Remarks:

- A more thorough introduct° to stochastic processes will be given in G. Binichou's 2nd semester option.
- We recall Bayes' rule: $P(B;A) = P(B|A) \times P(A)$. The Markov ppty can be derived by applying it w/ the assumption $P(x_n t_n | x_{n-1} t_{n-1}; \dots; x_0 t_0) = P(x_n t_n | x_{n-1} t_{n-1})$ (i.e., memorylessness).

The Markov ppty implies the Chapman-Kolmogorov equation:

$$P(x_2 t_2 | x_0 t_0) = \int dx_1 \; P(x_2 t_2 | x_1 t_1) \, P(x_1 t_1 | x_0 t_0)$$

2°) Fokker-Planck through a Kramers-Moyal expansion

We consider the pbility distribution $P(x,t)$ (forget about the explicit mention of the initial condition), and consider its evolution over a short time scale Δt. We choose Δt such that the displacement r of the particle over Δt is much smaller than l_{force}, and much smaller than the typical length scale over which $P(x,t)$ varies. Then:

$$P(x, t+\Delta t) = \int_{-\infty}^{+\infty} P(x, t+\Delta t | x', t) \; P(x', t) \; dx' \quad \text{(Chap-Kol)}$$

$$= \int_{-\infty}^{+\infty} P(x, t+\Delta t | x-r, t) \; P(x-r, t) \; dr$$

Note that $P(x'+r, t+\Delta t | x', t)$ is the probability of "jumping" by r over a time scale Δt starting from x'. We denote it by

$$W_{\Delta t}(x', r) = P(x'+r, t+\Delta t | x', t),$$

which varies over a length scale l_{force} in x'.

$$P(x, t+\Delta t) = \int_{-\infty}^{+\infty} W_{\Delta t}(x-r, r)\, P(x-r, t)\, dr$$

The first argument of both terms of the integrand varies over a typical length scale l_{force}; we thus expand for small $r \ll l_{force}$:

$$P(x, t+\Delta t) = \int_{-\infty}^{+\infty} \Big\{ W_{\Delta t}(x, r)\, P(x,t) + \sum_{n=1}^{+\infty} \frac{(-r)^n}{n!} \partial_x^n \big[W_{\Delta t}(x, r)\, P(x,t) \big] \Big\} dr$$

$$= P(x,t) \underbrace{\int_{-\infty}^{+\infty} W_{\Delta t}(x, r)\, dr}_{=1} + \sum_{n=1}^{+\infty} \frac{(-1)^n}{n!} \partial_x^n \big[a_{n,\Delta t}(x)\, P(x,t) \big]$$

Kramers-Moyal expansion

with $a_{n,\Delta t}(x) = \int_{-\infty}^{+\infty} r^n W_{\Delta t}(x, r)\, dr$ the "n^{th} jump moment"

We truncate to order 2:

$$P(x, t+\Delta t) - P(x,t) = -\partial_x \big[a_{1,\Delta t}(x)\, P(x,t) \big] + \frac{1}{2} \partial_x^2 \big[a_{2,\Delta t}(x)\, P(x,t) \big]$$

We now use the results of I 2°) to obtain the jump moments:

$$a_{1,\Delta t}(x_0) = \int_{-\infty}^{+\infty} r\, P(x_0 + r, \Delta t | x_0, 0) = \langle x(\Delta t) - x_0 \rangle = \mu F(x_0) \cdot \Delta t$$

$$a_{2,\Delta t}(x_0) = \langle (x(\Delta t) - x_0)^2 \rangle = 2D\Delta t + \underbrace{(\mu F(x_0) \Delta t)^2}_{\text{higher order in } \Delta t}$$

Rewriting the evolut° eqn to lowest order in Δt

$$\underline{\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x} \Big[\mu F(x) \cdot P(x,t) - D \frac{\partial P}{\partial x} \Big]}$$

Fokker - Planck equation

Notes:

- Can be rewritten as $\partial_t P = -\partial_x J$, with J a "probability current" with a convective part and a diffusive part given by Fick's law:
$$J = \mu F(x) \cdot P(x,t) - D\, \partial_x P(x,t)$$
- The case where the diffusion coefficient depends on space leads to complications in the interpretation of the Langevin equation (Itô-Stratonovich dilemma). This is beyond our current discussion but go check out Risken "The Fokker-Planck eqn..." Sec. 3.3, 3.4
- The eqn conserves probability: $\partial_t \int_{-\infty}^{+\infty} P(x,t)\,dx = \int \partial_x J = J(+\infty) - J(-\infty) = 0$
  Hence the probability distrib. is always normalized.

Exercises

- Justify the neglect of the higher Kramers-Moyal terms for $\Delta t \to 0$.
- Generalize the FP eqn for a multiple coordinate Langevin dynamics:
$$\dot{x}_i = \mu_{ij} F_j(\{x_k\}) + \xi_i(t) \qquad \text{w/ } \langle \xi_i(t) \rangle = 0, \quad \langle \xi_i(t) \xi_j(t') \rangle = 2 D_{ij}\, \delta(t-t')$$
$$\Rightarrow \quad \partial_t P(x_1, x_2 \dots x_n, t) = -\partial_{x_i}\left(\mu_{ij} F_j P - D_{ij}\, \partial_{x_j} P\right)$$
- Establish the Backwards Kolmogorov equation:
$$\partial_t P(x, t | x_0 t_0) = \mu F(x)\, \partial_{x_0} P(x t | x_0 t_0) + D\, \partial_{x_0}^2 P(x t | x_0 t_0)$$

3) Einstein's equation: a strange coincidence?

Assume our force derives from an equilibrium potential: $F(x) = -\partial_x U$
Then the current-less solution $J = -\mu(\partial_x U) \cdot P - D \partial_x P = 0 \Rightarrow P \propto e^{-\frac{\mu U(x)}{D}}$
is compatible w/ Boltzmann's distrib. iff $D = \mu k_B T$
(for many components: $\mu_{ij} (D^{-1})_{jk} = \frac{\delta_{ik}}{k_B T}$). Why is that?
That suggests that equilibrium dynamics is quite constrained / that not all dynamics are compatible w/ equilibrium.