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Magnetotransport in the two-dimensional Lorentz model: non-markovian Grad limit of the BBGKY hierarchy

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Abstract

We consider the classical two-dimensional Lorentz model of non-interacting charged point particles acted upon by a perpendicular uniform magnetic field. Particles are elastically scattered by hard disks randomly distributed in the motion plane. Trajectories of gas particles between collisions are arcs of cyclotron circles. We consider the Grad limit of the corresponding BBGKY hierarchy. The Lorentz model in magnetic field provides an example where the Grad limit of the solution for the one-particle density cannot be found by formally applying the limit to evolution equations. The resulting non-markovian kinetic equation is shown to coincide with the generalization of the Boltzmann equation originally proposed upon intuitive arguments.

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1. Introduction

Consider a point mass m carrying an electric charge e, called hereafter particle e or electron, propagating in two dimensions among hard disks of radius a. The disks play the role of fixed scattering centers of infinite mass, randomly distributed in the plane

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(including overlapping configurations) with a uniform number density n. A constant and uniform magnetic field $\mathbf{B}$ perpendicular to the plane of motion acts on particle e and makes it follow arcs of circular cyclotron orbits between elastic collisions with scatterers. A gas of such independent "electrons" represents the classical two-dimensional Lorentz model submitted to a magnetic field.

The magnetotransport in the Lorentz model has been studied in a series of papers starting in 1995 [1] when it has been discovered that the model provided an example of a dynamic system for which the Boltzmann kinetic equation was not valid in the Grad limit [2] for two reasons: the existence of unavoidable recollision processes and a finite probability of collisionless motion. The intuitive derivation of the adequate non-markovian kinetic equation, called by the authors the generalized Boltzmann equation (GBE), originally presented in Ref. [1] has been then described in a detailed way in Ref. [3] together with its analytic solution. However, two questions have been left open at this time:

1. How to construct a mathematically rigorous derivation of the GBE?
2. How to derive the GBE from the BBGKY hierarchy?

The answer to the first question can be hopefully found in the future along the lines described in Ref. [4]. The derivation presented therein, based on the Liouville equation, is as yet not complete from the mathematical point of view, as a number of assumptions still need to be proved.

The object of the present paper is to answer the second question by explaining clearly the connection of the GBE with the BBGKY hierarchy. In fact, the first step in this direction has been already made in Ref. [5] where the dynamics of the so-called "circling electrons", characterized by everlasting collisionless motion, has been rigorously derived from the hierarchy. Here, we show that the non-markovian GBE conjectured in Ref. [1] follows from the BBGKY hierarchy in the Grad limit

$$\lim_{\text{Grad}} \equiv \begin{cases} a \to 0, \ n \to \infty \,, \\ an = const. \text{ and } anR_c = const. \,, \end{cases} \tag{1}$$

where $R_c = vm/(e|\mathbf{B}|)$ is the cyclotron radius. Although the scatterers shrink to points in this limit, their number density increases in such a way that the mean free path of the electron $\Lambda = \frac{1}{2}na$ remains constant. Moreover, the ratio R_c/Λ is kept to a nonzero finite value in order to cancel the probability that electrons might be trapped around a finite cluster of scatterers [1]. The role of the dimensionless parameter tending to zero is played by the volume fraction ε occupied by the scatterers

$$\varepsilon = na^2 \to 0 \,. \tag{2}$$

In Section 2, we introduce the BBGKY hierarchy equations and we present the Grad limit of results derived in Ref. [5] for the "virtual" hierarchy for any value of ε. In Section 3 we discuss the specificity of the Grad limit in the case of the cyclotron motion: we argue why the Lorentz model in a magnetic field provides an example where one cannot obtain the solution for the one-particle density by formally taking the Grad limit in BBGKY equations according to a dimensional analysis (more details are given in Appendix A). Section 4 contains the complete derivation of the kinetic equation from the BBGKY hierarchy in the Grad limit, before and after the cyclotron period. In the final Section 5, the derived equation is shown to be identical with the GBE equation of Ref. [1] (details are presented in Appendix B).

2. BBGKY hierarchy

2.1. Equations for elastic collisions with hard disks

The aim of the kinetic theory is to determine the one-particle density $F_e(e;t)$ of electrons in the microscopic state $e \equiv (\mathbf{r},\mathbf{v})$ at time t, where $\mathbf{r}$ and $\mathbf{v}$ denote the position and velocity vectors, respectively. The evolution of $F_e(e;t)$ is coupled with that of the reduced distributions $F_{e,s}(e,1,\ldots,s;t)$, $s=1,2,\ldots$, representing the density of $(s+1)$-particle phase–space configurations in which the electron occupies the state e and the s disks are located at the points $j \equiv \mathbf{R}_j$, $j=1,\ldots,s$. The infinite system of coupled equations called the BBGKY hierarchy takes the form (see e.g. Ref. [6])

$$\left(\frac{\partial}{\partial t}+L_e\right)F_e(e;t)=n\int d1T(e,1)F_{e,1}(e,1;t)\ , \tag{3a}$$

$$\left(\frac{\partial}{\partial t}+L_e-T(e,1)\right)F_{e,1}(e,1;t)=n\int d2T(e,2)F_{e,2}(e,1,2;t)\ , \tag{3b}$$

$$\left(\frac{\partial}{\partial t}+L_e-\sum_{j=1}^{s}T(e,j)\right)F_{e,s}(e,1,\ldots,s;t)$$

$$=n\int d(s+1)T(e,s+1)F_{e,s+1}(e,1,\ldots,s,s+1;t) \quad s=2,\ldots\ . \tag{3c}$$

On the l.h.s of the equations there appears the generator of the free cyclotron motion

$$L_e=\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{r}}+\omega\left[\mathcal{R}\left(\frac{\pi}{2}\right)\cdot\mathbf{v}\right]\cdot\frac{\partial}{\partial\mathbf{v}}\ . \tag{4}$$

The matrix $\mathcal{R}(\pi/2)$ rotates the velocity $\mathbf{v}$ by angle $\pi/2$. The frequency ω is given by

$$\omega=2\pi/T_c=e|\mathbf{B}|/m\ , \tag{5}$$

where T_c denotes the period of the cyclotron motion.

The influence of elastic collisions on the state of the electron is taken into account in the hierarchy (3a)–(3c) by the hard-sphere collision operator [7][2] (see also Refs. [6,8])

$$T(e,j)=T^r(e,j)+T^v(e,j),\quad j=1,2,\ldots\ . \tag{6}$$

[2] In this article the notations $\bar{T}^r_-$ and $\bar{T}^v_-$ are used to denote the collision operators T^r and T^v of the present paper.

The expressions of $T^r(e,j)$ and $T^v(e,j)$ involve the unit vector

$$\hat{\boldsymbol{\sigma}}_j \equiv \frac{\mathbf{r}-\mathbf{R}_j}{|\mathbf{r}-\mathbf{R}_j|} \tag{7}$$

which is oriented from the center $\mathbf{R}_j$ of disk j toward the point occupied by the electron at the impact moment. The virtual part

$$T^v(e,j) = -\delta(|\mathbf{r}-\mathbf{R}_j| - a - 0^+)|\hat{\boldsymbol{\sigma}}_j \cdot \mathbf{v}|\theta(-\hat{\boldsymbol{\sigma}}_j \cdot \mathbf{v}) \tag{8}$$

describes the encounters with (incoming) pre-collision velocity $\mathbf{v}$. It is responsible for the disappearance of states with that velocity. The real part

$$T^r(e,j) = \delta(|\mathbf{r}-\mathbf{R}_j| - a - 0^+)|\hat{\boldsymbol{\sigma}}_j \cdot \mathbf{v}|\theta(\hat{\boldsymbol{\sigma}}_j \cdot \mathbf{v})b_{\hat{\boldsymbol{\sigma}}_j} \tag{9}$$

takes into account the collisions with pre-collisional velocity $\mathbf{v}-2(\mathbf{v}\cdot\hat{\boldsymbol{\sigma}}_j)\hat{\boldsymbol{\sigma}}_j$ which, after collision, restore the states with (outcoming) velocity $\mathbf{v}$. Indeed, $T^r(e,j)$ contains the rotation operator $b_{\hat{\boldsymbol{\sigma}}_j}$ describing the elastic collision law,

$$\begin{aligned} b_{\hat{\boldsymbol{\sigma}}_j} f(\mathbf{r},\mathbf{v}) &\equiv f(\mathbf{r},\mathbf{v}-2(\mathbf{v}\cdot\hat{\boldsymbol{\sigma}}_j)\hat{\boldsymbol{\sigma}}_j) \\ &= f(\mathbf{r},\mathscr{R}(-\psi_j)\cdot\mathbf{v})\,, \end{aligned} \tag{10}$$

where ψ_j denotes the scattering angle and $\mathscr{R}(-\psi_j)$ is the rotation operator of angle $-\psi_j$. We notice that, according to definitions (9) and (10), the effect of $T^r(e,1)$ can be rewritten as

$$\{T^r(e,j)f\}(\mathbf{r},\mathbf{v}) = \delta(|\mathbf{r}-\mathbf{R}_1| - a - 0^+)|\hat{\boldsymbol{\sigma}}_j \cdot \mathbf{v}'|\theta(-\hat{\boldsymbol{\sigma}}_j \cdot \mathbf{v}')f(\mathbf{r},\mathbf{v}')\,, \tag{11}$$

where $\mathbf{v}'$ is the incoming precollisional velocity $\mathbf{v}-2(\mathbf{v}\cdot\hat{\boldsymbol{\sigma}}_j)\hat{\boldsymbol{\sigma}}_j$. As a consequence of (8) and (11), the support of $\{T(e,j)f\}(\mathbf{r},\mathbf{v})$ is centered on the border of disk j in the position space while the velocity argument of f is an incoming velocity.

2.2. Grad limit of the one-particle density in the virtual hierarchy

In our previous work [5] we found the rigorous solution to the complete "virtual" hierarchy, obtained from the BBGKY equations (3a)–(3c) by retaining only the virtual parts $T^v(e,j)$'s of collision operators $T(e,j)$'s. From the physical point of view, solving the virtual hierarchy is equivalent to the determination of the probability of collisionless motion (see also the related problem of annihilation dynamics [9]). The very possibility of getting a rigorous solution was due to a remarkable property of the virtual hierarchy: it propagates in time the factorized structure (A.1) of the reduced distributions recalled in Appendix A.

After the Grad limit (1) has been taken, the exact probability density of collisionless motion $F_e^v(e;t)$ derived in Ref. [5] takes the simple form

$$\begin{aligned} F_e^v(e;t) &= \theta(T_c - 0^+ - t)\mathrm{e}^{-\nu t}F_e(\mathbf{r}_e(-t),\mathbf{v}_e(-t);t=0) \\ &\quad + \theta(t - T_c + 0^+)\mathrm{e}^{-\nu T_c}F_e(\mathbf{r}_e(-t),\mathbf{v}_e(-t);t=0)\,. \end{aligned} \tag{12}$$

In (12), θ stands for the unit Heaviside step function and

$$v = 2nav = -n \int d1 T^v(e,1) \tag{13}$$

denotes the constant collision frequency for a given velocity modulus $v = |\mathbf{v}|$. $\mathbf{r}_e(-t)$ is the free cyclotron motion of an electron which starts at point $\mathbf{r}$ with velocity $\mathbf{v}$ at time 0 and then evolves backward in time till instant $-t$,

$$(\mathbf{r}_e(-t), \mathbf{v}_e(-t)) = (\mathrm{e}^{-tL_e}\mathbf{r}, \mathrm{e}^{-tL_e}\mathbf{v}) . \tag{14}$$

The evolution contained in (12) is the following. During the first cyclotron period ($t < T_c$), the electron performs the free cyclotron motion with exponentially decreasing probability $\exp(-vt)$. For $t \geqslant T_c$ the continuation of the free circling becomes certain, and the probability factor is stabilized at the value $\exp(-vT_c)$. It is quite remarkable that this kind of evolution is contained in the hierarchy equations.

The distribution (12) represents only a component of the full solution of the BBGKY hierarchy. Adopting the terminology of Ref. [3], for $t \geqslant T_c$ we shall call it the distribution of circling electrons and denote it by $F_e^c(e;t)$. What remains to be determined is the distribution $F_e^w(e;t)$ of electrons scattered by hard disks and wandering among them through the plane.

3. Specificity of the Grad limit for cyclotron motion

For the Lorentz model *in the absence of external field*, L_e is replaced by the free straight-line propagator $L_e^0 = \mathbf{v} \cdot \partial/\partial\mathbf{r}$ and the Grad limit of the exact solution for the one-particle density $F_e^0(e;t)$ in the BBGKY hierarchy coincides with the solution for $F_e^B(e;t)$ in the so-called Boltzmann hierarchy [10,11] for straight-line motion. The Boltzmann hierarchy is a limit of the BBGKY hierarchy which is obtained by the following argument based on dimensional analysis. One notice that on the r.h.s. of the hierarchy equations (3a)–(3c), the collision operators are multiplied by n, whereas on the l.h.s. the number density does not appear. In the Grad limit (1), as the scattering cross section (proportional to a) vanishes while the scatterer density n becomes infinite, one neglects the collision operators on the l.h.s. of the BBGKY equations. (In Appendix A we recall the factorized solution of the Boltzmann hierarchy, which is different from the factorized solution propagated by the virtual hierarchy.) In the present section, we show why, in the case of cyclotronic motion, such a procedure would lead to erroneous results.

3.1. Grad limit of the first equation in the virtual hierarchy

In the case of the virtual hierarchy for cyclotronic motion, the Grad limit (12) of the exact solution $F_e^v(e;t)$ obeys the kinetic equation

$$\left(\frac{\partial}{\partial t} + L_e\right) F_e^v(e;t) = -v\theta(T_c - 0^+ - t) F_e^v(e;t) . \tag{15}$$

In Appendix A we show how the closed equation (15) can be retrieved directly from the first virtual hierarchy equation

$$\left(\frac{\partial}{\partial t} + L_e\right) F_e^v(e;t) = n \int d1 T^v(e,1) F_{e,1}^v(e,1;t) \tag{16}$$

by properly handling the Grad limit of correlation effects. We exhibit that (15) arises from (16) as a consequence of the exact relation

$$\lim_{\text{Grad}} \theta(-\hat{\boldsymbol{\sigma}}_1 \cdot \mathbf{v}) F_{e,1}^v(e,1;t)\Big|_{|\mathbf{r}-\mathbf{R}_1|=a+0^+} = \begin{cases} \lim_{\text{Grad}} \theta(-\hat{\boldsymbol{\sigma}}_1 \cdot \mathbf{v}) F_e^v(e;t)\big|_{|\mathbf{r}-\mathbf{R}_1|=a+0^+} & \text{if } t < T_c, \\ 0 & \text{if } t \geqslant T_c. \end{cases} \tag{17}$$

The crucial difference between the cyclotron and straight-line free motions lies in the two-body electron/hard-disk problem: a free trajectory starting from the surface of a hard disk brings the particle back to this surface in the case of the cyclotron evolution, whereas the particle following the free straight-line motion can never encounter the disk again. As discussed in Appendix A, this implies that the limit of an infinite T_c is to be taken in (17) when L_e is replaced by L_e^0. Then the r.h.s. of (15) is equal to $-\nu F_e^v(e;t)$ at any time: in the Grad limit $F_e^{v0}(e;t)$ obeys the "virtual" Boltzmann equation ,

$$\left(\frac{\partial}{\partial t} + L_e^0\right) F_e^{v0}(e;t) = -\nu F_e^{v0}(e;t) \, . \tag{18}$$

The solution of (18) is equal to $\exp(-\nu t) \times F_e(\mathbf{r}_e^0(-t), \mathbf{v}_e^0(-t); t=0)$ for all times, contrary to the Grad limit (12) for cyclotron motion, where the latter evolution is valid only during the time interval $[0, T_c]$. $((\mathbf{r}_e^0(-t), \mathbf{v}_e^0(-t))$ is defined as $(\mathbf{r}_e(-t), \mathbf{v}_e(-t))$ in (14) with L_e^0 in place of L_e.)

3.2. Failure of Boltzmann hierarchy for cyclotron motion

The very reason why, contrary to the case of straight-line motion, the Grad limit of $F_e(e,t)$ for the cyclotron motion cannot be obtained by formally taking the Grad limit in BBGKY equations (by neglecting collision operators on their l.h.s.) is the following. As can be seen for instance in a binary collision expansion [8], erasing collision operators on the l.h.s. of the BBGKY hierarchy is equivalent to neglecting two kinds of events in the r.h.s. of the first hierarchy equation (3a) obeyed by $F_e(e;t)$:

(a) recollisions with the same disk after free-motion evolution;
(b) recollisions with the same disk after scattering by another disk.

In the Grad limit events (b) give vanishing contributions to the r.h.s. of (3a) at any time (as recalled in Section 4 these contributions are of relative order na^2).

However, though recollisions (a) are forbidden by kinematics in the case of the straight-line free motion, in the case of circular cyclotron motion they become possible after a cyclotron period has elapsed. As a consequence, in the case of the virtual hierarchy, it is only for times shorter than the cyclotron period that the Grad limit (15) of the first equation coincides with the "virtual" Boltzmann equation (18) written in Appendix A for L_e^0.

3.3. Dynamical corrrelation effects

In other words, after the cyclotron period, two-body correlations arising from recollision with the same scatterer after free evolution play a role. The first hierarchy equation (3a) can be rewritten in terms of the two-body electron-scatterer correlation function,

$$G_{e,1}(e,1;t) = F_{e,1}(e,1;t) - F_e(e;t) \tag{19}$$

as

$$\left(\frac{\partial}{\partial t} + L_e\right) F_e(e;t) = n \int d1 T(e,1)\{F_e(e;t) + G_{e,1}(e,1;t)\} \ . \tag{20}$$

In the case of the virtual hierarchy, $G^v_{e,1}(e,1;t)$ contributes to the r.h.s. of the first hierarchy equation after T_c, as indicated by (17). For the hierarchy with full collision operators, we show in Section 4 that $G_{e,1}(e,1;t)$ also gives a non-vanishing contribution to the r.h.s. of (20).

Moreover, we notice that the influence of dynamical correlations upon the equation obeyed by $F_e(e;t)$ after T_c involves correlations with an arbitrary number of scatterers, as it is already the case in the virtual hierarchy. In particular, the contribution from correlations to the r.h.s. of the first hierarchy equation (20) cannot be calculated in the the so-called "ring approximation", where one cancels the contribution from correlations of order higher than 2. More precisely, in the ring approximation $G^{\text{ring}}_{e,1}(e,1;t)$ is calculated from the second hierarchy equation (3b) in which one neglects the contribution to the r.h.s. from the three-body correlation defined by the cluster decomposition (see e.g. Ref. [8])

$$G_{e,2}(e,1,2;t) = F_{e,2}(e,1,2;t) - F_{e,1}(e,1;t) - F_{e,1}(e,2;t) + F_e(e;t) \ . \tag{21}$$

Then, by arguments similar to those used in Section 4, the Grad limit of the contribution from $G^{\text{ring}}_{e,1}(e,1;t)$ to the r.h.s. of (20) is shown to read

$$\begin{aligned}&\lim_{\text{Grad}} n \int d1 T(e,1) G^{\text{ring}}_{e,1}(e,1;t)\\ &\quad = \lim_{\text{Grad}} n \int d1 T(e,1) \int_0^t d\tau \mathrm{e}^{-(t-\tau)[L_e - T(e,1)+v]} T(e,1) F^{\text{ring}}_e(e;\tau) \ .\end{aligned} \tag{22}$$

The ring approximation already fails in the case of the virtual hierarchy, because the r.h.s. of (22) written with virtual operators vanishes, which is in contradiction with

(15) for $t \geqslant T_c$. (The cancellation is due to the property (31) written for the virtual propagator: it is linked to the disappearance of the particle after its first hit on a scatterer in the annihilation dynamics.) In the Grad limit, we will show that the contribution from $G_{e,2}(e,1,2;t)$ disappears only in the r.h.s. of the equation obeyed by $F_e(e,t) - F_e^v(e,t)$ for $t \geqslant T_c$.

4. BBGKY hierarchy: deriving the kinetic equation in the Grad limit

4.1. Evolution before T_c

First we show that within the time interval $0 < t < T_c$ the probability density $F_e(e,1;t)$ in the Grad limit obeys the Boltzmann equation

$$\left(\frac{\partial}{\partial t} + L_e\right) F_e(e;t) \underset{0<t<T_c}{=} \lim_{\text{Grad}} n \int d1\, T(e,1) F_e(e;t) \; . \tag{23}$$

The reduction of the first hierarchy equation (20) to (23) is due to the relation

$$\lim_{\text{Grad}} T(e,1) G_{e,1}(e,1;t) \underset{0<t<T_c}{=} 0 \; . \tag{24}$$

The argument is the following. In $T(e,1)G_{e,1}(e,1;t)$, the electron arrives at time t at the surface of scatterer 1. (The condition imposed by $T(e,1)$, see (8) and (9).) Besides, the dynamic structure of $G_{e,1}(e,1;t)$ implies the appearance of at least one collisional configuration between the electron and the hard disk 1 in the past, i.e., within the time interval $[0,t]$. Indeed, the contribution to the two-particle density $F_{e,1}(e,1;t)$ in which particle e does not touch disk 1 is exactly the subtracted one-particle distribution $F_e(e;t)$ (see e.g. Chapter 11 in Ref. [8]). (Another argument is based on the representation of $G_{e,1}(e,1;t)$ in terms of the so-called irreducible n-body propagators and their expansions, similar to those used in Appendix B, in terms of the virtual propagator and the real part of the collision operator.) Therefore, $T(e,1)G_{e,1}(e,1;t)$ does not vanish only if the electron trajectory intersects disk 1 twice in the time interval $[0,t]$. Now, when the radius a vanishes—as it is the case in the Grad limit—the reappearance of e on the surface of 1 after free circling requires at least the time equal to the cyclotron period. It follows that the events which contribute to $T(e,1)G_{e,1}(e,1;t)$ for $t < T_c$ are only those where the electron is sent back on disk 1 by scattering with another disk. However, such a recollision process is eliminated in the Grad limit, because it is at least of order $\varepsilon = na^2$ for geometrical reasons (see e.g. Ref. [12] or [11]). (Indeed the collision frequency is proportional to nav, but a collision with disk 2 at instant t scatters the electron back to disk 1 which it has previously hit only for the fraction of disks 2 which are hit with an incident angle in some interval with width Ω. Ω varies as the angle α under which disk 1 is seen from disk 2 with curved beams whose curvature is equal to the cyclotron radius R_c. α is equal to a/R_c times a dimensionless function of time t.) This completes the proof of the asymptotic relation (24). We notice also that from Eq. (17), one directly finds the equality $T(e,1)G^v_{e,1}(e,1;t) = 0$ if $0 < t < T_c$, in accordance with the first part of the above argument.

As no pre-collisional correlations between the electron and the scattering disks can be created in the Grad limit for $0 < t < T_c$, the one-particle density $F_e(e;t)$ of the BBGKY hierarchy reduces in this period of time to the one-particle density of the Boltzmann hierarchy propagating the initial molecular chaos [11]. However, after T_c the situation becomes qualitatively different, beacuse recollisions with the same scatterer are now possible via free motion owing to the circular structure of trajectories. Their effect persists in the Grad limit and the correlations begin to play an important role.

4.2. Evolution after T_c

4.2.1. Circling electrons

According to (15), for $t \geqslant T_c$ the distribution $F_e^c(e;t) = F_e^v(e;t)$ of freely circling electrons satisfies the simple equation

$$\left(\frac{\partial}{\partial t} + L_e\right) F_e^c(e;t) \underset{t \geqslant T_c}{=} 0 \, . \tag{25}$$

Eq. (25) shows that for $t \geqslant T_c$ both sides of the first equation (16) of the virtual hierarchy vanish.

In fact, the virtual two-particle density $F_{e,1}^v(e,1;t)$ obeys a property stronger than a mere vanishing under the action of the virtual collision operator $T^v(e,1)$ for $t \geqslant T_c$: it is canceled also under the action of the full collision operator,

$$\lim_{\text{Grad}} T(e,1)\{F_e^v(e;t) + G_{e,1}^v(e,1;t)\} \underset{t \geqslant T_c}{=} 0 \, . \tag{26}$$

Eq. (26) is a consequence of the property of $\{T(e,1)f\}(\mathbf{r},\mathbf{v})$ recalled after (11) and of the exact relation (17). It implies that for $t \geqslant T_c$ collisions influence only wandering electrons.

4.2.2. Wandering electrons

Let us introduce the distribution of wandering electrons $F_e^w(e;t) \equiv [F_e - F_e^c](e;t)$. By using (25) and (26), the first hierarchy equation (20) can be rewritten in the form

$$\left(\frac{\partial}{\partial t} + L_e\right) F_e^w(e;t) \underset{t \geqslant T_c}{=} n \int d1\, T(e,1)\{F_e^w(e;t) + [G_{e,1} - G_{e,1}^v](e,1;t)\} \, . \tag{27}$$

In order to proceed, it is convenient to rewrite the second hierarchy equation (3b) in terms of the two-particle correlation function $G_{e,1}(e,1;t)$ and of the three-particle correlation function $G_{e,2}(e,1,2;t)$ defined in (19) and (21) respectively. One finds

$$\left(\frac{\partial}{\partial t} + L_e - T(e,1) + v\right) G_{e,1}(e,1;t) = T(e,1)F_e(e;t) + n\int d2\, T^r(e,2)G_{e,1}(e,1;t) + n\int d2\, T(e,2)G_{e,2}(e,1,2;t) \, . \tag{28}$$

(The collision frequency ν has been defined in (13).) We now subtract from (28) the corresponding equation of the virtual hierarchy and obtain

$$\left(\frac{\partial}{\partial t} + L_e - T(e,1) + \nu\right)[G_{e,1} - G^v_{e,1}](e,1;t)$$

$$= T(e,1)[F_e - F^v_e](e;t) + T^r(e,1)F^v_{e,1}(e,1;t) \tag{29a}$$

$$+ n\int d2T^r(e,2)[G_{e,1}(e,1;t) + G_{e,2}(e,1,2;t)] \tag{29b}$$

$$+ n\int d2T^v(e,2)[G_{e,2}(e,1,2;t) - G^v_{e,2}(e,1,2;t)]\,. \tag{29c}$$

Since at the initial moment $G_{e,1}(e,1;0) = G^v_{e,1}(e,1;0)$, the solution of (29) has the form of the time convolution of the two-body propagator $\exp(-t[L_e - T(e,1) + \nu])$ with the r.h.s. of (29). Upon inserting this solution into the r.h.s. of (27) we find the following terms:

(i) The contribution from term (29a) to the r.h.s. of (27) reads

$$n\int d1T(e,1)\int_0^t d\tau \mathrm{e}^{-(t-\tau)[L_e - T(e,1)+\nu]}$$

$$\times\{T(e,1)[F_e - F^v_e](e;\tau) + T^r(e,1)F^v_{e,1}(e,1;\tau)\}\,. \tag{30}$$

It can be analyzed by noticing that

$$\mathrm{e}^{-(t-\tau)[L_e - T(e,1)]}T^v(e,1) = 0\,. \tag{31}$$

Indeed, similarly to the action (14) of the free propagator $\exp(-tL_e)$, when the propagator $\exp(-t[L_e - T(e,1)])$ is applied to an initial condition $(\mathbf{r},\mathbf{v})$, it provides a solution of the two-body electron/hard-disk problem backward in time. In the latter motion, the electron, constrained to lie outside the scatterer, can arrive at the surface of the disk only with an outcoming velocity. This implies (31) in view of the definition (8), where $T^v(e,1)$ contains the factor $\theta(-\hat{\boldsymbol{\sigma}}_1\cdot\mathbf{v})$ which is non-zero only for an incoming velocity. Therefore, the part of the term $\{\ldots\}$ in (30) which gives a non-vanishing contribution is reduced to

$$T^r(e,1)\{[F_e - F^v_e](e;\tau) + F^v_{e,1}(e,1;\tau)\}\,. \tag{32}$$

Eq. (32) can be re-expressed in terms of the one-particle distributions $F_e(e;t)$ and $F^v_e(e;t)$ only, by using (11) and (17). Eventually, the contribution (30) from the term (29a) yields

$$n\int d1T(e,1)\int_0^t d\tau \mathrm{e}^{-(t-\tau)[L_e - T(e,1)+\nu]}$$

$$\times\, T^r(e,1)\{\theta(T_c - 0^+ - \tau)F_e(e;\tau) + \theta(\tau - T_c + 0^+)[F_e - F^v_e](e;\tau)\}\,. \tag{33}$$

The Grad limit of (33) is finite. The role of this term in the kinetic equation is discussed in Section 5.

(ii) The term (29b) yields the contribution

$$n\int d1T(e,1)\int_0^t d\tau \mathrm{e}^{-(t-\tau)[L_e-T(e,1)+v]}$$
$$\times n\int d2T^r(e,2)[G_{e,1}(e,1;\tau)+G_{e,2}(e,1,2;\tau)]\,. \tag{34}$$

Both correlation functions $G_{e,1}(e,1;\tau)$ and $G_{e,2}(e,1,2;\tau)$ require the occurrence of a collisional configuration with disk 1. Thus, the structure of (34) involves at least two encounters with 1 separated by a scattering—real collision—by another disk 2. Therefore, the term (34) is at least of order $\varepsilon = na^2$ and it vanishes in the Grad limit (the argument has already been used in the derivation of (24)).

(iii) The term (29c) yields the contribution

$$n\int d1T(e,1)\int_0^t d\tau \mathrm{e}^{-(t-\tau)[L_e-T(e,1)+v]}$$
$$\times n\int d2T^v(e,2)[G_{e,2}(e,1,2;\tau)-G^v_{e,2}(e,1,2;\tau)]\,. \tag{35}$$

The dynamic events building the three-particle correlation functions $G_{e,2}(e,1,2;\tau)$ and $G^v_{e,2}(e,1,2;\tau)$ require the occurrence of collisional configurations with both scatterers 1 and 2. Only *real* collisions with 1 and 2 contribute to the difference of these functions. The inspection of the structure of expression (35) shows that it involves two encounters with the same disk separated by a real collision with another disk. As in the case of term (34), such events do not contribute in the Grad limit.

The result of the above analysis is that the evolution of wandering electrons after the cyclotron period T_c is governed in the Grad limit by the kinetic equation

$$\left(\frac{\partial}{\partial t}+L_e\right)F^w_e(e;t)$$
$$\underset{t\geqslant T_c}{=}\lim_{\text{Grad}}\left\{n\int d1T(e,1)F^w_e(e;t)\right.$$
$$+n\int d1T(e,1)\int_0^{T_c-0^+} d\tau \mathrm{e}^{-(t-\tau)[L_e-T(e,1)+v]}T^r(e,1)F_e(e,1;\tau)$$
$$\left.+n\int d1T(e,1)\int_{T_c-0^+}^t d\tau \mathrm{e}^{-(t-\tau)[L_e-T(e,1)+v]}T^r(e,1)F^w_e(e,1;\tau)\right\}\,. \tag{36}$$

5. Magnetotransport in the Grad limit

The kinetic equations (23), for $t < T_c$, and (25) with (36), for $t \geqslant T_c$, provide a complete description of the evolution of the electronic density in the Grad limit. Eq. (23) can be rewritten in terms of the scattering angle ψ as

$$\left(\frac{\partial}{\partial t} + L_e\right) F_e(e;t) \underset{t<T_c}{=} nav \int_{-\pi}^{\pi} d\psi \left|\sin\frac{\psi}{2}\right| \{F_e(\mathbf{r}, \mathscr{R}(-\psi)\cdot\mathbf{v};t) - F_e(\mathbf{r},\mathbf{v};t)\} \; . \tag{37}$$

In (37), the rotation $\mathscr{R}(-\psi)$ gives the pre-collisional orientation to the electron velocity (see (10)). For times not exceeding the cyclotron period T_c the Boltzmann equation (23) governs the dynamics, because the manifestation of pre-collisional correlations requires the occurrence of recollisions with the same disk and the recollisions which survive in the Grad limit are only those which result from unperturbed cyclotron motion, without intermediate encounters with other scatterers. Clearly, in the limit of point disks, such dynamical events are impossible before the cyclotron period T_c is accomplished.

For times larger than T_c, Eq. (36) governs the evolution of wandering electrons subject to scattering processes, whereas at the same time the fraction $\exp(-\nu T_c)$ of the initial density is formed from electrons circling freely under the action of the magnetic field (25). The r.h.s. of Eq. (36) is entirely due to the creation of dynamical correlations. The non-markovian collision operator in (36) contains the propagator $\exp(-t[L_e - T(e,1) + \nu])$. When this propagator is applied to an initial condition, it provides a solution of the two-body electron/hard-disk problem, and, at the same time, takes into account the probability weight $\exp(-\nu t)$ of collisionless motion. The physical content of the memory term in (36) becomes thus clear once the motion of the electron recolliding with a fixed hard disk is understood (see e.g. Ref. [4]; see also Ref. [13] for a related problem in the presence of an electric field). The evaluation of the Grad limit in (36) yields the kinetic equation (for a detailed calculation see Appendix B)

$$\begin{aligned}\left(\frac{\partial}{\partial t} + L_e\right) F_e^w(e;t) \underset{t\geqslant T_c}{=} nav \int_{-\pi}^{\pi} d\psi \left|\sin\frac{\psi}{2}\right| \\ \times \Bigg\{\sum_{s=0}^{N-1} \mathrm{e}^{-\nu s T_c} \times [F^w(\mathbf{r}, \mathscr{R}(-(s+1)\psi)\cdot\mathbf{v}; t - sT_c) \\ -F^w(\mathbf{r}, \mathscr{R}(-s\psi)\cdot\mathbf{v}; t - sT_c)] + \mathrm{e}^{-\nu N T_c} \\ \times\ [F(\mathbf{r}, \mathscr{R}(-(N+1)\psi)\cdot\mathbf{v}; t - NT_c) - F(\mathbf{r}, \mathscr{R}(-N\psi)\cdot\mathbf{v}; t - NT_c)]\Bigg\}\ ,\end{aligned} \tag{38}$$

where N is the integer part of t/T_c: $t = NT_c + \delta t$ with $0 \leqslant \delta t < T_c$. In (38), rotation angles are multiples of the scattering angle ψ because, when the radius a of disk 1 vanishes, the direction of the electron velocity just before the $(n+1)$th collision coincides with its direction just after the nth collision.

Combining the Boltzmann equation (37) with (25) and (38), we recover the GBE of Ref. [3] where for simplicity only spatially homogeneous states have been considered. (The GBE has been written in Ref. [3] in a compact way by introducing the notation $F^G \equiv \theta(T_c - 0^+ - t)[F^c + F^w] + \theta(t - T_c + 0^+)F^w$). The authors of Ref. [3] say that the BBGKY hierarchy is "the royal road to *every* kinetic equation". In the present derivation of the GBE we have indicated the existence of this road. There remains an interesting and subtle question open: the structure of leading corrections to the GBE when the Grad parameter is small, $na^2 \ll 1$, but finite.

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Appendix A

In view of the derivation of the Grad limit for the first hierarchy equation with full collisional operators in Section 4, we show how the closed equation (15) obeyed by the Grad limit (12) of the exact solution for $F_e^v(e;t)$ can be retrieved directly from the first virtual hierarchy equation (16) by properly handling the Grad limit of correlation effects. The discussion leads to a comparison between the different factorized solutions for the virtual hierarchy on one hand and for the so-called Boltzmann hierarchy on the other hand.

A.1. Factorized solutions of the virtual hierarchy

In [5] the very possibility of getting a rigorous solution to the virtual hierarchy was due to a remarkable property of this hierarchy. Contrary to the real part $T^r(e,j)$, the virtual collision operator $T^v(e,j)$ does not alter the electron velocity, so that the virtual hierarchy propagates in time the factorized structure of reduced distributions of the form

$$F_{e,s}^v(e,1,\ldots,s;t) = F_e^v(e;t)\prod_{j=1}^{s} F^v(j;t|e) . \tag{A.1}$$

Thanks to factorization (A.1) the infinite virtual hierarchy is reduced to a set of two coupled equations. The conditional density $F^v(j;t|e)$ obeys the equation

$$\left(\frac{\partial}{\partial t} + L_e - T^v(e,j)\right) F^v(j;t|e) = 0 . \tag{A.2}$$

At the initial time $F^v(j;t=0|e) = \theta(|\mathbf{r} - \mathbf{R}_j| - a - 0^+)$ in the present Lorentz model where scatterers are randomly distributed with possible overlapping. The resolution of

(A.2) provides an exact closed equation for $F_e^v(e;t)$, whose solution is calculated in Ref. [5]. Eq. (12) is the Grad limit of the latter exact solution for $F_e^v(e;t)$.

Now we look directly at the Grad limit of the first virtual hierarchy equation (16), where correlation effects appear through $F_{e,1}^v(e,1;t) \equiv F_e^v(e,t) \times F^v(1;t|e)$. $F^v(1;t{=}0|e)$ is equal to 1 except inside disk 1 where it vanishes. The contribution from the interior of disk 1 to the r.h.s. of the first hierarchy equation (16) is at least of order na^2, and it vanishes in the Grad limit (1). Therefore, $F^v(1;t=0|e)$ can be replaced by 1 on the r.h.s. of this equation without changing the value of the integral limit. The same is true for all equations in the virtual hierarchy. In other words, we are allowed to write

$$\lim_{\mathrm{Grad}} F^v(j;t|e) = \left\{ \mathrm{e}^{-t[L_e - T^v(e,j)]} 1 \right\} (\mathbf{r}, \mathbf{v}) \tag{A.3}$$

on the r.h.s. of all these equations.

The explicit form of function (A.3) is studied in Ref. [5]. It is equal to 1 at any time t for nearly all $(\mathbf{r}, \mathbf{v})$. The exceptions are states $(\mathbf{r}, \mathbf{v})$'s such that an electron, starting at point $\mathbf{r}$ outside disk j with a velocity $\mathbf{v}$, has a free motion backward in time which leads it to the surface of disk j with an outgoing velocity after a finite time $\tau^*(\mathbf{r}, \mathbf{v}, \mathbf{R}_j)$. For these $(\mathbf{r}, \mathbf{v})$'s (A.3) is equal to $\theta(\tau^* - 0^+ - t)$: it vanishes for times t larger than $\tau^*(\mathbf{r}, \mathbf{v}, \mathbf{R}_j)$. The virtual collision operator $T^v(e,1)$ on the r.h.s. of the first equation (16) in the virtual hierarchy selects evolutions where the trajectory intersects the surface of disk 1 at point $\mathbf{r}$ with an incoming velocity $\mathbf{v}$. In this case $\tau^*(\mathbf{r}, \mathbf{v}, \mathbf{R})$ is finite and tends to the cyclotron period T_c when a vanishes. This yields the exact relation (17). When the latter relation is used in the first equation (16) of the virtual hierarchy, $F_e^v(e;t)$ proves to obey the closed equation (15). The kinetic equation (15) does coincide with the equation obeyed by (12).

A.2. Straight-line motion and Boltzmann hierarchy

In the absence of any external field, an electron starting from the disk surface with an incoming velocity will never hit it again under its straight-line motion backward in time. In the case of the annihilation (or virtual) dynamics, this implies that the limit of an infinite T_c is to be taken in (17) when L_e is replaced by L_e^0. Then, at any time t (17) becomes

$$\lim_{\mathrm{Grad}} \theta(-\hat{\boldsymbol{\sigma}}_1 \cdot \mathbf{v}) F_{e,1}^{v0}(e,1;t)\big|_{|\mathbf{r}-\mathbf{R}_1|=a+0^+} = \lim_{\mathrm{Grad}} \theta(-\hat{\boldsymbol{\sigma}}_1 \cdot \mathbf{v}) F_e^{v0}(e;t)\big|_{|\mathbf{r}-\mathbf{R}_1|=a+0^+} \,. \tag{A.4}$$

As a consequence, the r.h.s. of the first equation of the virtual hierarchy with L_e^0 in place of L_e becomes equal to $-vF_e^{v0}(e;t)$: $F_e^{v0}(e;t)$ obeys the "virtual" Boltzmann equation (18), contrary to $F_e^v(e;t)$ for the cyclotronic motion.

More generally when the full collision operators $T(e,j)$'s are considered, for the Lorentz model in the absence of external field, the Grad limit of the exact solution for the one-particle density $F_e(e;t)$ in the BBGKY hierarchy coincides with the solution for $F_e(e;t)$ in the so-called Boltzmann hierarchy. The latter is obtained by neglecting collision operators on the l.h.s. of the BBGKY equations, according to the dimensional analysis recalled at the beginning of Section 3. The corresponding Boltzmann hierarchy propagates in time the factorization of $F_{e,s}(e,1,\ldots,s;t)$ equal to

$$F^B_{e,s}(e,1,\ldots,s;t) = F^B_e(e;t)\prod_{j=1}^{s} F^B(j)\ . \tag{A.5}$$

The conditional density $F^B(j;t|e)$ is equal to the constant $F^B(j)$: the factorization (A.5) is different from the factorization (A.1) propagated by the exact virtual hierarchy. In the present model for hard-disk configurations, the conditional density $F^B(j)$ can be replaced by 1 in the Grad limit, as already explained before (A.3). Then the two-body reduced distribution function $F^B_{e,1}(e,j;t)$ coincides with the electron probability density $F^B_e(e;t)$ for any state $(\mathbf{r},\mathbf{v})$ and for any time t,

$$\lim_{\text{Grad}} F^B_{e;1}(e,1;t) = \lim_{\text{Grad}} F^B_e(e;t)\ . \tag{A.6}$$

As a consequence, in the Grad limit, the first equation of the Boltzmann hierarchy becomes

$$\left(\frac{\partial}{\partial t} + L^0_e\right) F^B_e(e;t) = n\int d1\, T(e,1) F^B_e(e;t)\ . \tag{A.7}$$

Eq. (A.7) coincides with the well-known Boltzmann equation. In the case where every collision operator $T(e,j)$ is replaced by its virtual part $T^v(e,j)$, (A.7) is reduced to (18).

We stress that, for the BBGKY hierarchy with full collision operators and free straight-line motion, property (A.4) rewritten for $F^0_{e,1}(e,1;t)$ and $F^0_e(e;t)$ is sufficient to ensure two properties: first, in the Grad limit the first hierarchy equation becomes a closed equation for the Grad limit of the exact solution $F^0_e(e;t)$; second, this closed equation coincides with the Boltzmann equation (A.7), namely with the Grad limit of the first equation in the Boltzmann hierarchy. The fact that property (A.4) exhibited for the virtual hierarchy is also a sufficient condition for the hierarchy with full collision operators when it is rewritten for $F^0_{e,1}(e,1;t)$ and $F^0_e(e;t)$ relies on the property recalled after (11): $\{T(e,1)f\}$ does not vanish only for position arguments of f on the border of disk 1 and for velocity arguments of f which are incoming.

However, though the Grad limit of the exact solution $F^0_e(e;t)$ of the BBGKY hierarchy coincides with the Grad limit of the exact solution $F^B_e(e;t)$ of the Boltzmann hierarchy, this is not the case for the Grad limits of the other reduced distributions $F^0_{e,s}(e,1,\ldots,s;t)$ and $F^B_{e,s}(e,1,\ldots,s;t)$ respectively. This can be checked in the case of virtual hierarchy: the Grad-limit factorized solution (A.1) and (A.3), also valid for L^0_e, does not coincide with the Grad-limit Boltzmann solution (A.5) and (A.6).

Appendix B

In the present appendix we calculate the Grad limit of the r.h.s. of (36). The propagator $\exp(-t[L_e - T(e,1)])$ describes the electron motion around the hard disk 1 in the absence of any other scatterer. Its explicit action can be retrieved from its expansion in terms of the "virtual" propagator $\exp(-t[L_e - T^v(e,1)])$ (studied in

Ref. [5]) and of the real collision operator $T^r(e,1)$,

$$e^{-t[L_e-T(e,1)]}=e^{-t[L_e-T^v(e,1)]}$$
$$+\sum_{s=1}^{+\infty}\underbrace{e^{-t[L_e-T^v(e,1)]}*T^r(e,1)e^{-t[L_e-T^v(e,1)]}*\cdots*T^r(e,1)}_{s\ \text{terms}}e^{-t[L_e-T^v(e,1)]}\ . \tag{B.1}$$

In (B.1) the convolution is defined as

$$f(t)*g(t)\equiv\int_0^t d\tau f(t-\tau)g(\tau)\ . \tag{B.2}$$

In order to calculate $T(e,1)\exp(-t[L_e-T^v(e,1)])$, we recall that $\{T(e,1)f\}(\mathbf{r},\mathbf{v})$ is a function which is non-zero only when the position argument of f is on the border of disk 1, while the velocity argument of f is incoming (see (11)). If the electron starts at $t=0$ at the surface of disk 1 with an incoming velocity and evolves backward in time under the action of $\exp(-t[L_e-T^v(e,1)])$, it touches disk 1 again (with an outcoming velocity) at a time which coincides with T_c when a vanishes. Then, according to the definition (10) of $T^r(e,1)$,

$$\lim_{a\to 0}\{\theta(-\hat{\boldsymbol{\sigma}}_1\cdot\mathbf{v})e^{-t[L_e-T^v(e,1)]}T^r(e,1)f\}(\mathbf{r},\mathbf{v})|_{|\mathbf{r}-\mathbf{R}_1|=a+0^+}$$
$$=\delta(t-T_c+0^+)\{\theta(-\hat{\boldsymbol{\sigma}}_1\cdot\mathbf{v})e^{-T_cL_e}b_{\hat{\boldsymbol{\sigma}}}f\}(\mathbf{r},\mathbf{v})|_{|\mathbf{r}-\mathbf{R}_1|=0^+}\ . \tag{B.3}$$

In (B.3), $\{\theta(-\hat{\boldsymbol{\sigma}}_1\cdot\mathbf{v})\exp(-T_cL_e)b_{\hat{\boldsymbol{\sigma}}}\}(\mathbf{r},\mathbf{v})|_{|\mathbf{r}-\mathbf{R}_1|=0^+}$ is again a state at the surface of disk 1 with an incoming velocity. Then (B.1) and (B.3) lead to

$$\lim_{a\to 0}T(e,1)e^{-(t-\tau)[L_e-T(e,1)]}T^r(e,1)$$
$$=\sum_{s=1}^{+\infty}\delta([t-\tau]-sT_c+0^+)T(e,1)\underbrace{e^{-T_cL_e}b_{\hat{\boldsymbol{\sigma}}_1}\cdots e^{-T_cL_e}b_{\hat{\boldsymbol{\sigma}}_1}}_{s\ \text{terms}}\ . \tag{B.4}$$

According to (B.4), the two contributions in the r.h.s. of (36) yield

$$\lim_{a\to 0}T(e,1)\int_0^{T_c-0^+}d\tau e^{-(t-\tau)[L_e-T(e,1)+v]}T^r(e,1)F_e(e,1;\tau)$$
$$=e^{-vNT_c}T(e,1)\underbrace{e^{-T_cL_e}b_{\hat{\boldsymbol{\sigma}}_1}\cdots e^{-T_cL_e}b_{\hat{\boldsymbol{\sigma}}_1}}_{N\ \text{terms}}F_e(e;t-NT_c)\ , \tag{B.5}$$

where N is the integer part of t/T_c, and

$$\lim_{a\to 0}T(e,1)\int_{T_c-0^+}^{t}d\tau e^{-(t-\tau)[L_e-T(e,1)+v]}T^r(e,1)F_e^w(e;\tau)$$
$$=\sum_{s=1}^{N-1}e^{-vsT_c}T(e,1)\underbrace{e^{-T_cL_e}b_{\hat{\boldsymbol{\sigma}}_1}\cdots e^{-T_cL_e}b_{\hat{\boldsymbol{\sigma}}_1}}_{s\ \text{terms}}F_e^w(e;t-sT_c)\ . \tag{B.6}$$

Moreover, under the motion backward in time during one cyclotron period, the free propagator $\exp(-T_c L_e)$, makes the position come back to its initial value and the velocity rotate with an angle 2π. The rotation angle of $b_{\hat{\boldsymbol{\sigma}}_1}$ is equal to minus scattering angle ψ (see (10)). Thus, the action of the s terms in the r.h.s. of (B.6) is just to make the velocity rotate with an angle $-s\psi$. According to the definition (6) of $T(e,1)$,

$$\lim_{\text{Grad}} n \int d1\, T(e,1) \underbrace{e^{-T_c L_e} b_{\hat{\boldsymbol{\sigma}}_1} \cdots e^{-T_c L_e} b_{\hat{\boldsymbol{\sigma}}_1}}_{s \text{ terms}} f(e; t - sT_c)$$

$$= nav \int_{-\pi}^{\pi} d\psi \left| \sin \frac{\psi}{2} \right|$$

$$\times \{ f(\mathbf{r}, \mathscr{R}(-(s+1)\psi) \cdot \mathbf{v}; t - sT_c) - f(\mathbf{r}, \mathscr{R}(-s\psi) \cdot \mathbf{v}; t - sT_c) \} . \tag{B.7}$$

When (B.5) and (B.6) are used together with the latter identity, (36) becomes (38).

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