intersection is zero. Indeed, using relation (13) we find $\cos\alpha_{t_{ct}} = 1$, so that $\alpha_{t_{ct}} = 0$. When a complete cyclotron rotation is accomplished

$$A_2^c(T_c) = 2avT_c - \pi a^2 . \tag{19}$$

Eventually, when $a < a_c$ the probability of collisionless motion corresponding to random distribution of scatterers outside the initial disc $S_2(0)$ is given by

$$P_2^c(t) = \theta(T_c - t)\exp(-2anvt)[1 + \theta(t - t_{tc})\exp(nI(t))] \\ + \theta(t - T_c)\exp[-2anvT_c + \pi na^2] , \tag{20}$$

where θ is a unit step function.

$a > a_c$: When the scatterer radius is larger than that of the cyclotron circle the rotating disc $S_2(\tau)$ (see Definition (2)) is always overlapping the initial region $S_2(0)$ (See Fig. 3). A complete cyclotron rotation is possible provided the disc of radius $(a + a_c)$ is free of scatterer centers. Taking into account the assumed absence of hard discs within $S_2(0)$ we find

$$A_2^c(T_c) = \pi[(a_c + a)^2 - a^2] . \tag{21}$$

The area $A_2^c(t)$ for $0 < t < T_c$—area of the spotted region in Fig. 3—may be calculated as the difference between $A_2^c(T_c)$ and the area of the hatchured domain in Fig. 3. The evaluation of the latter area requires solving the geometric problem shown in Fig. 4. Denote the area of a sector of a disc by $D(a, \phi)$, where a is the radius

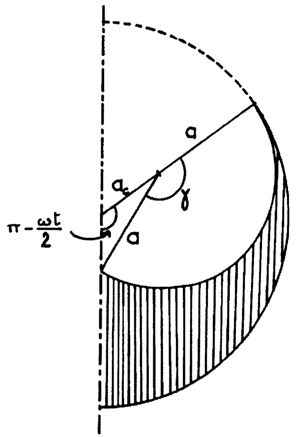

Fig. 4. $a > a_c$: geometrical elements used in evaluating the area relevant for the collisionless motion.

and ϕ the angle, respectively. We find

$$A_2^c(t) = A_2^c(T_c) - 2\left[D\left(a_c + a, \pi - \frac{\omega t}{2}\right) - D(a,\gamma) - \frac{1}{2}aa_c \sin\gamma\right] \tag{22}$$

together with

$$a_c \sin\frac{\omega t}{2} = a\sin\left(-\pi + \gamma + \frac{\omega t}{2}\right) . \tag{23}$$

The explicit formula for $A_2^c(t)$ corresponding to the above relations reads

$$A_2^c(t) = J(t) = \left(aa_c + \frac{1}{2}a_c^2\right)\omega t - \frac{1}{2}a_c^2\sin(\omega t) + a^2\mathrm{Arcsin}\left(\frac{a_c}{a}\sin\frac{\omega t}{2}\right) + aa_c\sin\frac{\omega t}{2}\sqrt{1-\left(\frac{a_c}{a}\sin\frac{\omega t}{2}\right)^2} , \tag{24}$$

which is valid for times $0 < t < T_c$ when $a_c < a$.

Eventually, when $a_c < a$ the probability of collisionless motion reads

$$P_2^c(t) = \theta(T_c - t)\exp(-nJ(t)) + \theta(t - T_c)\exp[-n\pi(a_c^2 + 2aa_c)] . \tag{25}$$

2.3. Everlasting unperturbed circling

As it has been noticed in the introduction, the cyclotron motion in two dimensions is expected to be characterized by a non-zero long-time limit of $P_2^c(t)$. In the present approach one finds from Eqs. (20) and (25) that for $t > T_c$

$$P_2^c(t) = P_2^c(T_c) = \begin{cases} \exp[-\pi an(4a_c - a)] & \text{for } a_c > a , \\ \exp[-\pi a_c n(2a + a_c)] & \text{for } a_c < a . \end{cases} \tag{26}$$

Owing to the finite length of the cyclotron circle there is thus a finite probability for particle e to continue for all times collisionless motion.

This phenomenon remains present in the Grad limit,

$$\lim_{Grad} \equiv \begin{cases} a \to 0, & n \to \infty\,, \\ \lambda_2 = 1/2an & =const\,, \end{cases} \tag{27}$$

where Eq. (26) reduces to

$$\lim_{Grad} P_2^c(t) = \exp\left(-2\pi a_c/\lambda_2\right). \tag{28}$$

Already the fact that $P_2^c(t)$ remains larger than zero in the long-time limit suffices to realize that the Boltzmann equation cannot be obtained in the Grad limit for the two-dimensional Lorentz gas in the presence of a magnetic field. This unusual situation has been studied extensively and the so-called "generalized Boltzmann equation" has been derived instead on intuitive grounds [2] and also from the Liouville equation [4].

3. Virtual collision operator $T^v(e, \mathbf{R})$

In order to prepare the formulation of the collisionless motion problem in the framework of the kinetic theory we study in this section the corresponding two-body problem.

3.1. Survival probability in the two-body problem

Consider a system composed of particle e and a single hard sphere of radius a located at point $\mathbf{R}$. Our aim is to determine the joint probability density $f_2(\mathbf{r}, \mathbf{v}, \mathbf{R}; t)$ for the occurence of configuration $(\mathbf{r}, \mathbf{v}, \mathbf{R})$ with $|\mathbf{r} - \mathbf{R}| > a$ at time t by collisionless motion. The reasoning presented below generalizes the method described in Ref. [5] to the case where the particle trajectories are not necessarily the free-motion straight lines. In fact, to assure that particle e did not touch the scatterer, we have to study its motion backward in time up to the initial moment $t = 0$. The state $(\mathbf{r}, \mathbf{v})$ plays the role of the initial state for the backward propagation. The position and velocity attained from it by unperturbed motion at time $(-\tau)$ will be denoted by $\mathbf{r}_e(-\tau; \mathbf{r}, \mathbf{v})$ and $\mathbf{v}_e(-\tau; \mathbf{r}, \mathbf{v})$, respectively. The abbreviated notation $\mathbf{r}_e(-\tau) = \mathbf{r}_e(-\tau; \mathbf{r}, \mathbf{v})$ and $\mathbf{v}_e(-\tau) = \mathbf{v}_e(-\tau; \mathbf{r}, \mathbf{v})$ will be also used, when not leading to confusion. The probability density $f_2(\mathbf{r}, \mathbf{v}, \mathbf{R}; t)$ can be written as

$$f_2(\mathbf{r}, \mathbf{v}, \mathbf{R}; t) = \theta(|\mathbf{r} - \mathbf{R}| - a)[1 - \eta(\mathbf{r}, \mathbf{v}, \mathbf{R}; t)] f_2(\mathbf{r}_e(-t), \mathbf{v}_e(-t), \mathbf{R}; 0). \tag{29}$$

The step function $\theta(|\mathbf{r} - \mathbf{R}| - a)$ ensures that particle e lies (at time t) outside the hard sphere, while $\eta(\mathbf{r}, \mathbf{v}, \mathbf{R}; t)$ denotes the characteristic function defined by

$$\eta(\mathbf{r}, \mathbf{v}, \mathbf{R}; t) = \begin{cases} 1 & \text{if a collision occurs within } [0, t]\,, \\ 0 & \text{for collisionless motion within } [0, t]\,. \end{cases} \tag{30}$$

The characteristic function η has the product form

$$\eta(\mathbf{r},\mathbf{v},\mathbf{R};t) \\ =\theta\left(a-\min_{\tau}|\,\mathbf{r}_e(-\tau)-\mathbf{R}|\right)\theta(t-\tau^*(\mathbf{r},\mathbf{v},\mathbf{R}))\theta(\mathbf{v}_e(-\tau^*)\cdot(\mathbf{r}_e(-\tau^*)-\mathbf{R}))\,. \tag{31}$$

The first θ-factor represents the geometric constraint requiring that the particle trajectory crosses the scattering sphere of radius a (the minimal distance of points $\mathbf{r}_e(-\tau)$ from the center $\mathbf{R}$ of the sphere must be less than a).

The argument of the second step function contains time $\tau^* > 0$ defined as the moment at which particle e touches the surface of the sphere for the first time in its backward motion. τ^* is thus the solution of the equation

$$(|\mathbf{r}_e(-\tau)-\mathbf{R}|)_{\tau=\tau^*}=a \tag{32}$$

satisfying the condition $\mathbf{v}_e(-\tau^*)\cdot(\mathbf{r}_e(-\tau^*)-\mathbf{R})>0$. Clearly, the free backward motion is possible only up to time $(-\tau^*)$. Notice that at $\tau=\tau^*$ the velocity of the particle encountering the scatterer surface for the first time points out of it.

3.2. Evolution of the two-body survival probability

The generator of the backward unperturbed motion L_e is the first-order differential operator determining the trajectory through

$$\left(\frac{\partial}{\partial t}+L_e\right)\mathbf{r}_e(-t;\mathbf{r},\mathbf{v})=0\,. \tag{33}$$

The operator L_e describes in general free motion combined with the effect of acceleration due to an external field. The generator of free motion has the form

$$L_e^0=\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{r}}\,, \tag{34}$$

whereas the acceleration effects involve the gradient in velocity space. In the case of the cyclotron motion

$$L_e^c=\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{r}}+\omega\left[\mathscr{R}\left(\frac{\pi}{2}\right)\cdot\mathbf{v}\right]\cdot\frac{\partial}{\partial\mathbf{v}}\,. \tag{35}$$

One can show that the evolution of the characteristic function η is governed by the equation

$$\left(\frac{\partial}{\partial t}+L_e\right)\eta(\mathbf{r},\mathbf{v};t)=0\,. \tag{36}$$

Indeed, L_e is the generator of the motion along the unperturbed trajectory. As the step function in (31) representing the geometric constraint for crossing the scatterer is constant along a given trajectory (satisfying (33)) the equation

$$L_e\theta\left(a-\min_{\tau}|\,\mathbf{r}_e(-\tau)-\mathbf{R}|\right)=0 \tag{37}$$

holds. Moreover, applying L_e to Eq. (32), which defines τ^* as an implicit function of variables $(\mathbf{r},\mathbf{v},\mathbf{R})$, we find

$$\begin{aligned} 0 &= L_e[|\mathbf{r}_e(-\tau^*(\mathbf{r},\mathbf{v},\mathbf{R});\mathbf{r},\mathbf{v})-\mathbf{R}|] \\ &= [L_e(|\mathbf{r}_e(-\tau;\mathbf{r},\mathbf{v})-\mathbf{R}|]_{\tau=\tau^*} + \left[\frac{\partial}{\partial\tau}|\mathbf{r}_e(-\tau;\mathbf{r},\mathbf{v})-\mathbf{R}|\right]_{\tau=\tau^*} L_e\tau^* \\ &= \left(\frac{\partial}{\partial\tau}|\mathbf{r}_e(-\tau;\mathbf{r},\mathbf{v})-\mathbf{R}|\right)_{\tau=\tau^*} (-1+L_e\tau^*)\,. \end{aligned} \tag{38}$$

Eq. (38) implies the relation

$$L_e\tau^*(\mathbf{r},\mathbf{v},\mathbf{R}) = 1\,. \tag{39}$$

The derived relations (37) and (39) prove the validity of the evolution Eq. (36).

Let us now apply the collisionless motion generator to the two-particle distribution (29). The relations (33) and (36) lead to the evolution equation

$$\begin{aligned} &\left(\frac{\partial}{\partial t}+L_e\right) f_2(\mathbf{r},\mathbf{v},\mathbf{R};t) \\ &\quad = [1-\eta(\mathbf{r},\mathbf{v},\mathbf{R};t)] f_2(\mathbf{r}_e(-t),\mathbf{v}_e(-t),\mathbf{R};0) L_e\theta(|\mathbf{r}-\mathbf{R}|-a-0^+)\,. \end{aligned} \tag{40}$$

The notation $(a+0^+)$ has been used when writing the argument of the θ-function to account for the fact that the surface of the scatterer can be reached from outside only. In L_e only the part L_e^0 acts on $\theta(|\mathbf{r}-\mathbf{R}|-a-0^+)$ yielding

$$L_e\theta(|\mathbf{r}-\mathbf{R}|-a-0^+) = \frac{1}{a}[\mathbf{v}\cdot(\mathbf{r}-\mathbf{R})]\delta(|\mathbf{r}-\mathbf{R}|-a-0^+)\,. \tag{41}$$

The δ-distribution selects points $\mathbf{r}$ lying at distance $(a+0^+)$, just outside the sphere. In the case of free motion, the condition for the absence of collisions within $[0,t]$ when $|\mathbf{r}-\mathbf{R}|=a$ reduces to the requirement that particle e goes away from the sphere moving backward in time. This defines the orientation of its velocity which must point inside the sphere at contact

$$1-\eta^0(\mathbf{r},\mathbf{v},\mathbf{R};t)|_{|\mathbf{r}-\mathbf{R}|=a} = \theta(-\mathbf{v}\cdot(\mathbf{r}-\mathbf{R}))\,. \tag{42}$$

In the case of the cyclotron motion the above requirement must be completed by a supplementary condition

$$t < \tau^{**}(v,\hat{\mathbf{v}}\cdot(\mathbf{r}-\mathbf{R})/a)\,, \tag{43}$$

where τ^{**} is the time needed to cover the arc of the cyclotron circle lying outside the sphere (particle e starts at the surface with the velocity pointing inside the sphere and moves backward in time). We find

$$1-\eta^c(\mathbf{r},\mathbf{v},\mathbf{R};t)|_{|\mathbf{r}-\mathbf{R}|=a} = \theta(-\mathbf{v}\cdot(\mathbf{r}-\mathbf{R}))\theta(\tau^{**}-t)\,. \tag{44}$$

The free motion case (42) can be looked upon as the limiting case of Eq. (44) for $\tau^{**}\to\infty$.

Inserting the above results into Eq. (40) we make appear in the right-hand side the product

$$\theta(\tau^{**} - t)f_2(\mathbf{r}_e(-t), \mathbf{v}_e(-t), \mathbf{R}; 0)\,, \tag{45}$$

which can be identified with $f_2(\mathbf{r}, \mathbf{v}, \mathbf{R}; t)$, as no collision could occur within the time interval $[0, t]$. From now on we introduce the notation $e \equiv (\mathbf{r}, \mathbf{v})$ and we rewrite (40) in the final form as

$$\left(\frac{\partial}{\partial t} + L_e\right) f_2(e, \mathbf{R}; t) = T^v(e, \mathbf{R})f_2(e, \mathbf{R}; t)\,, \tag{46}$$

defining the so called *virtual collision operator* by

$$T^v(e, \mathbf{R}) = \mathbf{v} \cdot (\mathbf{r} - \mathbf{R})\theta(-\mathbf{v} \cdot (\mathbf{r} - \mathbf{R}))\delta(|\mathbf{r} - \mathbf{R}| - a)/a\,. \tag{47}$$

Notice that the operator $T^v(e, \mathbf{R})$ selects velocities $\mathbf{v}$ which point inside the scatterer at its surface.

3.3. Explicit virtual two-body propagator

The above results permit us to give a precise meaning to the virtual two-body propagator $\exp\{-t[L_e - T_v(e, \mathbf{R})]\}$, used to write a formal solution to the evolution Eq. (46). Define the space of initial probability distributions by requiring that they give a non-zero weight only when particle e is outside the sphere centered at $\mathbf{R}$, so that

$$f_2(e, \mathbf{R}; 0) = \theta(|\mathbf{r} - \mathbf{R}| - a)f_2(e, \mathbf{R}; 0)\,. \tag{48}$$

Then the time evolution of $f_2(\mathbf{r}, \mathbf{v}, \mathbf{R}; t)$ can be written as

$$f_2(t) = \exp\{-t[L_e - T_v(e, \mathbf{R})]\}f_2(t = 0)\,. \tag{49}$$

We have shown that

$$\begin{aligned}&[\exp\{-t[L_e - T_v(e, \mathbf{R})]\}f_2(t = 0)](\mathbf{r}, \mathbf{v}, \mathbf{R})\\ &\quad\equiv \theta(|\mathbf{r} - \mathbf{R}| - a)[1 - \eta(\mathbf{r}, \mathbf{v}, \mathbf{R}; t)]f_2(\mathbf{r}_e(-t; \mathbf{r}, \mathbf{v}), \mathbf{v}_e(-t; \mathbf{r}, \mathbf{v}), \mathbf{R}; 0)\,.\end{aligned} \tag{50}$$

The characteristic function η can be explicitly written in the cases of free or cyclotron motion. For that purpose one must first determine the geometric conditions for the existence of an intersection between the unperturbed trajectory and the scattering sphere and then determine the first collision time τ^*.

In the case of the free motion (in dimension $d = 2$ or 3) the characteristic function η^0 reads (see Ref. [5])

$$\eta^0(e, \mathbf{R}; t) = \theta(t - \tau_0^*(e, \mathbf{R}))\theta(\mathbf{v} \cdot (\mathbf{r} - \mathbf{R}))\theta\left(va - \sqrt{v^2|\mathbf{r} - \mathbf{R}|^2 - [\mathbf{v} \cdot (\mathbf{r} - \mathbf{R})]^2}\right)\,, \tag{51}$$

where the collision time τ_0^* is given by

$$v^2\tau_0^*(e,\mathbf{R}) = \mathbf{v}\cdot(\mathbf{r}-\mathbf{R}) - \sqrt{v^2a^2 - v^2|\mathbf{r}-\mathbf{R}|^2 + [\mathbf{v}\cdot(\mathbf{r}-\mathbf{R})]^2}\,. \tag{52}$$

In the case of the two-dimensional cyclotron rotation η takes the form

$$\eta^c(e,\mathbf{R};t) = \theta(t-\tau_c^*(e,\mathbf{R}))\theta(a+a_c-|\mathbf{r}_c-\mathbf{R}|)\theta(|\mathbf{r}_c-\mathbf{R}|-a_c+a)\,. \tag{53}$$

The last two θ-factors in (53) represent the geometric constraints for the existence of intersection between the cyclotron circle and the scatterer. The very last θ-factor equals everywhere 1 when $a > a_c$. The determination of the collision time τ_c^* requires the evaluation of the length of the arc which particle e has to cover when moving backward in time before it encounters the surface of the hard disc. This geometric problem can be solved in a straightforward way. The analytic expression for τ_c^* will not be needed in the following.

4. BBGKY virtual hierarchy

The evolution law (46) can be directly used to write down the equation of motion for collisionless propagation of particle e between N scatterers distributed within a volume Ω. Denote by $f^{\Omega}_{e,N}(e,1,\ldots,N;t)$ the $(N+1)$-body probability density for the occurence of configuration $(\mathbf{r},\mathbf{v},\mathbf{R}_1,\ldots,\mathbf{R}_N)$ at time t (the shorthand notation $j\equiv\mathbf{R}_j$, $j=1,2,\ldots$ will be used hereafter). The collisionless propagation followed by instantaneous disappearance of the particle at its first encounter with a scatterer is described by the equation

$$\left(\frac{\partial}{\partial t} + L_e - \sum_{j=1}^{N} T^v(e,j)\right) f^{\Omega}_{e,N}(e,1,\ldots,N;t) = 0\,. \tag{54}$$

The complete sum of the virtual collision operators is needed to check when and where the collision occurs, and to annihilate particle e at this moment. We consider here the normalized initial state of the form

$$f^{\Omega}_{e,N}(e,1,\ldots,N;0) = f_e(\mathbf{r},\mathbf{v};0)\prod_{j=1}^{N}\left[\frac{\theta(|\mathbf{r}-\mathbf{R}_j|-a)}{\Omega-|S_d|}\right]\,, \tag{55}$$

where $|S_d|$ is the volume of a d-dimensional sphere.

Denoting the thermodynamic limit by

$$\lim_{\infty} \equiv \left(N\to\infty, n=\frac{N}{\Omega}=const\right)\,, \tag{56}$$

we define the reduced distributions $F_{e,s}$ representing the densities of $(s+1)$-particle states $(e,1,\ldots,s)$ as

$$F_{e,s}(e,1,\ldots,s;t) = \lim_{\infty}\Omega^s\int \mathrm{d}(s+1)\ldots\int \mathrm{d}N f^{\Omega}_{e,N}(e,1,\ldots,s,s+1,\ldots,N;t)\,. \tag{57}$$

In particular $F_{e,0}(e;t)$ denotes the probability density for the occurence of one-particle states e at time t. By standard methods (see e.g. Ref. [6]) one can derive from (54) the *virtual BBGKY hierarchy* satisfied by the reduced distributions $F_{e,s}$. One finds an infinite hierarchy of the form

$$\left(\frac{\partial}{\partial t}+L_e\right)F_e(e;t)=n\int \mathrm{d}1T^v(e,1)F_{e,1}(e,1;t)\,,$$

$$\left(\frac{\partial}{\partial t}+L_e-\sum_{j=1}^{s}T^v(e,j)\right)F_{e,s}(e,1,\ldots,s;t)$$

$$=n\int \mathrm{d}(s+1)T^v(e,s+1)F_{e,s+1}(e,1,\ldots,s,s+1;t),\quad s=1,2,\ldots\,. \tag{58}$$

In the first equation the simplified notation $F_e(e;t)=F_{e,0}(e;t)$ has been used. We turn now to the construction of the solution of the initial value problem for the above hierarchy, corresponding to the assumed initial state (55).

4.1. Reduction of the hierarchy

From Eqs. (55) and (57) we find that the state of the system at time $t=0$ is described by the reduced distributions of the form

$$F_{e,s}(e,1,\ldots,s;0)=F_e(e;0)\prod_{j=1}^{s}F(j;0|e)\,, \tag{59}$$

where

$$F(j;0|e)=\theta(|\mathbf{r}-\mathbf{R}_j|-a),\quad j=1,\ldots,s \tag{60}$$

represent the conditional density of scatterers at point $j=\mathbf{R}_j$ when particle e is known to be at point $\mathbf{r}$ with a velocity $\mathbf{v}$.

We now prove that the virtual hierarchy (58) does not create additional dynamic correlations and propagates the factorized structure (59) of the reduced distributions in the course of time. To this end we insert into (58) the conjecture

$$F_{e,s}(e,1,\ldots,s;t)=F_e(e;t)\prod_{j=1}^{s}F(j;t|e)\,. \tag{61}$$

Eliminating $(\partial/\partial t+L_e)F_e(e;t)$ with the help of the first equation of the hierarchy we find the relation

$$\left[\prod_{j=1}^{s}F(j;t|e)\right]n\int \mathrm{d}1'T^v(e,1')F_e(e;t)F(1';t|e)$$

$$+F_e(e;t)\left[\frac{\partial}{\partial t}+L_e-\sum_{j=1}^{s}T^v(e,j)\right]\prod_{j=1}^{s}F(j;t|e)$$

$$=n\int \mathrm{d}(s+1)T_v(e,s+1)F_e(e;t)\prod_{j=1}^{s+1}F(j;t|e)\,. \tag{62}$$

The first term on the left-hand side and the term on the right-hand side cancel out. The whole infinite hierarchy (58) reduces thus to the system of two coupled equations

$$\left(\frac{\partial}{\partial t}+L_e\right)F_e(e;t)=nF_e(e;t)\int \mathrm{d}1T^v(e,1)F(1;t|e)\,, \tag{63}$$

$$\left[\frac{\partial}{\partial t}+L_e-T^v(e,1)\right]F(1;t|e)=0\,. \tag{64}$$

As we have already noticed, this remarkable rigorous reduction of the virtual hierarchy reflects the fact that the annihilation dynamics of a single particle does not create dynamical correlations of some new kind different from that present in the initial state (59).

4.2. Solving the hierarchy equations

In order to solve Eq. (64) we use our main result (50) concerning the two-particle annihilation dynamics. We obtain

$$\begin{aligned}F(j;t|e)&=[\exp\{-t[L_e-T_v(e,j)]\}F(t=0)](j,e)\\&=\theta(|\mathbf{r}-\mathbf{R}_j|-a)[1-\eta(\mathbf{r},\mathbf{v},\mathbf{R}_j;t)]F(\mathbf{R}_j;0|\mathbf{r}_e(-t;\mathbf{r},\mathbf{v}),\mathbf{v}_e(-t;\mathbf{r},\mathbf{v}))\\&=\theta(|\mathbf{r}-\mathbf{R}_j|-a)[1-\eta(\mathbf{r},\mathbf{v},\mathbf{R}_j;t)]\theta(|\mathbf{r}_e(-t;\mathbf{r},\mathbf{v})-\mathbf{R}_j|-a)\,.\end{aligned} \tag{65}$$

In writing the last equality the explicit form of the initial reduced distributions (59) has been used.

Then, by inserting (65) into (63) and using the definition (47) of $T^v(e,1)$, we find a closed equation for the one-particle distribution F_e. It reads

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+L_e\right)F_e(e;t)\\&\quad=nF_e(e;t)\int \mathrm{d}\mathbf{R}\,\theta(-\mathbf{v}\cdot(\mathbf{r}-\mathbf{R}))[\mathbf{v}\cdot(\mathbf{r}-\mathbf{R})]\frac{1}{a}\delta(|\mathbf{r}-\mathbf{R}|-a)\\&\qquad\times[1-\eta(\mathbf{r},\mathbf{v},\mathbf{R};t)]\theta(|\mathbf{r}_e(-t;\mathbf{r},\mathbf{v})-\mathbf{R}|-a)\,.\end{aligned} \tag{66}$$

By definition (see (31)) the factor $(1-\eta)$ is different from zero only up to the moment of the first contact with the scatterer. Hence, the last step function in the right-hand side of (66) multiplied by $(1-\eta)$ equals identically 1. The delta distribution in (66) restricts the integration to the surface of the sphere of radius a. And the first θ-factor restricts it further to the hemisphere $\mathbf{v}\cdot(\mathbf{r}-\mathbf{R})<0$. With the use of (44) Eq. (66)

can be thus rewritten as

$$\left(\frac{\partial}{\partial t} + L_e\right) F_e(e;t) = -nF_e(e;t) \int \mathrm{d}\Sigma_d(\hat{\mathbf{n}})\theta(-\mathbf{v}\cdot\hat{\mathbf{n}})|\mathbf{v}\cdot\hat{\mathbf{n}}|\theta(\tau^{**}(v,\mathbf{v}\cdot\hat{\mathbf{n}}) - t)\,. \tag{67}$$

In (67) there appears time τ^{**} which has already been defined in equation (43), and a unit vector perpendicular to the surface of the absorbing sphere $\hat{\mathbf{n}} = (\mathbf{r} - \mathbf{R})/a$. $d\Sigma_d(\hat{\mathbf{n}}) = a^{d-1}d\hat{\mathbf{n}}$ denotes the measure of the surface area with orientation $\hat{\mathbf{n}}$ on a d-dimensional sphere of radius a and center at $\mathbf{r}$. The integral in the right-hand side of (67) yields a function of variables v and t only. So, putting

$$W_d(v;t) = \int \mathrm{d}\Sigma_d(\hat{\mathbf{n}})\theta(-\mathbf{v}\cdot\hat{\mathbf{n}})|\mathbf{v}\cdot\hat{\mathbf{n}}|\theta(\tau^{**}(v,\mathbf{v}\cdot\hat{\mathbf{n}}) - t)\,, \tag{68}$$

we find that the solution of (67) has the form

$$F_e(\mathbf{r},\mathbf{v};t) = \exp\left[-n\int_0^t \mathrm{d}\tau W_d(v;\tau)\right] F_e(\mathbf{r}_e(-t;\mathbf{r},\mathbf{v}),\mathbf{v}_e(-t;\mathbf{r},\mathbf{v});0)\,. \tag{69}$$

The consistency with the results of our geometric considerations in Section 2 requires that

$$W_d(v;t) = \frac{\partial}{\partial t}A_d(v;t)\,. \tag{70}$$

The validity of the above relation can be checked by an explicit evaluation of $W_d(v;t)$. We shall not present these calculations here. Let us just notice that the product $|\mathbf{v}\cdot\hat{\mathbf{n}}|d\Sigma_d(\hat{\mathbf{n}})$ restricted in (67) to the hemisphere $\mathbf{v}\cdot\hat{\mathbf{n}} < 0$ represents exactly the measure of the volume covered by a moving surface element of the sphere $S_d(t)$ per unit time (see Section 2). In the case of the free motion τ^{**} is infinite so that $\theta(\tau^{**} - t) = 1$, and the derivation of (70) is straightforward. In the case of the cyclotron motion τ^{**} is finite (see (43)), depending on the orientation $\hat{\mathbf{n}}$ of the surface element with respect to the particle velocity. The role of the $\theta(\tau^{**} - t)$-factor in (68) is then to introduce a cutoff for the volume covered by a given surface element in its backward motion once it touches the surface of the initially occupied spherical volume.

The important identification (70) permits us to write finally the solution of (67) as

$$F_e(\mathbf{r},\mathbf{v};t) = \exp(-nA_d(v;t))F_e(\mathbf{r}_e(-t;\mathbf{r},\mathbf{v}),\mathbf{v}_e(-t;\mathbf{r},\mathbf{v});0)\,. \tag{71}$$

The explicit form of the one-particle density F_e can be thus obtained with the help of the results established in the geometric considerations of Section 2.

In particular, in the case of the cyclotron motion with $a_c > a$, we get the solution

$$\begin{aligned} F_e(\mathbf{r},\mathbf{v};t) = \{&\theta(T_c - t)\exp[-2anvt + \theta(t - t_{tc})nI(t)] \\ &+ \theta(t - T_c)\exp[-2anvT_c + \pi na^2]\}F_e(\mathbf{r}_e(-t),\mathbf{v}_e(-t));0)\,. \end{aligned} \tag{72}$$

Eq. (72) contains information about the so-called "circling electrons", discussed in Ref. [2]. Indeed, the norm of the one-particle distribution evolves only during one cyclotron period, and acquires then a constant value $\exp[-2anvT_c + \pi na^2]$, representing the probability weight for the presence of particle e performing unperturbed cyclotron rotation at any time.

5. Concluding comments

The most important result of our study is the rigorous solution of the BBGKY hierarchy (58) for the annihilation dynamics of a point electric charge e propagating among immobile hard spheres (the Lorentz model). Both the presence and the absence of an external magnetic field have been considered. By definition, the annihilation dynamics lets the particle propagate up to the moment of its first encounter with a scatterer. At this moment the particle instantaneously disappears from the system. The surface of the scattering spheres can be thus looked upon as an absorbing manifold.

As a pedagogical introduction to the subject we presented in Section 2 a simple geometric argument which sufficed to calculate the probability of collisionless motion by relating it to the volume covered by a sphere moving along the unperturbed particle trajectory. However, our real goal was to develop an approach based on the kinetic theory. We thus considered the probability of collisionless motion in the two-body system (particle and a single absorbing sphere) in order to clarify the meaning of the so-called *virtual collision operator* T^v, known from the kinetic theory of hard-sphere fluids [7].[2] We showed that the action of the associated virtual propagator (50) on physically admissible initial states did faithfully represent the effect of annihilation dynamics. This result seems to us useful, especially for further studies of the mean free path effects in the kinetic theory of hard-sphere systems.

Once the appropriate collision operator has been derived, we could consider the corresponding *virtual BBGKY hierarchy* (58). The main observation about the hierarchy was the fact that it propagated the initial factorization (59) (see also (61)) of the reduced distributions without creating additional correlations. Owing to this remarkable property, the hierarchy could be rigorously reduced to the system of two coupled Eqs. (63) and (64). Using then the previously derived action of the virtual propagator we made appear in Eq. (63) the geometric factor evaluated in Section 2, arriving in this way at a complete solution (65) and (71) of the hierarchy (58).

The reduction of the hierarchy could be applied in the present study owing to the fact that the state of the system possessed already the required factorized structure (61) at the initial moment. Were a different type of correlations present at $t = 0$, we could not use the two coupled Eqs. (63), (64) to write down the solution of the hierarchy. However, it is probable that in the long-time limit the dynamics builds up dynamical correlations consistent with the factorization (61), whereas all deviations from it disappear. Whether the reduced densities asymptotically factorize into products of two particle conditional densities remains here an open very interesting question.

In the case of the two-dimensional cyclotron motion, we found a finite probability for an endless unperturbed rotation, a phenomenon never derived to our knowledge from the BBGKY hierarchy. Our result (72), considered in the Grad limit (27), represents

[2] In this article the notation $\bar{T}^{v}_{-}$ is used to denote the virtual collision operator T^v of the present paper.

a step toward a systematic derivation of the kinetic equation for magnetotransport—studied in Refs. [2,4]—from the hierarchy equations.

It would be certainly interesting to extend the analysis of this paper to the Lorentz model with non-overlapping absorbing spheres. However, the presence of spatial correlations between the spheres rules out the factorized structure of the initial state (59) assumed here, which makes the problem qualitatively different. Also the generalization to the case of an inhomogeneous absorbing medium would be of interest.

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