Hydrodynamics of a superfluid

1.1) $\vec{g}$ is the total momentum density of the fluid. It is a conserved quantity in the absence of external body forces, and is therefore a hydrodynamic variable.

1.2) $\vec{w}$ is the relative velocity of fluid b with respect to fluid a.

1.3) Both are odd.

1.4) Let $\vec{g}^a = \rho^a \vec{v}^a$ and $\vec{g}^b = \rho^b \vec{v}^b$. Then $\vec{g} = \rho^a \vec{v}^a + \rho^b \vec{v}^b$ and $\vec{w} = \vec{v}_b - \vec{v}_a$

Then
$$\vec{g} + \rho_a \vec{w} = \rho \vec{v}_b$$
$$\vec{g} - \rho_b \vec{w} = \rho \vec{v}_a$$

So
$$\vec{g}_a = \frac{\rho_a}{\rho}(\vec{g} - \rho_b \vec{w})$$
$$\vec{g}_b = \frac{\rho_b}{\rho}(\vec{g} + \rho_a \vec{w})$$

Thus $$f = \frac{\rho_a}{2\rho^2}(\vec{g} - \rho_b \vec{w})^2 + \frac{\rho_b}{2\rho^2}(\vec{g} + \rho_a \vec{w})^2$$

And
$$\vec{A}^g = \frac{\rho_a}{\rho^2}(\vec{g} - \rho_b \vec{w}) + \frac{\rho_b}{\rho^2}(\vec{g} + \rho_a \vec{w})$$
$$\underline{\vec{A}^g = \frac{\vec{g}}{\rho} = \vec{v}}$$

1.5) $\vec{A}^g$ is the center-of-mass velocity of the fluid.

1.6) $\vec{A}^w = -\frac{\rho_a \rho_b}{\rho^2}\left(\vec{g} - \rho_b \vec{w}\right) + \frac{\rho_a \rho_b}{\rho^2}\left(\vec{g} + \rho_a \vec{w}\right)$

$$\underline{\vec{A}^w = \frac{\rho^a \rho^b}{\rho} \vec{w}}$$

1.7) Since $\vec{v}_b' = \vec{v}_b - \vec{U}$ and $\vec{v}_a' = \vec{v}_a - \vec{U}$, we have $\underline{\vec{w}' = \vec{w}}$

1.8) $\begin{cases} \vec{r}' = \vec{r} - \vec{U}.t \\ t' = t \end{cases} \Rightarrow \begin{cases} \vec{r} = \vec{r}' + \vec{U}.t' \\ t = t' \end{cases}$

So $\partial_{\vec{r}'} = \left.\frac{\partial \vec{r}}{\partial \vec{r}'}\right|_{t'} \cdot \partial_{\vec{r}} + \left.\frac{\partial t}{\partial \vec{r}'}\right|_{t'} \cdot \partial_t = \vec{\nabla}$

$$\underline{\partial_{t'}} = \left.\frac{\partial \vec{r}}{\partial t'}\right|_{\vec{r}'} \cdot \partial_{\vec{r}} + \left.\frac{\partial t}{\partial t'}\right|_{\vec{r}'} \cdot \partial_t \underline{= \vec{U}.\vec{\nabla} + \partial_t}$$

1.9) Using Eq (4):

$$\partial_{t'} \vec{w}' = \vec{\Phi}\left(\frac{\vec{g}}{\rho} - \vec{U}, \frac{\rho^a \rho^b}{\rho} \vec{w}\right)$$

$$\partial_t \vec{w} + \underbrace{(\vec{U}.\vec{\nabla})\vec{w}}_{\text{negligible}} = \vec{\Phi}\left(\frac{\vec{g}}{\rho} - \vec{U}, \frac{\rho^a \rho^b}{\rho} \vec{w}\right)$$

So $\forall \vec{U}$ $\vec{\Phi}(\vec{A}^g - \vec{U}, \vec{A}^w) = \vec{\Phi}(\vec{A}^g, \vec{A}^w)$

Therefore $\underline{\Phi \text{ is independent on } \vec{A}^g}$ to 0^{th} order in $\vec{\nabla}$.

1.10) For a fluid at rest $\vec{w} = \vec{0}$ and doesn't change in time, so $\underline{\vec{\Phi}(\vec{A}^w = \vec{0}) = 0}$. To first order (Taylor expansion):

$$\vec{\Phi}(\vec{A}^w) = -\alpha \vec{A}^w,$$

where α must be a scalar to preserve rotational invariance.

1.11) α is a friction coefficient between the two fluids. $\alpha > 0$ to ensure the stability of the $\vec{w} = \vec{0}$ state, or equivalently the positivity of entropy production.

1.12) Eq. (5) yields

$$\partial_t \vec{w} = -\alpha \frac{\ell^a \ell^b}{\ell} \vec{w},$$

Therefore $\vec{w}$ relaxes to $\vec{0}$ in a non-hydro time $\tau = \frac{\ell}{\alpha \ell_{ab}}$ (indpt on system size). Thus it is not a hydrodynamic variable.

1.13) $\Pi_{ij} = -\sigma_{ij}$ where σ is the total stress tensor.

1.14) $\vec{g}$ is odd under time reversal. $g_i g_j$ is even under time reversal. Since it has the opposite time signature to $\vec{g}$, it is a reactive term:

$$\Pi_{ij}^{reac} = \frac{g_i g_j}{\ell}$$

1.15) Since $\vec{A}^w$ (or equivalently $\vec{w}$) relaxes in non-hydrodynamic time, it is slaved to $\vec{g}$ and is not an independent hydrodynamic force.

1.16) Equation (7) is a dual expansion in powers of $\vec{\nabla}$ and the magnitude of the hydrodynamic force $\vec{A}^g$.

1.17) We use the inversion symmetry: $\vec{r}' = -\vec{r}$. Under this operation $\vec{g}' = -\vec{g}$, $\vec{\nabla}' = -\vec{\nabla}$.

Looking at the $\vec{g}$-conservation eqn:

$$\partial_t \vec{g} = -\vec{\nabla}\cdot\Pi$$

changes sign — changes sign

thus Π must retain the same sign. Writing:

$$\Pi^{diss} = -\gamma A^g_k - \bar{\gamma}\,\underbrace{\nabla A^{\ell}_{k\ell}}$$

changes sign — does not chg sign

Therefore $\underline{\gamma_{ijk} = 0}$

1.18) We have

$$\Pi^{reac}_{ij} = \frac{g_i g_j}{\rho} + P\,\delta_{ij}$$

$$\Pi^{diss}_{ij} = -\bar{\gamma}^1\,\delta_{ij}\,\nabla_k v_k - \bar{\gamma}^2\,\nabla_i v_j - \bar{\gamma}^3\,\nabla_j v_i$$

Therefore

$$\partial_t g_i = -\nabla_j \Pi_{ij}$$

$$\Leftrightarrow \partial_t(\rho v_i) + \nabla_j(\rho v_i v_j) = -\nabla_i P + \bar{\gamma}^1\,\nabla_i\nabla_k v_k + \bar{\gamma}^2\,\nabla_j\nabla_i v_j + \bar{\gamma}^3\,\nabla_j\nabla_j v_i$$

$$\Leftrightarrow \overbrace{\left(\partial_t\rho + \nabla_j(\rho v_j)\right)}^{=0} v_i + \rho\left(\partial_t v_i + v_j\nabla_j v_i\right) = -\nabla_i P + (\bar{\gamma}^1 + \bar{\gamma}^2)\,\nabla_i\nabla_j v_j + \bar{\gamma}^3\,\nabla_j\nabla_j v_i$$

Denoting $\eta_{bulk} = \bar{\gamma}^1 + \bar{\gamma}^2$ and $\eta_{shear} = \bar{\gamma}^3$ this yields

$$\underline{\rho\left[\partial_t\vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right] = -\vec{\nabla}P + \eta_{bulk}\,\vec{\nabla}(\vec{\nabla}\cdot\vec{v}) + \eta_{shear}\,\Delta\vec{v}}$$

1.19) This is the Navier-Stokes equation.

1.20) Due to their mutual friction the two fluids quickly become co-moving and thereafter behave as a single mixed fluid of fixed composition.

2.1) $A_i^g = \left.\frac{\delta F}{\delta g(\vec r)}\right|_\varphi = \left.\frac{\partial f}{\partial g(\vec r)}\right|_{\vec v^s}$

And $f = \frac{\rho+\rho^n}{2}(v^s)^2 - \frac{(\vec g - \rho^s \vec v^s)^2}{2\rho^n}$

So $\underline{\vec A^g} = -\frac{\vec g - \rho^s \vec v^s}{\rho^n} = -\frac{\vec g^n}{\rho^n} \underline{= -\vec v^n}$

2.2) $A^\varphi = \frac{\delta}{\delta\varphi}\int \frac{\rho+\rho_n}{2}(\vec\nabla\varphi)^2 - \frac{1}{2\rho_n}(\vec g - \rho_s\vec\nabla\varphi)^2\, d\vec r^3$

$$= -(\rho+\rho_n)\Delta\varphi - \frac{\rho_s}{\rho_n}\vec\nabla\cdot(\vec g - \rho_s\vec\nabla\varphi)$$

$$\underline{A^\varphi = -(\rho+\rho_n)\,\vec\nabla\cdot\vec v^s - \rho_s\,\vec\nabla\cdot\vec v^n}$$

2.3) $\underline{\Psi^{diss} = \underbrace{-\gamma^{\varphi\varphi}A^\varphi}_{\text{1st order in }\nabla} \underbrace{-\gamma_i^{\varphi g}A_i^g}_{\text{0th order}} - \bar\gamma_{ij}^{\varphi g}\underbrace{\nabla_i A_j^g}_{\text{1st order}}}$

2.4) There is <u>no isotropic vector</u> $\gamma_i^{\varphi g}$ and the <u>only isotropic rank 2 tensor is the identity</u>. Thus

$$\underline{\Psi^{diss} = -\gamma^{\varphi\varphi}A^\varphi - \bar\gamma^{\varphi g}\,\nabla_i A_i^g}$$

2.5) Ψ^{diss} is a linear combination of the $\vec\nabla\cdot\vec v^x$. Since $\vec\nabla'\cdot\vec v^{x\prime} = \vec\nabla\cdot(\vec v^x - \vec U) = \vec\nabla\cdot\vec v^x$, it is <u>invariant under a Galilean transformation.</u>

2.6) We must have

$$\vec v^{s\prime} = \vec v^s - \vec U$$

$$\Leftrightarrow \vec\nabla'\varphi' = \vec\nabla\varphi - \vec U$$

$$\Leftrightarrow \vec\nabla\varphi' = \vec\nabla(\varphi - \vec U\cdot\vec r)$$

Integrating over space, we find

$$\varphi' = \varphi \; \boxed{\underbrace{-\vec{U}\cdot\vec{r}}_{=f(\vec{r})} + g(t)}$$

~~2.7) P^s/ρ^s is a scalar field which physically depends on the local density and temperature of the fluid (the Lagrange multiplier setup corresponds to the incompressible limit of this physical situation). Thus $\frac{P^s}{\rho^s}$ is invariant under a Galilean transformation.~~

2.7) Since Ψ^{diss} is invariant we only have to ensure that

$$\partial_{t'}\varphi' - \Psi^{mac'} = \partial_t \varphi - \Psi^{mac}$$

And so we compute

$$\partial_{t'}\varphi' - \Psi^{mac'} = \partial_t(\varphi - \vec{U}\cdot\vec{r} + g) + \vec{U}\cdot\vec{\nabla}(\varphi - \vec{U}\cdot\vec{r}) - \lambda(\vec{\nabla}\varphi - \vec{U})^2 - \frac{P^s}{\rho^s}$$

$$= \underbrace{\partial_t\varphi - \lambda(\vec{\nabla}\varphi)^2 - \frac{P^s}{\rho^s}}_{= \partial_t\varphi - \Psi^{mac}} + \partial_t g + \vec{U}\cdot(\vec{v}^s - \vec{U}) + 2\lambda(\vec{v}^s\cdot\vec{U}) - \lambda\vec{U}^2$$

So $\forall(\vec{v}^s, \vec{U})$ we must have

$$\partial_t g + \vec{U}\cdot\vec{v}^s - \vec{U}^2 + 2\lambda\vec{U}\cdot\vec{v}^s - \lambda\vec{U}^2 = 0$$

2.8) The only solution is $\lambda = -\frac{1}{2}$ and $\partial_t g = +\frac{U^2}{2}$

So we have

$$\boxed{\begin{aligned} \Psi^{mac} &= -\frac{(v^s)^2}{2} + \frac{P^s}{\rho^s} \\ \varphi' &= \varphi - \vec{U}\cdot\vec{r} + \frac{U^2 t}{2} \end{aligned}}$$

2.9) $\partial_t \varphi = -\frac{(v^s)^2}{2} \qquad - \gamma^{\varphi\varphi} A^\varphi - \bar{\gamma}^{\varphi g}\, \vec{\nabla}\cdot\vec{A}^g$

Take the gradient:

$$\partial_t \vec{v}^s = -\vec{\nabla}\frac{(v^s)^2}{2} + \gamma^{\varphi\varphi}(\rho+\rho^n)\,\vec{\nabla}(\vec{\nabla}\cdot\vec{v}^s)$$
$$+ \gamma^{\varphi\varphi}\rho^s\,\vec{\nabla}(\vec{\nabla}\cdot\vec{v}^n)$$
$$+ \bar{\gamma}^{\varphi g}\,\vec{\nabla}(\vec{\nabla}\cdot\vec{v}^n)$$

$$\Rightarrow \partial_t \vec{v}^s + \vec{\nabla}\frac{(v^s)^2}{2} = \mu^s\,\vec{\nabla}(\vec{\nabla}\cdot\vec{v}^s) + \mu^n\,\vec{\nabla}(\vec{\nabla}\cdot\vec{v}^n)$$

with $\underline{\mu^s = \gamma^{\varphi\varphi}(\rho+\rho^n)}$

$\underline{\mu^n = \gamma^{\varphi\varphi}\rho^s + \bar{\gamma}^{\varphi g}}$

2.10) $\underline{\Pi^{diss}_{ij} = -\gamma^{g\varphi}_{ij} A^\varphi - \gamma^{gg}_{ijk} A^g_k - \bar{\gamma}^{gg}_{ijkl} \nabla_k A^g_k}$

2.11) Π is invariant under $\vec{r}\to-\vec{r}$ and A^g_k changes sign $\Longrightarrow \gamma^{gg}_{ijk} = 0$

2.12) $\underline{\Pi^{diss}_{ij} = -\gamma^{g\varphi} A^\varphi \delta_{ij} - \bar{\gamma}^1 \delta_{ij}\,\vec{\nabla}\cdot\vec{A}^g - \bar{\gamma}^2 \nabla_i A^g_j - \bar{\gamma}^3 \nabla_j A^g_i}$

2.13) $$\partial_t \vec{g} + \vec{\nabla}\left(\frac{gg}{\rho}\right) = -\vec{\nabla}P + \gamma^{g\varphi}\,\vec{\nabla}A^\varphi$$
$$-(\bar{\gamma}^1+\bar{\gamma}^2)\,\vec{\nabla}(\vec{\nabla}\cdot\vec{v}^n)$$
$$-\bar{\gamma}^3\,\Delta\vec{v}^n$$
$$= -\vec{\nabla}P - \gamma^{g\varphi}(\rho+\rho^n)\,\vec{\nabla}(\vec{\nabla}\cdot\vec{v}^s)$$
$$-(\bar{\gamma}^1+\bar{\gamma}^2+\gamma^{g\varphi}\rho^s)\,\vec{\nabla}(\vec{\nabla}\cdot\vec{v}^n)$$
$$-\bar{\gamma}^3\,\Delta\vec{v}^n.$$

and so $\underline{\eta^n_{shear} = -\bar{\gamma}^3}$ $\quad \underline{\tilde{\eta}^n_{bulk} = -(\bar{\gamma}_1+\bar{\gamma}^2+\gamma^{g\varphi}\rho^s)}$ $\quad \underline{\tilde{\eta}^s_{bulk} = -\gamma^{g\varphi}(\rho+\rho^n)}$

2.15) Onsager relation – assumes reversible microscopic dynamics.

2.16) $\gamma^{gq} = \bar{\gamma}^{qg}$. Thus

$$\bar{\gamma}^{qg} = \mu^n - \gamma^{qq}\rho^s$$
$$= \mu^n - \frac{\rho^s}{\rho+\rho^n}\mu^s$$
$$= -\frac{\tilde{\eta}^s_{bulk}}{\rho+\rho^n}$$

$$\tilde{\eta}^s_{bulk} = \rho^s\mu^s - (\rho+\rho^n)\mu^n \Rightarrow \underline{\eta^s_{bulk} = -(\rho+\rho^n)\mu^n}$$

2.16)
$$\partial_t \vec{g} - \rho^s \partial_t \vec{v}^s = \rho^n \partial_t \vec{v}^n$$
$$= -\vec{\nabla}\cdot P + \eta^n_{shear}\,\Delta \vec{v}^n + \left(\tilde{\eta}^n_{bulk} - \rho^s\mu^n\right)\vec{\nabla}\left(\vec{\nabla}\cdot\vec{v}^n\right) + \left(\tilde{\eta}^s_{bulk} - \rho^s\mu^s\right)\vec{\nabla}\left(\vec{\nabla}\cdot\vec{v}^s\right)$$

$$\underline{\rho^n \partial_t \vec{v}^n = -\vec{\nabla}\cdot P + \eta^n_{shear}\,\Delta\vec{v}^n + \eta^n_{bulk}\,\vec{\nabla}\left(\vec{\nabla}\cdot\vec{v}^n\right) + \eta^s_{bulk}\,\vec{\nabla}\left(\vec{\nabla}\cdot\vec{v}^s\right)}$$

2.17) The tensorial character of q and $\vec{g}$ forbids the appearance of a friction between the two fluids. Thus both $\vec{v}^n$ and $\vec{v}^s$ retain their hydrodynamic character, and must be included in the final equations.

2.18) The superfluid flow always derives from a gradient and thus never involves shear.

3.1) $\underline{\vec{v}^a = \vec{v}^b}$, because $\vec{w} = \vec{v}^b - \vec{v}^a$ is non-hydrodynamic

3.2) $\frac{\vec{g}}{\rho} = \vec{v} = \vec{v}_a = \vec{v}_b$. Navier-Stokes:

$$\rho\left(\partial_t \vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right) = -\vec{\nabla}P + \eta\Delta\vec{v} + \eta_b\vec{\nabla}(\vec{\nabla}\cdot\vec{v})$$

We postulate $v_y = v_z = 0$ and

$$v_x = -\frac{A}{2}\left(z - \frac{d}{2}\right)\left(z + \frac{d}{2}\right)$$

So $\partial_t \vec{v} = \vec{0}$ $(\vec{v}\cdot\vec{\nabla})\vec{v} = \vec{0}$ and $\vec{\nabla}\cdot\vec{v} = 0$. We are left with

$$\eta\Delta v_x = -a$$

$$\Leftrightarrow \quad -\eta A = -a$$

So $\underline{\vec{v} = \frac{\vec{\nabla}P}{2\eta}\left(\frac{d}{2} - z\right)\left(\frac{d}{2} + z\right)}$

3.3) $Q = \frac{1}{d}\int_{-\frac{d}{2}}^{\frac{d}{2}} \rho v_x \, dz$

$= \frac{|\vec{\nabla}P|}{2\eta d}\rho\int_{-\frac{d}{2}}^{\frac{d}{2}}\left(\frac{d^2}{4} - z^2\right)dz$

$= \frac{d^2|\vec{\nabla}P|}{16\eta}\rho\int_{-1}^{1}(1-\zeta^2)d\zeta$

$= \frac{d^2|\vec{\nabla}P|}{16\eta}\rho \; 2\times\left(1 - \frac{1}{3}\right)$

$\vec{Q} = -\frac{d^2\rho}{12\eta}\vec{\nabla}P$

So $\underline{R = \frac{12\eta}{\rho d^2}}$

3.4) $T\sigma = -A^q\left(-\gamma^{qq}A^q - \bar{\gamma}^{qg}\,\vec{\nabla}\cdot\vec{A}^g\right)$
$\qquad - \nabla_j A_i^g\left(-\gamma^{gq}A^q\delta_{ij} - \bar{\gamma}^1\delta_{ij}\,\vec{\nabla}\cdot\vec{A}^g - \bar{\gamma}^2\nabla_i A_j^g - \bar{\gamma}^3\nabla_j A_i^g\right)$

$$\underline{T\sigma = \gamma^{qq}(A^q)^2 + 2\underbrace{\gamma^{gq}}_{\text{using Onsager}}A^q\,\vec{\nabla}\cdot\vec{A}^g + \bar{\gamma}^1\left(\vec{\nabla}\cdot\vec{A}^g\right)^2}$$
$$\underline{+\bar{\gamma}^2\,\nabla_i A_j^g\,\nabla_j A_i^g + \bar{\gamma}^3\,\nabla_i A_j^g\,\nabla_i A_j^g}$$

3.5) $\underline{\sigma \geq 0}$ after the second principle of thermodynam

3.6) If $\underline{\vec{v}^n = \vec{0}}$ and $\vec{v}^s = \overrightarrow{\text{constant}}$ then σ vanishes. If $\vec{v}^s$ is in the $x\,y$ plane the boundary conditions are additionally satisfied. Finally these satisfy the flow equations. For this plug flow of superfluid we have $\vec{Q} = \rho^s\vec{v}^s$ and so:

$$\underline{\vec{v}^s = \frac{\vec{Q}}{\rho^s}}$$

3.7) No entropy is produced, therefore the flow is reversible.

3.8) $\rho^n\Big[\underbrace{\partial_t\vec{v}^n}_{=\vec{0}} + \underbrace{(\vec{v}^n\cdot\vec{\nabla})\vec{v}^n}_{\text{omitted in Sec. 2}\;=\vec{0}}\Big] = -\vec{\nabla}P + \eta_s^n\underbrace{\Delta\vec{v}}_{=\vec{0}} + \underbrace{\eta_b^n\vec{\nabla}(\vec{\nabla}\cdot\vec{v})}_{=\vec{0}} + \eta_b^s\underbrace{\vec{\nabla}(\vec{\nabla}\cdot\vec{v})}_{=\vec{0}}$

Therefore $\underline{\vec{\nabla}P = \vec{0}}$

3.9) $\underline{R = \frac{0}{Q} = 0}$: the superfluid flows without any viscous resistance.

3.10) No: <u>the normal flow velocity at the top and bottom plates would have been different, forcing a dissipative flow.</u>