When Random Walkers Help Solving Intriguing Integrals

Satya N. Majumdar and Emmanuel Trizac

LPTMS, CNRS, Université Paris-Sud, Université Paris-Saclay, 91405 Orsay, France

We revisit a family of integrals that delude intuition and that recently appeared in mathematical literature in connection with computer algebra package verification. We show that the remarkable properties displayed by these integrals become transparent when formulated in the language of random walks. In turn, the random walk view naturally leads to a plethora of nontrivial generalizations that are worked out. Related complex identities are also derived, without the need of explicit calculation. The crux of our treatment lies in a causality argument where a message that travels at finite speed signals the existence of a boundary.

Introduction.—While intuition and experimentation are both crucial in mathematical works, inductive thinking may be spectacularly misguided in some cases. A celebrated illustration of the dangers of pattern extrapolation is provided by the question of circle division by chords [1]: Consider n points on the circumference of a circle and join every pair of points by a chord such that at any point inside the circle at most two chords can intersect. How many regions S_n does the circle get divided into? By simple drawing, one sees that $S_1 = 1$ (by convention), $S_2 = 2$, $S_3 = 4$, $S_4 = 8$, $S_5 = 16$. At this point one may naively guess that for general n, $S_n = 2^{n-1}$. That is wrong. It turns out that $S_6 = 31$. Indeed, the correct answer is $S_n = \binom{n}{4} + \binom{n}{2} + 1$, which happens to coincide with the sequence 2^{n-1} up to $n = 5$, but starts differing from it for $n = 6$ onwards.

Our interest goes here to a lesser known such problem, and the surprising behavior of integrals of the type

$$I_N = \int_{-\infty}^{\infty} \prod_{n=1}^{N} \operatorname{sinc}\left(\frac{k}{2n-1}\right) dk, \qquad (1)$$

$$J_N = \int_{-\infty}^{\infty} \cos(k) \prod_{n=1}^{N} \operatorname{sinc}\left(\frac{k}{2n-1}\right) dk, \qquad (2)$$

where $\operatorname{sinc}(x) = \sin(x)/x$ denotes the cardinal sine function [2,3]. We do not dwell on the prevalence of the sinc function in mathematics (geometry, spectral analysis,…) and physics (signal processing, optics,…), see, e.g., [5]. It was shown that $I_1 = I_2 = I_3 = I_4 = I_5 = I_6 = I_7 = \pi$, whereas $I_N < \pi$, for all $N \geq 8$ [6]. In the latter situation, the difference $\pi - I_N$ is minute, less than 10^{-10} for $N = 8$, which was first realized numerically, and attributed to a bug in the software [6]. A related phenomenon was observed for the J family: $J_N = \pi/2$ for $N \leq 56$, but $J_N < \pi/2$ for all

$N \geq 57$ [7]. A theorem shown in [6] rationalizes this matter of fact: it states that provided $\sum_{n=2}^{N} |a_n| < |a_1|$,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \prod_{n=1}^{N} \operatorname{sinc}(a_n k) dk = \frac{1}{2|a_1|}. \qquad (3)$$

Without loss of generality, one can choose the coefficients a_n to be positive real quantities, and a_1 can then be taken as the largest of them. Given that $\sum_{n=2}^{7} 1/(2n-1) < 1$ while $\sum_{n=2}^{8} 1/(2n-1) > 1$, this explains the behavior of the I family, for which $a_1 = 1$. The J family falls under the same argument [8]. When the equality in (3) breaks, the explicit integrals could be computed. The corresponding values, related to the volume of hypercubes, cut by parallel hyperplanes, is immaterial for our purposes [6]. Our goal is rather to provide a transparent understanding of the statement (3). To this end, we will show that the language of random walks, and physical intuition not only provide a natural framework towards understanding this change of behavior, but also lead to relevant and interesting generalizations, thereby offering an explicit and effort-free calculation of complex multidimensional integrals [9]. At the heart of our approach lies a *causality* argument, formulated in terms of a message that signals the existence of a boundary.

Random walkers in a finite or infinite "world".—We start by considering a random walk making N steps on a line, $x_N = \sum_{n=1}^{N} \eta_n$ starting at $x_0 = 0$, where η_n is uniformly distributed in $[-a_n, a_n]$ and the η_n's are independent random variables. The probability density function (PDF) of each η_n is thus a rectangle function, with simple characteristic function (Fourier-Transform) $\langle e^{ik\eta_n} \rangle = \operatorname{sinc}(a_n k)$. The characteristic function of x_N, sum of independent increments, thus reads

$$\langle e^{ikx_N} \rangle = \prod_{n=1}^{N} \langle e^{ik\eta_n} \rangle = \prod_{n=1}^{N} \operatorname{sinc}(a_n k), \qquad (4)$$

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which allows us to write its PDF as the inverse Fourier transform

$$p_N(x_N) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \prod_{n=1}^{N} \text{sinc}(a_n k) e^{-ikx_N} \qquad (5)$$

$$\Rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} \prod_{n=1}^{N} \text{sinc}(a_n k) dk = p_N(0). \qquad (6)$$

The I_N integral under scrutiny is thus isomorphic to $p_N(0)$, i.e., the probability density of the random walk to be back at the origin, while starting from the origin ($x_0 = 0$).

To proceed further, it is useful to reinterpret $p_N(0)$ as follows. Consider a large (infinite actually) number of independent random walkers, all starting at $x_0 = 0$. Then $p_N(0)$ is just the fraction of walkers at $x = 0$ after N steps, i.e., the density of this gas of independent particles at the origin after N steps. After step 1, their density is uniform in $[-a_1, a_1]$ so that $p_1(0) = 1/(2a_1)$ (incidentally meaning that $I_1 = \pi$). A second step is then made, with amplitude $a_2 < a_1$. Because the jump a_2 is finite and $a_2 < a_1$, it is clear that all the walkers that were, after step 1, in the range $[-(a_1 - a_2), a_1 - a_2]$ will not leave their step-1 domain $[-a_1, a_1]$ following the second step. Only walkers near the two edges, e.g., those in the range $[a_1 - a_2, a_1]$ or $[-a_1, -a_1 + a_2]$ may leave the step-1 domain after the second jump. Hence, the walkers in $[-(a_1 - a_2), a_1 - a_2]$ do not "see" the edges of the step-1 domain—for them, it is as if the system was infinite with uniform density $1/(2a_1)$. In such an "infinite world," the gain and loss contribution balancing those walkers leaving the origin and those reaching it after the second step do cancel: $p_1(0) = p_2(0)$. This is illustrated in Fig. 1 where one can appreciate that the flatness of the density near the origin is preserved, although in a range that diminishes with the number N of steps performed. The argument does not depend on the left-right symmetry of the random steps [11]. In other words, we invoke causality and the boundedness of the steps to state that the only possibility for $p_N(0)$ to be affected by a new step is when walkers having started from the edges at $\pm a_1$ do reach the origin. Those "messengers" carry the information that the "world" is not infinite, which in turn impinges on $p_N(0)$. If $\sum_{n=2}^{N} a_n < a_1$, the distance traveled by the messengers is not sufficient to reach the origin, and $p_N(0) = 1/2a_1$ is N independent. We therefore recover statement (3) [12]. Besides, while our random walk argument directly applies to I_N integrals, it also is relevant for the J_N type, as explained in the Supplemental Material [13]. It is, nevertheless, necessary here to supplement the analysis with a new property, the left-right symmetry of the random steps. The key feature becomes the preservation of the edge density $p_N(a_1)$ under performing random steps, while it pertained to the preservation of $p_N(0)$ when treating I_N integrals [13].
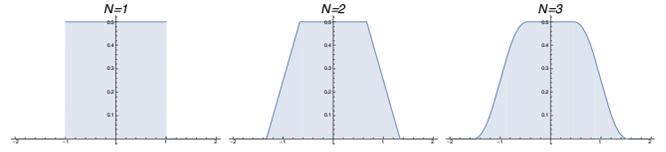

FIG. 1. Probability density function of the random walk x_N after $N = 1$, $N = 2$, and $N = 3$ steps, as indicated. Here, the amplitudes of the steps are $a_1 = 1$, $a_2 = 1/3$, $a_3 = 1/5$, in line with the definition of I integrals in Eq. (1). The density at the origin is invariant (equal to $1/2$), which means that $I_1 = I_2 = I_3 = \pi$. For, respectively, $N = 1$, 2 and 3, the extension of the flat region near the origin is $2a_1$, $2(a_1 - a_2)$, $2(a_1 - a_2 - a_3)$, due to the progression of the walkers arriving from the boundaries (so-called "messengers" in the main text). Note that the density at $x = 1$ is preserved as well, and half that at the origin (leaving aside the $N = 1$ case).

The random walk reformulation provides us with an immediate generalization. Consider, for instance, the case where the first step is of Pearson's type [14] (i.e., of a fixed amplitude a_1, ending at $\pm a_1$), while the subsequent steps are again uniform as before with $\eta_n \in [-a_n, a_n]$ for $n \geq 2$, then $p_N(0) = 0$ for $\sum_{n=2}^{N} a_n < a_1$. In this case, the characteristic function after N steps is given by $\langle e^{ikx_n} \rangle = \cos(ka_1) \prod_{n=2}^{N} \text{sinc}(a_n k)$. Then our causality argument tells us that $p_N(0) = 0$ for $\sum_{n=2}^{N} a_n < a_1$. Evidently, the origin remains void of walkers, until the messengers arrive at $x = 0$. Provided that $\sum_{n=2}^{N} a_n < a_1$, this implies [15]:

$$\int_{-\infty}^{\infty} \cos(a_1 k) \prod_{n=2}^{N} \text{sinc}(a_n k) dk = 0. \qquad (7)$$

This identity can be recovered by invoking a different random walk sharing with the previous one steps 2, 3, ..., n, but not the first step [13].

One-dimensional generalizations.—A natural extension of the above results consists in considering that all steps except the first are arbitrary, but of finite range. The corresponding PDFs are therefore of finite support. The argument now involves the associated characteristic functions, that we denote $\hat{\mathcal{F}}_n(k)$. These $\hat{\mathcal{F}}_n(k)$ are defined as the Fourier transforms of PDFs $\mathcal{F}_n(x)$ that have a finite support, taken for convenience to be unity. Provided $\sum_{n=2}^{N} a_n < a_1$, one can write

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \text{sinc}(a_1 k) \prod_{n=2}^{N} \hat{\mathcal{F}}_n(a_n k) dk = \frac{1}{2a_1}. \qquad (8)$$

Under the same condition and taking once more advantage of causality, we obtain [15]

$$\int_{-\infty}^{\infty} \cos(a_1 k) \prod_{n=2}^{N} \hat{\mathcal{F}}_n(a_n k) dk = 0, \qquad (9)$$

where again all amplitudes a_n are considered positive. Some particular cases have been addressed in earlier studies [6,10], but we stress that many more are subsumed under Eqs. (8) and (9). The task amounts to establishing a catalog of eligible $\hat{\mathcal{F}}_n(k)$. It is not our purpose here, and we simply mention some emblematic such functions: the Bessel functions $J_0(k)$, $J_1(k)/k$ and more generally $J_\nu(k)/k^\nu$ for $\nu > -1/2$, $(1 - \cos k)/k^2$, $(k - \sin k)/k^3$ and a number of hypergeometric functions. To generate candidates, advantage can be taken from the study of hyperuniform systems, that feature potentials of bounded Fourier transform; see, e.g., [16].

Beyond dimension one.—A second natural extension of previous considerations consists in considering d-dimensional random walks, with $d > 1$. A straightforward calculation shows that the counterparts of the one dimensional $\mathrm{sinc}(ka)$ and $\cos(ka)$ functions are given as follows. For a one-step walk with jump $\boldsymbol{\eta}$, chosen respectively (i) uniformly *within* a d-dimensional sphere of radius a and (ii) uniformly *on* the surface of the same sphere (Pearson's type jump), the associated characteristic functions of the jumps are

$$\langle e^{i\boldsymbol{k}\cdot\boldsymbol{\eta}} \rangle = \begin{cases} \dfrac{(2\pi)^{d/2}}{V_d} \dfrac{J_{d/2}(ka)}{(ka)^{d/2}} & \text{(i)} \\[2ex] \dfrac{(2\pi)^{d/2}}{S_d} \dfrac{J_{d/2-1}(ka)}{(ka)^{d/2-1}} & \text{(ii)} \end{cases} \qquad (10)$$

where $V_d = \pi^{d/2}/\Gamma(d/2 + 1)$ and $S_d = dV_d$ are, respectively, the volume and surface of a d-dimensional unit sphere. For $d = 1$, using $J_{1/2}(z) = \sqrt{2/\pi z} \sin(z)$ and $J_{-1/2}(z) = \sqrt{2/\pi z} \cos(z)$, one recovers, respectively, $\mathrm{sinc}(ka)$ and $\cos(ka)$. A new catalog of functions $\hat{\mathcal{F}}_n^{(d)}(\boldsymbol{k})$ can then be established, such that their d-dimensional Fourier transform is of bounded support (several interesting candidates can also be found in [16], a rather generic one being the hypergeometric function $_1F_2\{[(d+m)/2]; (d/2); 1 + [(m+d)/2], -k^2\}$, where m is some arbitrary parameter). Knowing the eligible building blocks $\hat{\mathcal{F}}_n^{(d)}(\boldsymbol{k})$, one can write upon setting $k = |\boldsymbol{k}|$

$$\int_{\mathbb{R}^d} \frac{J_{d/2}(a_1 k)}{k^{d/2}} \prod_{n=2}^{N} \hat{\mathcal{F}}_n^{(d)}(a_n \boldsymbol{k}) d^d \boldsymbol{k} = \left(\frac{2\pi}{a_1}\right)^{d/2}, \qquad (11)$$

provided $\sum_{n=2}^{N} a_n < a_1$.

A nontrivial identity follows from Eq. (11) by considering a special case. Choose the nth step uniformly from a d_n-dimensional sphere of radius b_n (for $n \geq 1$). Consequently, using $\hat{\mathcal{F}}_n^{(d_n)}(a_n \boldsymbol{k}) \propto J_{d_n/2}(b_n k)/k^{d_n/2}$ in Eq. (11), we get upon setting $d_n = 2\mu_n$ the following identity

$$\int_0^\infty dk k^{\mu_1 - 1} J_{\mu_1}(b_1 k) \prod_{j=2}^{N} \frac{J_{\mu_j}(b_j k)}{k^{\mu_j}}$$
$$= \frac{2^{\mu_1 - 1 - \sum_{j=2}^{N} \mu_j}}{b_1^{\mu_1}} \frac{\Gamma(\mu_1)}{\prod_{j=2}^{N} \Gamma(1 + \mu_j)} \prod_{j=2}^{N} b_j^{\mu_j} \qquad (12)$$

provided $\sum_{j=2}^{N} b_j < b_1$ [17]. Not surprisingly, with a Pearson first step that depopulates the origin and for $\sum_{n=2}^{N} a_n < a_1$:

$$\int_{\mathbb{R}^d} \frac{J_{d/2-1}(a_1 k)}{k^{d/2-1}} \prod_{n=2}^{N} \hat{\mathcal{F}}_n^{(d)}(a_n \boldsymbol{k}) d^d \boldsymbol{k} = 0. \qquad (13)$$

Mixing dimensions, we note that the steps $n \geq 2$ can be of any type provided the associated Fourier transform is bounded: in (11) and (13), the building blocks $\hat{\mathcal{F}}_n^{(d)}(\boldsymbol{k})$ can be some $\hat{\mathcal{F}}_n^{(d')}(\boldsymbol{k})$, borrowed from a lower-dimensional catalog with $d' < d$. In doing so, we generate a wealth of complex integrals. Some of the simplest are known [18,19], for instance

$$\int_0^\infty k^{\nu-\mu+1} J_\nu(a_1 k) J_\mu(a_2 k) \cos(a_3 k) \mathrm{sinc}(a_4 k) = 0 \qquad (14)$$

for $a_2 + a_3 + a_4 < a_1$, which follows from (13) with $\nu = d/2 - 1$, $\hat{\mathcal{F}}_{n=2}^{(d)}(\boldsymbol{k}) = J_\mu(k)/k^\mu$ and that appears under section 6.711.2 in [18], when $a_4 = 0$ [20]. Yet, infinitely many other identities that are subsumed in (11) or (13) are complex and seemingly unknown.

It is worth stressing that in some cases the random walk reformulation may not immediately lead to an explicit result; however it may nevertheless offer a direct means of calculation. As an example, we find the following nontrivial identity

$$\int_{\mathbb{R}^2} dk_x dk_y \cos(k_x) \cos(k_y) J_0(bk) J_0(ak)$$
$$= \frac{4}{\sqrt{[(a+b)^2 - 2][2 - (a-b)^2]}}, \qquad (15)$$

with $k = \sqrt{k_x^2 + k_y^2}$. The above result holds for $\sqrt{2} \in [|b - a|, b + a]$ (and, say, $a > 0$, $b > 0$); otherwise, the integral vanishes. The proof is provided in the Supplemental Material [13]. We outline here the main steps. Consider a 2D random walk, starting at the origin $\boldsymbol{0}$ and making four successive steps: a first jump ± 1 along the x direction, a second jump ± 1 along the y direction, then a Pearson jump on the circle of radius b with a final fourth step distributed as the third, but with a radius a and say $a < b$, see Fig. 2. Then the lhs of Eq. (15), using the results from line (ii) of Eq. (10) ($d = 1$ for the first two steps and $d = 2$ for the last two steps), is precisely $4\pi^2 p_4(\boldsymbol{0})$, where $p_4(\boldsymbol{0})$ denotes the
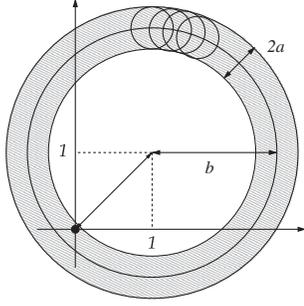
FIG. 2. Sketch of the random walk geometry considered. After step 2, the walker is on one of four corners of a square, chosen to be (1,1) on the graph. The third step is uniform on the large circle with radius b, while the fourth is uniform on a smaller circle with radius a. The walker can then end up, nonuniformly, at any point within the shaded annulus of inner radius $b - a$ and outer radius $b + a$. Since the origin lies in that region, the integral considered in nonvanishing.

density at the origin **0** after four steps. This density can, in turn, be computed by elementary means, see the Supplementary Material [13], leading to the rhs of Eq. (15). If $\sqrt{2} \notin [b - a, b + a]$, the random walk which is at a distance $\sqrt{2}$ from the origin after step 2 cannot be back to be origin after step four, exploring an annulus with inner radius $b - a$ and outer radius $b + a$ (see Fig. 2). Equally tractable is the case where the fourth step is not Pearson but uniform within the disc of radius a [for the fourth step we use (i) of Eq. (10) with $d = 2$], leading to

$$\int_{\mathbb{R}^2} dk_x dk_y \cos(k_x)\cos(k_y)J_0(bk)\frac{J_1(ak)}{k}$$
$$= \frac{2}{a}\arccos\left(\frac{2 + b^2 - a^2}{2\sqrt{2}b}\right). \qquad (16)$$

The lhs can be made more complex without sacrificing the possibility of an explicit calculation of $p_4(\mathbf{0})$.

When sums and integrals coincide.—We now turn to a distinct problem, that bears a similarity with the previous ones, after suitable reformulation. There was some interest recently in identities of the form [21,22]

$$\int_{-\infty}^{\infty} \prod_{n=1}^{N} \text{sinc}(a_n k) dk = \sum_{k=-\infty}^{\infty} \prod_{n=1}^{N} \text{sinc}(a_n k) \qquad (17)$$

which hold provided $\sum_{n=1}^{N} a_n < 2\pi$ [23]. The latter condition can be compared to that applying to (3), $\sum_{n=2}^{N} a_n < a_1$, that can be rewritten as $\sum_{n=1}^{N} a_n < 2a_1$. The difference between the two criteria, where the same quantity is bounded either by 2π or by $2a_1$, indicates that the identity (17) cannot be reduced to any of the previous arguments. Yet, the random walk reformulation also is insightful to show, and understand, relation (17). The idea is to compare two populations of random walkers, one on

the infinite line (case F, for "flat"), and the other on the unit circle (case C). Both populations, starting from the origin, undergo the same random jumps. Provided that the front runners (the random walkers having traveled the greater distance from the origin) did not travel round the circle in case C, moving on a flat line or on a finite circle is immaterial. The corresponding condition reads $\sum_{n=1}^{N} a_n < 2\pi$. When this inequality is fulfilled, the probability density of the walkers at the origin is thus the same in cases C and F. Expressing the PDF as either a Fourier series for case C or a Fourier transform for case F, we then get

$$\int_{-\infty}^{\infty} \prod_{n=1}^{N} \hat{\mathcal{F}}_n(a_n k) dk = \sum_{k=-\infty}^{\infty} \prod_{n=1}^{N} \hat{\mathcal{F}}_n(a_n k), \qquad (18)$$

which includes (17) and many other cognate relations [24]. As above, the $\hat{\mathcal{F}}_n$ refer to arbitrary functions, the Fourier transform of which are bounded with unit support. Loosely speaking, Eq. (18) can thus be viewed as the "flat world equation."

Conclusion and discussion.—We have proposed a random walk interpretation of a curious phenomenon, exhibited by integrals of types (1) and (2). The underlying physical image is that of an ensemble of random walkers starting from the origin, and performing a first step so as to populate uniformly the interval $[-a_1, a_1]$. The walkers then undergo a series of N smaller steps with respective amplitudes a_n. If the maximal span of these steps cannot lead walkers from the edge (i.e., at $x = \pm a_1$) back to the origin $x = 0$, then the walkers near $x = 0$ have a fixed density [given by the lhs in Eq. (3)], specified by the first step and thus equal to $1/(2a_1)$. In pictorial terms, the walkers near the origin cannot know they live in a finite world, unless the messengers starting from the confines at $\pm a_1$ reach them. This may never happen if $\sum_{n=2}^{\infty} a_n < a_1$, in which case an equality like (3) will hold at all orders N [25]. It is interesting to note here that the model of random walks with shrinking steps directly applies to physico-chemical problems such as line broadening for single molecule spectroscopy in disordered media [26,27]. The random walk reformulation naturally leads to nontrivial extensions, since it is irrelevant that the steps $n = 2, 3$, etc., be uniformly distributed, provided the first one (labeled $n = 1$) has the desired property [uniform to lead to (3) or Pearson to lead to (7)] and that the subsequent steps ($n = 1, 2, \ldots$) are bounded. Generalizations in higher dimensions appear of particular interest, and provide calculation-free results that would otherwise require considerable effort and ingenuity.

While the mathematical problem at stake deceives intuition, we have shown that physical arguments may take over. Physics' insight takes the form of a causality rule, where a message travels from a boundary. Applied to random walkers undergoing jumps of bounded amplitude,

this yields a clear account for the change of behavior of a class of multidimensional integrals [of the types (11) and (13)]. The random walk picture also allows for simple calculation of complex multidimensional integrals [such as in (15) and (16)]. Besides, related probabilistic ideas can be generalized to compute other classes of integrals that are otherwise hard to obtain. Assume for instance that a 1D walker starts with a Pearson jump of amplitude $\pm a_1$, so that $\langle |x_1| \rangle = a_1$. The boundedness of subsequent steps and the causality rule mean that $\langle |x_N| \rangle$ remain at a_1, as long as $\sum_{n=2}^{N} a_n \leq a_1$, i.e., as long as the message starting from $x = a_1$ does not hit $x = 0$. This implies

$$\int_0^\infty \frac{1}{k^2} \left(1 - \cos(ka_1) \prod_{n=2}^{N} \hat{\mathcal{F}}_n(a_n k) \right) dk = \frac{\pi}{2} a_1 \quad (19)$$

after a straightforward reexpression of $\langle |x_N| \rangle$ presented in [13], Sec. III. It thus appears that the curious phenomenon at work for Borwein integrals is more general and applies to a much broader class of complex integrals and discrete sums.

ACKNOWLEDGMENTS

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[2] We consider throughout that $\mathrm{sinc}(0) = 1$. Its remarkable properties led Whittaker to specify sinc as "a function of royal blood in the family of entire functions, whose distinguished properties separate it from its bourgeois brethren." The denomination "cardinal" seems to originate from this author [E. T. Whittaker, Proc. Roy. Soc. Edinburgh **35**, 181 (1915)].

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[8] Noticing that $\cos(k)\mathrm{sinc}(k) = \mathrm{sinc}(2k)$, one should here consider $a_1 = 2$ instead of $a_1 = 1$. Then, $N = 56$ is the threshold such that $\sum_{n=2}^{56} 1/(2n-1) < 2$ while $\sum_{n=2}^{57} 1/(2n-1) > 2$.

[9] Our approach shares with previous ones [6,7,10] the explicit usage of Fourier transforms. Yet, it differs in spirit in its

[10] G. Almkvist and J. Gustavsson, arXiv:1405.1265v2.

[11] A Fokker-Planck equation establishes this fact, but it proves relevant for future extensions to remain at a more intuitive level of understanding.

[12] We reiterate here that the walk is commutative, as much as the products appearing in (1); we single out the first step ($n = 1$) as being that with the largest amplitude among the a_n's.

[13] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.123.020201 for (a) a proof of the symmetry argument allowing to understand the behavior of J_N type of integrals, (b) a proof of Eqs. (15) and (16), (c) an extension of our causality argument to a different situation with walkers starting a given distance away from a wall, and finally (d) the argument showing that the building blocks behind our integrals or series do not need to be the Fourier transforms of nonnegative functions. In other words, the $\hat{\mathcal{F}}_n^{(d)}(\boldsymbol{k})$ do not need to be *characteristic* functions, it is sufficient that their Fourier transform be bounded.

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[15] Instead of $\cos(k)$, we could use any function having a Fourier-Transform void of k component for $-1 < k < 1$. such as $[\cos(k) - k\sin(k)]/(1 - k^2)$.

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