Hydrodynamics of a superfluid

2020-2021 Statistical Mechanics exam – ICFP M2 – Soft Matter & Biophysics track

The use of written documents as well as electronic devices including cell phones and pocket calculators is prohibited. Write your answers to each problem on separate sheets of papers, as they will be corrected by different people. Vectors are denoted in a bold font in the following.

The concepts of generalized hydrodynamics constrain the behavior of large-scale flows in many systems, including some where quantum effects play a significant role. In this problem, we derive the hydrodynamics equations for superfluid helium II. Interestingly, this discussion requires almost no knowledge of quantum mechanics, and can be conducted simply by accepting that the quantum phase of a Bose liquid plays the role of a broken symmetry variable, and that the gradient of this phase is proportional to the velocity of the superfluid flow which drives mass and momentum transport.

At low temperatures helium II can be described as a superfluid "background" that is in its quantum ground state, within which quantized excitations (mainly phonons at low temperature) propagate. The excitations can be thought of as a classical gas of particles that live within the superfluid. It is therefore reasonable, if not completely accurate, to describe helium II as a mixture of two fluids: a superfluid fraction, and a "normal fluid" made of quasi-particles that behave as if they had a mass.

To familiarize ourselves with fluid mixtures, in Sec. 1 we first derive a system of simplified hydrodynamic equations for a mixture of two classical fluids. In Sec. 2, we contrast these results with those obtained by introducing the aforementioned broken symmetry variable to describe helium II [1, 2, 3]. Finally in Sec. 3, we discuss the resulting superfluid hydrodynamic equations to two geometries to demonstrate some of the peculiarities pertaining to its flows.

All fluids considered here are isotropic, achiral and translationally invariant. The sections are presented in order of increasing difficulty. While Sec. 2 is formally independent from Sec. 1, a good understanding of the latter is very helpful in solving the former. Both section lay the necessary foundation to tackle Sec. 3.

1 Classical two-fluid system

We consider a three-dimensional fluid made of two components a and b. For simplicity we consider their respective mass densities ρ^a and ρ^b as constant and homogeneous in space. This may stem from a mechanism whereby the teo components are locally produced and evacuated, as in a Malthusian flock, or simply because of their individual incompressibility. We denote $\rho = \rho^a + \rho^b$ in the following, and the temperature is also held constant throughout the system. We assume that the local state of the system is well described by the momentum densities of the two components, respectively denoted by $\mathbf{g}^a(\mathbf{r}, t)$ and $\mathbf{g}^b(\mathbf{r}, t)$, where $\mathbf{r}$ denotes the position and t denotes time. The free energy density of the system consequently reads

$$f = \frac{(\mathbf{g}^a)^2}{2\rho^a} + \frac{(\mathbf{g}^b)^2}{2\rho^b}, \tag{1}$$

which is simply the sum of the kinetic energy densities of the a and b fractions.

In the following we describe our fluid with two fields, namely $\mathbf{g}(\mathbf{r},t) = \mathbf{g}^a + \mathbf{g}^b$ and $\mathbf{w}(\mathbf{r},t) = \mathbf{g}^b/\rho^b - \mathbf{g}^a/\rho^a$.

1.1 What is the physical meaning of $\mathbf{g}$? Why is it a hydrodynamic variable?

1.2 Propose a physical interpretation of $\mathbf{w}$.

1.3 What are the signatures of $\mathbf{g}$ and $\mathbf{w}$ under time reversal?

1.4 Write the thermodynamic force $\mathbf{A}^g$ conjugate to $\mathbf{g}$ defined through

$$A_i^g = \left.\frac{\partial f}{\partial g_i}\right|_{\mathbf{w}} \tag{2}$$

as a function of $\mathbf{g}$, $\mathbf{w}$. When performing this calculation do pay attention to what is being held constant in the derivative.

1.5 What is the physical interpretation of $\mathbf{A}^g$?

1.6 Write the thermodynamic force $\mathbf{A}^w$ conjugate to $\mathbf{w}$ as a function of $\mathbf{g}$, $\mathbf{w}$.

1.7 The general form of the evolution equation for $\mathbf{w}$ reads

$$\partial_t \mathbf{w} = \Phi(\mathbf{A}^g, \mathbf{A}^w). \tag{3}$$

The function Φ however takes a simpler form in a Galilean invariant system. Remembering that in a frame moving with constant velocity $\mathbf{U}$ relative to the lab frame, $\mathbf{g}$ is transformed to $\mathbf{g}' = \mathbf{g} - \rho\mathbf{U}$, how is $\mathbf{w}$ transformed under the same Galilean transformation?

1.8 How is the time derivative $\partial_{t'}$ of the new reference frame related to the old?

1.9 Galilean invariance demands that

$$\partial_{t'} \mathbf{w}' = \Phi(\mathbf{A}^{g\prime}, \mathbf{A}^{w\prime}), \tag{4}$$

where the function Φ is the same as in Eq. (3). Combine Eqs. (3) and (4) to show that to zeroth order in gradient Φ depends on only one of its arguments.

1.10 Give a brief physical motivation for a linearized form

$$\partial_t \mathbf{w} = -\alpha \mathbf{A}^?, \tag{5}$$

where $\mathbf{A}^?$ is the argument that Φ does depend on.

1.11 Give a physical interpretation for the proportionality coefficient α. What is its sign and why?

1.12 Is $\mathbf{w}$ a hydrodynamic variable and why?

1.13 The conservation equation for $\mathbf{g}$ is

$$\partial_t g_i = -\nabla_j \Pi_{ij}. \tag{6}$$

The momentum current tensor Π is closely related to a quantity routinely used in mechanics and hydrodynamics. Name this quantity and write down the relation.

1.14 This tensor can be decomposed into two parts as $\Pi_{ij} = \Pi_{ij}^{\text{reac}} + \Pi_{ij}^{\text{diss}}$. One of these parts is equal to $g_i g_j/\rho$ for a "Mathusian" fluid. Which of the two is it and how do you know?

1.15 The other part of the tensor can be expanded as

$$-\gamma_{ijk} A_k^g - \bar{\gamma}_{ijkl} \nabla_k A_l^g. \tag{7}$$

Pursuant to rotational symmetry, the most general possible forms of the two constant tensors present in this expansion are

$$\gamma_{ijk} = \text{constant} \times \epsilon_{ijk} \tag{8a}$$

$$\bar{\gamma}_{ijkl} = \text{constant} \times \delta_{ij}\delta_{kl} + \text{constant} \times \delta_{ik}\delta_{jl} + \text{constant} \times \delta_{il}\delta_{jk}, \tag{8b}$$

where ϵ is the completely antisymmetic tensor, which has $\epsilon_{123} = 1$ and is antisymmetic under any exchange of indices, and δ is the Kroenecker symbol. In the following feel free to give reasonable names to the constants featured in these expressions. Why is A^w absent from the expansion of Eq. (7)?

1.16 In what small parameter(s) is the expansion of Eq. (7) made?

1.17 Use a symmetry of the fluid to show that one of the terms of Eq. (7) vanishes.

1.18 In a fluid that is incompressible the reactive part of the tensor Π_{ij} is augmented by a term $P\delta_{ij}$ where P is a Lagrange multiplier that enforces the incompressibility condition $\nabla_i g_i = 0$. In such a system, write the evolution equation for $\mathbf{v} = \mathbf{g}/\rho$, the center-of-mass velocity of the fluid.

1.19 Which equation characteristic of the physics of a single fluid do you recognize?

1.20 Explain physically why this two-fluid system is described by a single-fluid equation.

2 Hydrodynamics of a superfluid with constant densities

We describe our superfluid as a mixture of two fluids each with a constant mass density, which we respectively denote as ρ^n (**n**ormal fluid fraction) and ρ^s (**s**uperfluid fraction). The two fractions have variable momentum densities $\mathbf{g}^n(\mathbf{r}, t)$ and $\mathbf{g}^s(\mathbf{r}, t)$, both of which are odd under time reversal symmetry, and the temperature is assumed constant[1]. We moreover define the total mass density $\rho = \rho^n + \rho^s$ (which is constant), the total momentum density $\mathbf{g}(\mathbf{r}, t) = \mathbf{g}^n + \mathbf{g}^s$ as well as the normal and superfluid velocities $\mathbf{v}^n(\mathbf{r}, t) = \mathbf{g}^n/\rho^n$ and $\mathbf{v}^s(\mathbf{r}, t) = \mathbf{g}^s/\rho^s$. The main input we need from quantum mechanics is the following: the superfluid fraction of the fluid is characterized by a quantum phase $\phi(\mathbf{r}, t)$. This phase plays the role of a broken symmetry variable, which dictates the local superfluid velocity through the relation $\mathbf{v}^s = \nabla\phi$ (here we omit a proportionality coefficient $\hbar/m$ without any consequence for our discussion). This constrains the superfluid flow to be a potential flow (*i.e.*, $\nabla \times \mathbf{v}^s = \mathbf{0}$), a condition responsible for much of its original behavior.

In the following we develop a hydrodynamic description based on the conserved quantity $\mathbf{g}$ and the broken symmetry variable ϕ. Microscopic considerations (see chapter 3 of Ref. [4]) dictate a free energy functional

$$F = E - TS = \int \left[\frac{\rho + \rho^n}{2} (\nabla\phi)^2 - \frac{(\mathbf{g}^n)^2}{2\rho^n} \right] \mathrm{d}^3\mathbf{r}, \tag{9}$$

where the unusual sign in front of the second term of the integrand has an entropic origin.

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[1] The following also implicitly assumes that the specific volumes of the superfluid and normal fractions are identical, but you do not need to worry about this to solve the problem.

2.1 Compute the thermodynamic force $\mathbf{A}^g$ conjugate to the total momentum density of the fluid as a function of $\mathbf{v}^n$ and $\mathbf{v}^s$. Again, pay attention to what is being held constant.

2.2 Show that the thermodynamic force

$$A^{\phi}(\mathbf{r}) = \left.\frac{\delta F}{\delta\phi(\mathbf{r})}\right|_{\mathbf{g}} \tag{10}$$

conjugate to the quantum phase is a linear combination of $\nabla \cdot \mathbf{v}^s$ and $\nabla \cdot \mathbf{v}^n$, and give the proportionality coefficients. Be mindful of the functional derivative in its definition.

2.3 The most general form for the evolution equation of ϕ is

$$\partial_t \phi = \Psi(\mathbf{A}^g, A^{\phi}) = \Psi^{\mathrm{reac}}(\mathbf{A}^g, A^{\phi}) + \Psi^{\mathrm{diss}}(\mathbf{A}^g, A^{\phi}). \tag{11}$$

Write Ψ^{diss} as the most general possible expansion to first order in the thermodynamic forces $\mathbf{A}^g$, A^{ϕ} and to first order in gradient. To determine the order of the different terms, consider that $\mathbf{v}^n$ and $\mathbf{v}^s$ are of the same order, implying that both ∇v^n and ∇v^s terms are to be regarded as first order in gradient. This expansion involves unknown tensors of phenomenological coefficients of ranks up to two.

2.4 Invoke rotational symmetry (no detailed demonstration needed) to simplify this dependence down to the point where it involves only two scalar phenomenological coefficients $\gamma^{\phi\phi}$ and $\bar{\gamma}^{\phi g}$. Note that the quantum phase ϕ is unchanged by a spatial rotation.

2.5 How does the expression that you derived for Ψ^{diss} transform under a Galilean change of frame? You may use the fact that the velocities $\mathbf{v}^n$ and $\mathbf{v}^s$ transform in the usual way, namely $\mathbf{v}^{x\prime} = \mathbf{v}^x - \mathbf{U}$ with $x \in \{n, s\}$.

2.6 The reactive term in Eq. (11) is determined by Galilean invariance. Using the transformation formula for $\mathbf{v}^s$ and the fact that $\mathbf{v}^s = \nabla\phi$, show that

$$\phi' = \phi + f(\mathbf{r}) + g(t), \tag{12}$$

where you will specify the function $f(\mathbf{r})$ but leave the integration constant $g(t)$ unknown for now.

2.7 We propose the form

$$\Psi^{\mathrm{reac}} = \lambda (v^s)^2. \tag{13}$$

Show that with the right choices of λ and $g(t)$ Eq. (13) ensures that Eq. (11) is Galilean invariant. What is the required value of λ?

2.8 What is the required form of $g(t)$?

2.9 Use Eq. (11) to show that

$$\partial_t \mathbf{v}^s - \lambda \nabla (v^s)^2 = \mu^s \nabla(\nabla \cdot \mathbf{v}^s) + \mu^n \nabla(\nabla \cdot \mathbf{v}^n). \tag{14}$$

Express the viscosity-like constants μ^n and μ^s as functions of $\gamma^{\phi\phi}$ and $\bar{\gamma}^{\phi g}$.

2.10 The conservation equation for $\mathbf{g}$ is identical to Eq. (6), although the tensors Π_{ij}^{reac} and Π_{ij}^{diss} are now functions of the forces associated with the superfluid. Write down Π_{ij}^{diss} as the most general expansion in terms of the forces $\mathbf{A}^g$, A^{ϕ} to the same order in gradient and used for the expansion of Ψ^{diss}. This involves tensors of phenomenological coefficients up to rank 4.

2.11 Use inversion symmetry $\mathbf{r} \to -\mathbf{r}$ to eliminate one of the terms.

2.12 Use rotational invariance and the expressions of Eq. (8) to whittle down the crowd of remaining phenomenological coefficients to just four scalars. Remember that the only isotropic tensor of rank two in three dimensions is the unit tensor.

2.13 Using $\Pi_{ij}^{\text{reac}} = g_i g_j/\rho + P\delta_{ij}$, where the overall pressure P is a Lagrange multiplier enforcing the overall incompressibility condition, show that

$$\partial g_i + \nabla_j \frac{g_i g_j}{\rho} = -\nabla_i P + \eta_{\text{shear}}^n \Delta v_i^n + \tilde{\eta}_{\text{bulk}}^n \nabla_i (\nabla \cdot \mathbf{v}^n) + \tilde{\eta}_{\text{bulk}}^s \nabla_i (\nabla \cdot \mathbf{v}^s) \tag{15}$$

Give the expressions of the three independent viscosity-like constants η_{shear}^n, $\tilde{\eta}_{\text{bulk}}^n$ and $\tilde{\eta}_{\text{bulk}}^s$.

2.14 Drop all terms of order $(v^x)^2$ for simplicity and write down the full evolution equation for $\mathbf{v}^n$ as a function of $\mathbf{v}^n$ and $\mathbf{v}^s$. This should look nice once you define $\eta_{\text{bulk}}^n = \tilde{\eta}_{\text{bulk}}^n - \rho^s \mu^n$ and $\eta_{\text{bulk}}^s = \tilde{\eta}_{\text{bulk}}^n - \rho^s \mu^s$.

2.15 What is the name of the physical relationship that relates μ^n and η_{bulk}^s? What is the main assumption underlying it?

2.16 What is the mathematical expression of this relation?

2.17 In contrast with the situation of Sec. 1, the hydrodynamic equations for the superfluid cannot be written down as those of a single fluid. By taking a look at the difference between the evolution equations derived for $\mathbf{w}$ and $\mathbf{v}^s$, explain this difference.

2.18 Why is there no shear viscosity-like coefficient for the superfluid flow?

3 Superfluid flow

Here we consider the steady-state flow of the two fluids studied above along the direction x between two parallel plates located in $z = \pm d/2$. Make sure you are confident in the results of the previous sections before tackling this one.

3.1 For the classical mixture studied in Sec. 1, what relation exists between the velocities $\mathbf{v}^a$ and $\mathbf{v}^b$ of the two fluids as long as d is larger than any molecular length scale?

3.2 Give the steady-state flow profile of the classical fluid mixture for a constant pressure gradient $\nabla P = -a\hat{\mathbf{x}}$ assuming no-slip boundary conditions at the wall. You may assume a Poiseuille flow profile and simply show that it satisfies the boundary conditions and the flow equation.

3.3 Give the relationship between the pressure gradient and the average mass flux per unit surface $\mathbf{Q}$ for this flow. What is the value of the hydrodynamic resistance $R = |\nabla P|/|\mathbf{Q}|$?

3.4 In the case of the helium II flow, the quantum nature of the fluid implies that there is a no-slip boundary condition at the wall for the tangential component of $\mathbf{v}^n$, but that the component of $\mathbf{v}^s$ tangential to the wall is unconstrained. The component of the overall mass flux $\mathbf{g}$ perpendicular to the wall must additionally vanish. In cases such as this one where the boundary conditions are insufficient to determine the full structure of the flow, the missing boundary conditions are determined by minimizing the entropy production in the fluid. As discussed in the lectures, this entropy production σ is given by

$$T\sigma = -A^{\phi}\Psi^{\text{diss}} - \Pi_{ij}^{\text{diss}}\nabla_j A_i^g. \tag{16}$$

Express σ as a function of the thermodynamic forces A^{ϕ} and $\mathbf{A}^g$ and the phenomenological coefficients you introduced in your flux-force relations.

3.5 What is the smallest possible value for σ and why? Note that this condition is enforced by certain relations between the phenomenological coefficients which we will not detail here.

3.6 Rather than imposing a pressure gradient, here we choose to look for a flow with a given mass flux $\mathbf{Q} = Q\hat{\mathbf{x}}$. By looking at your expression of the entropy production, postulate a very simple flow profile with a non-vanishing mass flux that satisfies the flow equations and the boundary conditions while realizing the absolute minimum of the entropy production.

3.7 What does this realization of the lower bound imply for the reversibility of the flow?

3.8 Going back to the flow equations derived in Sec. 2, what is the pressure gradient for this flow?

3.9 Deduce the value of the hydrodynamic resistance for the flow of the superfluid between two parallel plates.

3.10 Would we have reached a similar conclusion if we had considered a shear flow?

References

[1] Lev Landau. Theory of the superfluidity of helium II. *Phys. Rev.*, 60:356, 1941.

[2] L. D. Landau and E M Lifshitz. *Fluid Mechanics*, volume 6 of *Course of Theoretical Physics*. Pergamon Press, Oxford, second edition, 1987.

[3] P M Chaikin and T C Lubensky. *Principles of Condensed Matter Physics*. Cambridge University Press, Cambridge, 1995.

[4] L D Landau and E M Lifshitz. *Statistical Physics Part 1*, volume 5 of *Course of Theoretical Physics*. Pergamon Press, Oxford, third edition, 1980.