J. Physique Lett. **45** (1984) L-707 - L-712 15 JUILLET **1984**, PAGE L-707

Classification
Physics Abstracts
75.50K

Self-averaging correlation functions in the mean field theory of spin glasses

M. Mézard and G. Parisi (*)

Laboratoire de Physique Théorique de l'Ecole Normale Supérieure (+),
24, rue Lhomond, 75231 Paris Cedex 05, France

(*Reçu le 25 avril 1984, accepté le 28 mai 1984*)

Résumé. — Dans le modèle des verres de spin où les interactions ont une portée infinie, nous considérons le spin alterné σ_λ associé à un vecteur propre donné de la matrice des couplages. Nous montrons que la moyenne thermique de σ_λ^2 est une grandeur qui ne fluctue pas avec les couplages, et nous la calculons.

Abstract. — In the infinite range spin glass model, we consider the staggered spin σ_λ associated with a given eigenvector of the interaction matrix. We show that the thermal average of σ_λ^2 is a self-averaging quantity and we compute it.

The low temperature behaviour of spin glasses is a long-standing problem which still remains a source of surprises. One of the recent discoveries in the theory of spin glasses is the lack of self-averageness [1, 2]. Certain quantities, like the weights and overlaps between equilibrium states or the magnetic susceptibility, fluctuate with the realizations of the random couplings, even in the thermodynamic limit. On the other hand, some physical quantities like the free energy, the internal energy and the magnetization are self-averaging.

These results have been obtained in the mean field theory, that is in the infinite range model of Sherrington and Kirkpatrick (S.K.) [3] where the N Ising spins σ_i interact with one another through couplings J_{ij}. The J_{ij} are independent random variables with Gaussian distribution of zero mean and variance $1/\sqrt{N}$. In this approach, the average over the couplings is calculated with the replica method [5], and the non-self-averageness is closely related to the replica symmetry breaking (RSB). Hence the breaking of ergodicity and the fluctuations relative to the realizations of the couplings are tightly mixed and it is highly desirable to disentangle these two kinds of effects.

(*) Address : INFN, Frascati and Universita di Roma II, Tor Vergata, Roma, Italy.
(+) Laboratoire Propre du Centre National de la Recherche Scientifique, associé à l'Ecole Normale Supérieure et à l'Université de Paris Sud.

In this paper, we attempt a first step in this direction. We introduce the set of gauge invariant extensive two-point functions :

$$E_p = \frac{1}{N} \sum_{i_1,\dots,i_{p+1}} \langle \sigma_{i_1} J_{i_1 i_2} \dots J_{i_p i_{p+1}} \sigma_{i_{p+1}} \rangle$$

$$= \frac{1}{N} \sum_{i,j} \langle \sigma_i (J^p)_{ij} \sigma_j \rangle \tag{1}$$

($\langle\ \rangle$ denotes the thermal average while we shall use the notation $\overline{(\ )}$ for the averages over the couplings). We shall show that these functions E_p are self-averaging, and we shall compute them explicitly, within the RSB scheme of reference [4], in terms of the average order parameter function $Q(x)$.

This result is most easily described in terms of the staggered spin states : let $| \lambda \rangle$ be the eigenstates of the random $N \times N$ matrix J, with eigenvalues λ, and let $| \sigma \rangle$ denote the N-dimensional vector of the spins (with $\sigma_i \equiv \langle i | \sigma \rangle$). The staggered spin states are $\sigma_\lambda \equiv \langle \lambda | \sigma \rangle$. From the definition (1) of E_p, we have :

$$E_p = \int \mathrm{d}\lambda \, \rho(\lambda) \, \lambda^p \, g(\lambda) \tag{2}$$

where $\rho(\lambda)$ is the density of eigenvalues around λ, which is given for large N by [9] :

$$\rho(\lambda) = \frac{1}{2\pi}\sqrt{4 - \lambda^2} \tag{3}$$

and $g(\lambda)$ is the staggered correlation function :

$$g(\lambda) = \langle \sigma_\lambda^2 \rangle = \sum_{i,j} \langle \lambda | i \rangle \langle \lambda | j \rangle \langle \sigma_i \sigma_j \rangle . \tag{4}$$

From the results on the E_p, we shall deduce that $g(\lambda)$ are self-averaging, and we shall give their expressions in terms of λ and $Q(x)$ (we shall always use the RSB scheme of [4]).

We now turn to the computation of the average values $\overline{E}_p$. $\overline{E}_1$ is equal to minus twice the internal energy. There are two basic steps in its computation :

— by introducing n replicas of the system and letting $n \to 0$, one can get rid of the partition function Z in the denominator of (1);

— the average over the couplings can then be done. One takes care of the explicit J_{ij} factor in (1) through an integration by parts, using :

$$J_{ij} \, \mathrm{e}^{-\frac{N}{2} J_{ij}^2} = - \frac{1}{N} \frac{\partial}{\partial J_{ij}} \mathrm{e}^{-\frac{N}{2} J_{ij}^2} . \tag{5}$$

This gives

$$\overline{E}_1 = \lim_{n \to 0} \frac{1}{n} \sum_{a,b=1}^{n} \overline{\mathrm{Tr}_{\{\sigma\}} \left(\frac{\beta}{N^2} \sum_{i \neq j} \sigma_i^a \sigma_i^b \sigma_j^a \sigma_j^b \exp\left[- \beta \sum_{c=1}^{n} H(\sigma_c) \right] \right)}$$

$$= \beta \lim_{n \to 0} \frac{1}{n} \mathrm{Tr}\, Q^2 \tag{6}$$

where Q_{ab} is the $n \times n$ order parameter matrix, taken here with $Q_{aa} = 1$, and $H(\sigma_c)$ is the S.K.

Hamiltonian for replica c. For n going to zero, one finds the standard result :

$$\bar{E}_1 = \beta\left(1 - \int_0^1 Q^2(x)\,\mathrm{d}x\right). \tag{7}$$

The strategy to compute the other $\bar{E}_p$'s is the same. One transforms the string $J_{i_1 i_2} \dots J_{i_p i_{p+1}}$ in (1) into a sum $\sum\limits_{a_1} \dots \sum\limits_{a_p} (\sigma_{i_1}^{a_1} \sigma_{i_2}^{a_1}) \dots (\sigma_{i_p}^{a_p} \sigma_{i_{p+1}}^{a_p})$ through an integration by parts, using the identity (5). One must, however, be slightly more careful since this identity does not hold if a given link (i, j) appears several times in the sequence $(i_1, i_2) \dots (i_p, i_{p+1})$. For instance, for a double link l one should use :

$$J_l^2 \, \mathrm{e}^{-\frac{N}{2} J_l^2} = \frac{1}{N^2}\left(\frac{\partial^2}{\partial J_l^2} + N\right) \mathrm{e}^{-\frac{N}{2} J_l^2}. \tag{8}$$

Then one can perform a simple power-counting of the N factors to see what kind of diagrams dominates :

— a diagram where all the p links are distinct is of order N^0; there are $p + 1$ summations on the sites giving a N^{p+1}, p links giving a $1/N^p$ from (5), and a global $1/N$ factor in the definition (1) of E_p. Its precise contribution is

$$X_p = \lim_{n \to 0} \frac{1}{n} \operatorname{Tr}(\beta^p Q^{p+1}) \tag{9}$$

— a diagram with one double link has $p - 2$ ordinary links giving a $1/N^{p-2}$, one double link with an « anomalous » contribution from (8) going as $1/N$, and in general it has $(p + 1) - 2$ summations on sites, so it gives a total contribution of order $1/N$. The only case when such a diagram contributes is when the double link introduces only one constraint on the sites, in which case there are $(p + 1) - 1$ summations on sites only. This happens when the two identical links are neighbours in the sequence $(i_1, i_2) \dots (i_p, i_{p+1})$. Such a diagram gives a contribution X_{p-2} to $\bar{E}_p$.

The same power-counting argument shows that diagrams with links of order three or more are always negligible. So, the general formula for $\bar{E}_p$ is :

$$\bar{E}_p = \sum_{q=0}^{p} c_{p,q} X_q \tag{10}$$

where $c_{p,q}$ is the number of distinct diagrams with $p - q$ neighbouring double links. One finds, in this way :

$$\bar{E}_2 = X_2 + X_0, \quad \bar{E}_3 = X_3 + 2 X_1, \dots \tag{11}$$

These formulae can be inverted and give

$$X_p = \sum_{q=0}^{p} S_{p,q} \bar{E}_q. \tag{12}$$

Using the diagrammatic arguments of [6], we find that the $S_{p,q}$ are well known coefficients : $S_p(x) = \sum\limits_{q=0}^{p} S_{p,q} x^q$ are Chebyshev polynomials of the second kind, a set of polynomials orthogonal on $[-2, 2]$ with the measure $\sqrt{1 - x^2/4}$ [7].

From (12) one implicitly knows the $\overline{E}_p$'s as functions of the X_q's defined in (9), that is as functions of the eigenvalue spectrum of the 0×0 Q_{ab} matrix. It turns out that this spectrum can also be computed. We introduce the generating function $K(u) = \dfrac{1}{\sqrt{\det(1 + uQ)}}$. This is nothing but the partition function for n scalar fields ϕ_a interacting through the Hamiltonian

$$H = \frac{1}{2} \sum_{a,b} \phi_a(\delta_{ab} + Q_{ab}) \phi_b$$

$$K(u) = \int \prod_a \frac{\mathrm{d}\phi_a}{\sqrt{2\pi}} \mathrm{e}^{-H} . \tag{13}$$

This partition function can be computed to all orders in the RSB, using the methods of references [4, 8]. Taking the limit $n \to 0$ and expanding in powers of u, we find :

$$X_p = \beta^p(1 - I)^{p+1} + (p + 1) \beta^p(1 - I)^p Q(0) - \\ - \int_0^1 \frac{\mathrm{d}x}{x^2} \beta^p \{ [1 - I + \Delta(x)]^{p+1} - [1 - I]^{p+1} \} \tag{14}$$

where

$$I = \int_0^1 Q(x) \,\mathrm{d}x \qquad \Delta(x) = \int_0^x Q(x') \,\mathrm{d}x' - xQ(x) . \tag{15}$$

The values of the averages $\overline{E}_p$ are known from formulae (12) and (14).

The self-averageness of E_p can be demonstrated through an explicit computation of the fluctuations; with the replica trick one must average over the couplings a product of two strings :

$$\overline{E_p^2} = \lim_{n\to 0} \frac{1}{n(n-1)} \sum_{a\neq a'} \frac{1}{N^2} \sum_{i,j} \sum_{i',j'} \overline{\mathrm{Tr}_{\{\sigma\}}\left(\sigma_i^a (J^p)_{ij} \sigma_j^a \sigma_{i'}^{a'} (J^p)_{i'j'} \sigma_{j'}^{a'} \exp\left[- \beta \sum_{c=1}^{n} H(\sigma_c) \right]\right)} . \tag{16}$$

The simplest term appears when all the links are distinct. Using (5), one gets for this term :

$$\lim_{n\to 0} \frac{1}{n(n-1)} \sum_{a\neq a'} \beta^{2p} (Q^{p+1})_{aa} (Q^{p+1})_{a'a'} = X_p^2 . \tag{17}$$

As before, the corrections come from the multiple links. The power counting of N shows that the double links involving one link of each chain are always non-leading. The only terms which contribute for $N \to \infty$ are the double links in each individual chain. This gives, precisely :

$$\overline{E_p^2} = \sum_{q=0}^{p} \sum_{q'=0}^{p} c_{p,q} c_{p,q'} X_q X_{q'} = \overline{E}_p^2 . \tag{18}$$

We can state these results in terms of the staggered correlation functions $g(\lambda)$ defined in (4) : obviously, these functions are self-averaging, and we know all their moments. These can be inverted; from (2) and (12) we have :

$$X_p = \int_{-2}^{2} \mathrm{d}\lambda \frac{1}{\pi} \sqrt{1 - \frac{\lambda^2}{4}} S_p(\lambda) g(\lambda) . \tag{19}$$

So the X_p's are nothing but the coefficients in the expansion of $g(\lambda)$ on the Chebyshev polynomials.

The final result for $g(\lambda)$ is deduced from the explicit expressions (14) of X_p. For simplicity we quote the result in zero magnetic field (using $q(0) = 0$ and $\beta(1 - I) = 1$) :

$$g(\lambda) = \frac{1 - Q_{\mathrm{E.A.}}}{1 - \beta\lambda(1 - Q_{\mathrm{E.A.}}) + \beta^2(1 - Q_{\mathrm{E.A.}})^2} + \int_0^1 \mathrm{d}x \frac{\mathrm{d}Q}{\mathrm{d}x} \frac{1 - [1 + \beta\Delta(x)]^2}{[1 - \lambda[1 + \beta\Delta(x)] + [1 + \beta\Delta(x)]^2]^2}. \tag{20}$$

The function $\rho(\lambda)\, g(\lambda)$ is plotted in figure 1, using for $Q(x)$ the ansatz of reference [10] and the approximate expression $Q_{\mathrm{E.A.}}(T) = 1 - 2\left(\frac{T}{T_c}\right)^2 + \left(\frac{T}{T_c}\right)^3$.

It has been already mentioned several times [11-13] that the spin glass transition is associated with the appearance of a non-zero staggered magnetization, although it is debated whether the mode $\lambda = 2$ is macroscopically populated. It appears now that the staggered magnetization is less interesting than the staggered function $g(\lambda)$, since it is not self-averaging. (This can be seen from its first moment $\frac{1}{N}\sum_{i,j} \langle \sigma_i \rangle J_{ij} \langle \sigma_j \rangle$ which fluctuates with the couplings). However, $h(\lambda) = \overline{\langle \sigma_\lambda \rangle^2}$ can be computed in the same way as g. The only difference is that X_p should be replaced by :

$$Y_p = \lim_{n \to 0} \frac{1}{n(n-1)} \sum_{a \neq b} \beta^p (Q^{p+1})_{ab} = X_p - \beta^p(1 - I)^{p+1} . \tag{21}$$

This gives in zero field :

$$h(\lambda) = \overline{\langle \sigma_\lambda \rangle^2} = g(\lambda) - \frac{T}{2 - \lambda}. \tag{22}$$

From (20) and (22), we find that both $g(\lambda)$ and $h(\lambda)$ diverge at $\lambda \to 2$ as $(2 - \lambda)^{-5/4}$. Such a behaviour for the staggered magnetization $h(\lambda)$ has already been found in a different approach [13]. However, the products $\rho(\lambda)\, g(\lambda)$ and $\rho(\lambda)\, h(\lambda)$ exhibit integrable divergencies. There is no macroscopic condensation in any of the eigenmodes.

To conclude, we have obtained general expressions for the staggered spin correlation function and the staggered magnetization. It is surprising that $\langle \sigma_\lambda^2 \rangle$ turns out to be self-averaging : this might give some new kind of information on random matrices. Indeed, there is some combination of the components of the eigenvectors of a large random matrix which tends towards a

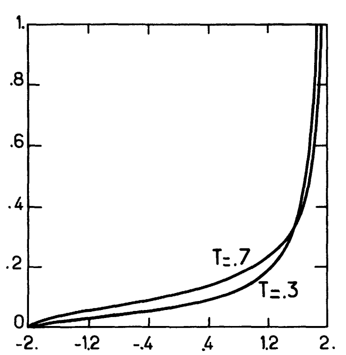

Fig. 1. — Plot of the staggered correlation function $\langle \sigma_\lambda^2 \rangle$ times the density of eigenvalues $\frac{1}{2\pi}\sqrt{4 - \lambda^2}$ *versus* λ, for the temperatures $T = 0.3$ and $T = 0.7$.

well defined limit when $N \to \infty$. We must recognize however, that this information is still very indirect since it involves the correlation function (for instance at $T = 0$) in the S.K. model having this matrix of couplings.

Acknowledgments.

We wish to thank N. Sourlas, G. Toulouse and M. Virasoro for useful conversations.

References

[1] MÉZARD, M., PARISI, G., SOURLAS, N., TOULOUSE, G., VIRASORO, M., *Phys. Rev. Lett.* **52** (1984) 1156 ; and preprint LPTENS 83/40, to be published in *J. Physique*.
[2] YOUNG, A. P., BRAY, A., MOORE, M., *J. Phys. C* **17** (1984) L 149.
[3] SHERRINGTON, D., KIRKPATRICK, S., *Phys. Rev. Lett.* **32** (1975) 1792.
[4] PARISI, G., *Phys. Rev. Lett.* **43** (1979) 1754 ; and *J. Phys. A* **13** (1980) L 115, 1101, 1887.
[5] EDWARDS, S. F., ANDERSON, P. W., *J. Phys. F* **5** (1975) 965.
[6] BRÉZIN, E., ITZYKSON, C., PARISI, G., ZUBER, J. B., *Commun. Math. Phys.* **59** (1978) 35.
[7] ABRAMOWITZ, M., STEGUN, I. A., ed., *Handbook of Mathematical Functions*, 9th printing (Dover Publications, Inc., New York).
[8] DUPLANTIER, B., *J. Phys. A* **14** (1981) 283.
[9] MEHTA, M. L., *Random Matrices* (Academic Press) 1967.
[10] VANNIMENUS, J., TOULOUSE, G., PARISI, G., *J. Physique* **42** (1981) 565.
[11] ANDERSON, P. W., in Les Houches 1978, *Ill Condensed Matter* (North Holland) ;
HERTZ, J. A., FLEISHMAN, L., ANDERSON, P. W., *Phys. Rev. Lett.* **43** (1979) 942.
[12] SOMPOLINSKY, H., *Phys. Rev. B* **23** (1981) 1371.
[13] DASGUPTA, C., SOMPOLINSKY, H., *Phys. Rev. B* **27** (1983) 4511.