



# Universal free energy correction for the two-dimensional one-component plasma

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## Abstract

The universal finite-size correction to the free energy of a Coulomb system is checked in the special case of a two-dimensional one-component plasma on a sphere. The correction is related to the known second moment of the short-range part of the direct correlation function for a planar system. © 2000 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

Two-dimensional Coulomb systems are models which have attracted some attention. On a two-dimensional manifold, a Coulomb system is a system made of particles interacting through the corresponding Coulomb potential plus perhaps some short-range interaction. In a plane, the Coulomb interaction energy of two particles of charges  $q$  and  $q'$ , separated by a distance  $r$ , is defined as  $-qq' \ln(r/L)$ , where  $L$  is some (irrelevant) length.

Some time ago, it has been shown that the free energy of such systems has a universal finite-size correction [1] very similar (except for its sign) to the one which occurs in a system with short-range forces at a critical point [2]: for a finite Coulomb system of characteristic size  $R$ , the free energy  $F$  has the large- $R$  behaviour

$$\beta F = AR^2 + BR + \frac{\chi}{6} \ln R + \dots, \quad (1)$$

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where  $\beta$  is the inverse temperature.  $A$  and  $B$  are non-universal constants describing the bulk and boundary contributions, respectively.  $(\chi/6) \ln R$  is the universal correction, depending only on the Euler number  $\chi$  which describes the topology of the manifold on which the system lives. However, the general derivation [1] of (1) had some heuristic features. The purpose of the present paper is to check (1) in the special case of a one-component plasma on the surface of a sphere, by a different method.

The one-component plasma is a system made of one species of point-particles of charge  $q$  in a uniform neutralizing background. On a sphere of radius  $R$ , the interaction between two particles can be chosen [3,4] as  $-q^2 \ln[(2R/L)\sin(\psi/2)]$ , where  $\psi$  is the angular distance (seen from the sphere centre) between the two particles. There are also particle–background and background–background interactions. A dimensionless coupling constant is  $\Gamma = \beta q^2$ .

A sphere has no boundaries and its Euler number is  $\chi = 2$ . Furthermore, for a given particle density,  $R^2$  is proportional to the number of particles  $N$ . Thus, expansion (1) becomes

$$\beta F = CN + \frac{1}{6} \ln N + \dots, \quad (2)$$

where  $C$  is a constant. The model is exactly solvable [3] when  $\Gamma = 2$  and it can be checked [1] that (2) is obeyed in that case. Also, exact calculations [5] for finite values of  $N$  at  $\Gamma = 4$  and  $\Gamma = 6$  are well fitted by (2).

The present derivation of (2) relies on a recent result [6] about the direct correlation function  $c(r)$  of the plane one-component plasma. By a diagrammatic analysis, it has been shown in Ref. [6] that the second moment of the short-range part  $c_{SR}(r)$  has the simple value

$$n^2 \int c_{SR}(r) r^2 d^2\mathbf{r} = \frac{1}{12\pi}. \quad (3)$$

A remarkable feature of (3) is its universality, in the sense that it is independent of the coupling constant  $\Gamma$ . It will now be shown how (3) leads to (2). More specifically, we show how (3) implies the finite-size correction to the chemical potential  $\mu = \partial F / \partial N$ :

$$\beta\mu = \beta\mu_\infty + \frac{1}{6N} + \dots. \quad (4)$$

This derivation bears some similarity with another one about the two-component plasma [7,8].

## 2. Density functional theory approach

We consider the OCP of average density  $n_s$  on the sphere of radius  $R$  (with a corresponding number of particles  $N = 4\pi R^2 n_s$ ). Introducing the stereographic projection of the sphere onto the plane  $\mathcal{P}$  tangent to its south pole (see Fig. 1), we map the homogeneous OCP on the sphere onto a modified inhomogeneous plasma on the plane,

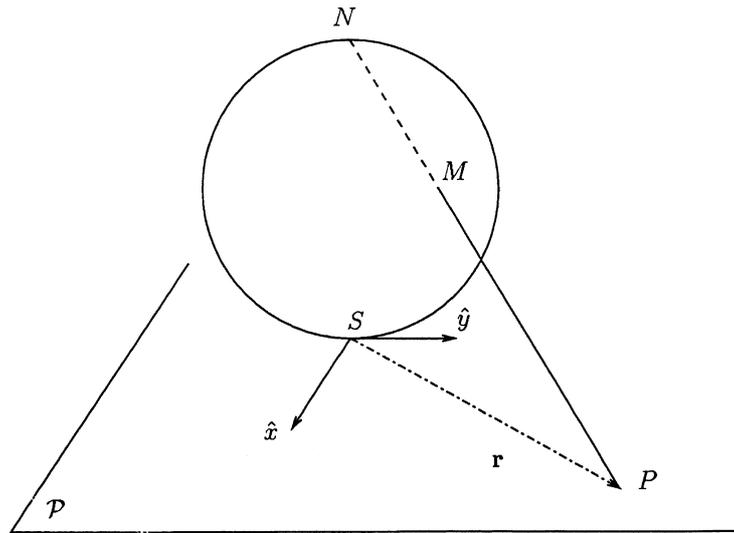


Fig. 1. Stereographic projection from the North pole onto the plane  $\mathcal{P}$ . A point  $M$  on the sphere is projected onto  $P$ , with Cartesian coordinates  $\mathbf{r}$  ( $\mathbf{r} = \mathbf{0}$  at the South pole  $S$ ).

with local particle density

$$n(\mathbf{r}) = n_s \left( 1 + \frac{r^2}{4R^2} \right)^{-2} . \tag{5}$$

In terms of planar coordinates  $\mathbf{r}_1$  and  $\mathbf{r}_2$ , the interaction potential between the two particles on the sphere with angular distance  $\psi_{12}$  can be written as the sum of the planar two-dimensional Coulomb potential  $v_p(\mathbf{r}, \mathbf{r}') = -q^2 \ln[|\mathbf{r} - \mathbf{r}'|/L]$  and one-body terms since:

$$-\ln \left[ \frac{2R}{L} \sin \left( \frac{\psi_{12}}{2} \right) \right] = -\ln \left[ \frac{|\mathbf{r}_1 - \mathbf{r}_2|}{L} \right] + \frac{1}{2} \ln \left( 1 + \frac{r_1^2}{4R^2} \right) + \frac{1}{2} \ln \left( 1 + \frac{r_2^2}{4R^2} \right) . \tag{6}$$

The two one-body terms appearing on the right-hand side of Eq. (6), as well as the metric, create a central potential and we can consider the projected planar system as a OCP interacting through the standard pair potential  $v_p$ , in an external one-body central potential  $V_R^N(r)$ . The latter acts as a confining mechanism ensuring the proper density given by Eq. (5), and its detailed form need not be precised. Without background, the free energy  $\mathcal{F}'$  for the set of particles with pair potential  $v_p$  can be formally expanded in a Mayer diagrammatic representation [6], with a leading term  $(\frac{1}{2}) \int n(\mathbf{r}) v_p(\mathbf{r}, \mathbf{r}') n(\mathbf{r}') d^2\mathbf{r} d^2\mathbf{r}'$  for the excess (over ideal) part of  $\mathcal{F}'$ . The presence of a neutralizing background cancels this mean-field electrostatic term and the intrinsic free energy functional of the inhomogeneous OCP becomes

$$\mathcal{F}[n(\mathbf{r})] = \mathcal{F}'[n(\mathbf{r})] - \frac{1}{2} \int n(\mathbf{r}) v_p(\mathbf{r}, \mathbf{r}') n(\mathbf{r}') d^2\mathbf{r} d^2\mathbf{r}' . \tag{7}$$

The local chemical potential reads

$$\mu(\mathbf{r}) = \frac{\delta \mathcal{F}[n]}{\delta n(\mathbf{r})} = \frac{\delta \mathcal{F}'[n]}{\delta n(\mathbf{r})} - \int v_p(\mathbf{r}, \mathbf{r}') n(\mathbf{r}') d^2 \mathbf{r}' \quad (8)$$

and the second functional derivative of  $\mathcal{F}$  yields:

$$\beta \frac{\delta \mu(\mathbf{r})}{\delta n(\mathbf{r}')} = \frac{\delta^2 \beta \mathcal{F}'[n]}{\delta n(\mathbf{r}) \delta n(\mathbf{r}')} - \beta v_p(\mathbf{r}, \mathbf{r}') \quad (9)$$

$$= -c(\mathbf{r}, \mathbf{r}') + \frac{\delta(\mathbf{r} - \mathbf{r}')}{n(\mathbf{r})} - \beta v_p(\mathbf{r}, \mathbf{r}') \quad (10)$$

$$= -c_{SR}(\mathbf{r}, \mathbf{r}') + \frac{\delta(\mathbf{r} - \mathbf{r}')}{n(\mathbf{r})}, \quad (11)$$

where the variations of the excess contribution to  $\mathcal{F}'$  give rise to the usual direct correlation function [9], having a short-range part given by

$$c_{SR}(\mathbf{r}, \mathbf{r}') = c(\mathbf{r}, \mathbf{r}') + \beta v_p(\mathbf{r}, \mathbf{r}'). \quad (12)$$

Note that the chemical potential of the OCP on the sphere coincides with  $\mu(0)$  for the optimum density profile (5).

Eq. (11) emphasizes the short-range dependence of the chemical potential on a density perturbation. Consequently,  $\mu(0)$  is the same in a finite  $N$ -particle OCP in the central potential  $V_R^N(r)$  and in the limit  $N \rightarrow \infty$  with an external potential  $V_R^\infty(r)$  ensuring the same density variation around the origin as expression (5), namely:

$$n(\mathbf{r}) = n_s \left( 1 - \frac{r^2}{2R^2} \right) + \dots \quad (13)$$

For the purpose of the present analysis, it is sufficient to truncate (5) after second order in  $r$ , as it becomes clear below. The knowledge of the finite-size correction to the chemical potential for the OCP on the sphere then amounts to computing the shift  $\delta\mu(0)$  induced by switching  $V_R^\infty(r)$  starting from the infinite homogeneous planar OCP with density  $n_s$  (corresponding to the stereographic projection of the “spherical” plasma in the thermodynamic limit  $R \rightarrow \infty$ ). The density variation caused by the addition of  $V_R^\infty(r)$  reads  $\delta n(\mathbf{r}) \simeq -n_s r^2 / (2R^2)$  and induces the shift

$$\beta \delta\mu(0) = \int \left[ -c_{SR}(\mathbf{r}) + \frac{\delta(\mathbf{r})}{n(\mathbf{r})} \right] \delta n(\mathbf{r}) d^2 \mathbf{r} \quad (14)$$

$$= \frac{n_s}{2R^2} \int c_{SR}(r) r^2 d^2 \mathbf{r}, \quad (15)$$

where the direct correlation function to be considered is that of the homogeneous reference planar OCP. From sum rule (3), we finally obtain

$$\beta \delta\mu(0) = \frac{1}{24\pi n_s R^2} = \frac{1}{6N}, \quad (16)$$

and Eq. (4) is recovered.

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