Supplementary materials for "Thermal bath Engineering for Swift Equilibration"

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We examine here an alternative protocol (termed "Route 2" for conciseness), that uses directly the position of the trap center as the additional degree of freedom, and not the effective temperature as in TESE. Though the experimental implementation of the effective temperature relies on an engineering of the trap center position, the two protocols differ: within Route 2, the trap does no longer follows a white noise signal. Route 2 is defined as follows: at initial time, the stiffness of the trap is changed abruptly from κ_i to $\kappa_f < \kappa_i$, while a second trap of stiffness $\kappa_{\max}$ is added. Whereas the first trap remains fixed until the end of the protocol, the center of the second will follow the trajectory $x_0(t) = f(t)\delta^{(i)}$. The shape $f(t)$ of this trajectory is fixed, but its amplitude $\delta^{(i)}$ is a random variable, centered and of standard deviation δ, whose values are uncorrelated between two realizations of the protocol. As explained in the main text, the statistics we refer to concern the ensemble of realizations of the protocol. As the particle density function remains Gaussian during the whole transformation, we only focus on the first and second moment of the position of the particle. This position is denoted by $x^{(i)}(t)$ for the ith realization, and $\langle . \rangle_i$ refers to the ensemble average on the realizations of the protocol.

For realization i, the colloid's position obeys Langevin equation

$$\dot{x}^{(i)} = -\frac{\kappa_f}{\nu} x^{(i)} - \frac{\kappa_{\max}}{\nu}\left(x^{(i)} - x_0^{(i)}(t)\right) + \sqrt{\frac{2k_BT}{\nu}}\xi^{(i)}(t) \tag{1}$$

and defining $\kappa_f + \kappa_{\max} = \kappa'$, $\frac{\nu}{\kappa'} = \tau'$ and $\frac{\nu}{\kappa_{\max}} = \tau_m$, its solution reads

$$x^{(i)}(t) = x^{(i)}(0)e^{-\frac{t}{\tau'}} + \int_0^t du \frac{x_0^{(i)}(u)}{\tau_m} e^{-\frac{t-u}{\tau'}} + \int_0^t du \sqrt{\frac{2k_BT}{\nu}}\xi^{(i)}(u)e^{-\frac{t-u}{\tau'}}, \tag{2}$$

where the initial position of the particle $x^{(i)}(0)$ is a random variable with the initial equilibrium Gaussian distribution (centered, of variance k_BT/κ_i). To be successful, a protocol needs to meet the following requirements on its first two moments

$$\langle x^{(i)}(t_f)\rangle_i = 0 \tag{3}$$

$$\langle x^{(i)}(t_f)^2\rangle_i = \frac{k_BT}{\kappa_f} \tag{4}$$

$$\frac{d}{dt}\langle x^{(i)}(t)^2\rangle_i\Big|_{t=t_f} = 0. \tag{5}$$

The second equality ensures that the system reaches the desired equilibrium, and the last one that it remains subsequently at equilibrium. While the first condition trivially holds if $\langle \delta^{(i)}\rangle_i = 0$, we need to engineer the shape $f(t)$ of the trap trajectory and its amplitude δ to fulfill the last two. Defining

$$F(t) = \int_0^t du f(u) e^{-\frac{t-u}{\tau'}}, \tag{6}$$

we have

$$\langle x^{(i)}(t)^2\rangle_i = \frac{k_BT}{\kappa_i}e^{-\frac{2t}{\tau'}} + \frac{\delta^2}{\tau_m^2}F^2(t) + \frac{k_BT}{\kappa'}\left(1 - e^{-\frac{2t}{\tau'}}\right). \tag{7}$$

Conditions (4) and (5) can be rephrased as

$$\frac{\delta^2}{\tau_m^2}F(t_f)\dot{F}(t_f) = k_BTe^{-\frac{2t_f}{\tau'}}\left(\frac{1}{\kappa_i\tau'} - \frac{1}{\nu}\right) \tag{8}$$

and

$$\frac{\dot{F}(t_f)}{F(t_f)} = \frac{\frac{1}{\tau'} - \frac{1}{\tau_i}}{e^{\frac{2t_f}{\tau'}}\left(\frac{1}{\chi} - \frac{\tau'}{\tau_i}\right) + \frac{\tau'}{\tau_i} - 1}. \tag{9}$$

Once we choose the final value $F(t_f)$, then

$$\dot{F}(t_f) = \frac{1 - \frac{\tau'}{\tau_i}}{e^{\frac{2t_f}{\tau'}}\left(\frac{1}{\chi} - \frac{\tau'}{\tau_i}\right) + \frac{\tau'}{\tau_i} - 1}\frac{F(t_f)}{\tau'}, \tag{10}$$

and

$$\delta^2 = \frac{k_B T}{\kappa_i}\left(\frac{\tau_m}{F^2(t_f)}\right)^2\left[\frac{1}{\chi} - \frac{\tau'}{\tau_i} + \left(\frac{\tau'}{\tau_i} - 1\right)e^{-\frac{2t_f}{\tau'}}\right]. \tag{11}$$

To fully determine the protocol, we just need to choose a shape for $F(t)$ that takes the good values $F(t_f)$ and $\dot{F}(t_f)$ at final time, for example a polynomial form

$$F(t) = A(t/t_f)^2 + B(t/t_f)^3, \tag{12}$$

and to deduce $f(t)$ from it

$$f(t) = \dot{F}(t) + F(t)/\tau'. \tag{13}$$

Note that according to equation (8), in general, $F(t_f)$ and $\dot{F}(t_f)$ have the same sign, as $\kappa' = \kappa_f + \kappa_{\max}$ is likely to be greater than κ_i for experimental reasons (the trap must not be too loose), and then $\tau' < \tau_i$. This means in particular that $f(t_f)$ cannot be zero, as can be seen from equation (13), and that it is impossible to bring the second trap back at the origin at the end of the protocol. As a consequence, the protocol cannot be carried out by only one trap, contrarily to our protocol that only needed one trap.

Finally, let us evaluate the standard deviation of the trap displacement $f(t)\delta$ when choosing a reasonable value for $F(t_f)$, for example $F(t_f) = \tau'$. Keeping our values of the parameters ($\tau_i = 4.7$ ms, $\chi = 0.44$, $t_f = 1$ ms, $\kappa_{\max} = 6\kappa_i$ so $\tau_m = \nu/\kappa_{\max} = \tau_i/6 \simeq 0.78$ ms and $\tau' = (1/\tau_f + 1/\tau_m)^{-1} \simeq 0.73$ ms), we obtain a standard deviation on the order of $3\sigma_i$, to be compared with our experimental $12\sigma_i$ (cf Fig. 1 of the main text). The shorter the protocol, the bigger the amplitude of the trap displacement. So authorizing up to the displacement of $12\sigma_i$ that we used in our experiment, this alternative protocol can be realized approximately ten times faster than ours, the other parameters being fixed, but requires two independent traps where TESE only requires one.

To conclude, Route 2 is fully operational, if not simpler than our protocol, but is restricted to a single particle system, and does not lend itself to a generalization to N non interacting trapped Brownian objects. A similar comment can be made to the TESE protocol, in the experimental implementation we proposed. Yet, the TESE idea in its generality does not suffer from the same shortcoming. The key is to be able to emulate a *bona fide* effective temperature, see our comment in the concluding section of the main text.

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