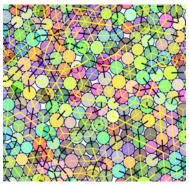

IPa school on disorder in complex systems

Introduction to phase transitions and critical phenomena

Tutorial 3

prepared with M. Lenz and F. van Wijland

Real space renormalization of an Ising model

The purpose of this set of problems is to learn how to implement the renormalization group ideas on simple physical systems, such as interacting spins.

1 Warm-up on the one-dimensional Ising chain

A one-dimensional Ising model has Hamiltonian $H = -J\sum_{i=1}^{N}\sigma_i\sigma_{i+1}$, where $J > 0$, $\sigma_i = \pm 1$ and where $i = 1, \ldots, N$, with $\sigma_{N+1} \equiv \sigma_1$ (periodic boundary conditions are used). The lattice spacing a is taken to be unity. The partition function of the Ising chain is denoted by $Z(K, N, a)$, where $K = \beta J$.

1) Prove that for three spins σ, σ' and σ'', one can always write

$$\sum_{\sigma'} \mathrm{e}^{K\sigma\sigma' + K\sigma'\sigma''} = A\mathrm{e}^{K'\sigma\sigma''} \tag{1}$$

where A and K' are functions of K. Show that one can write $\tanh K' = (\tanh K)^2$.

2) Use the identity in **1** to show that $Z(K, N, a) = A^{N/2}Z(K', N' = N/2, a')$. What is a'? How would you interpret $\ln A$?

3) The calculation in **2** can be interpreted as a renormalization procedure with a scaling factor b. What is b? By iterating the procedure, one obtains a recursion relation on the coupling K_n after n steps ($K_0 = K = \beta J$, $K_1 = K'$, etc.). What are the fixed points of the recursion relation? What is their physical meaning? Why are the terms "high-temperature" or "low-temperature" used when describing these fixed points (here and throughout, the temperature has however been kept a constant)?

4) We define the dimensionles correlation length as $\widetilde{\xi} = \xi/a$. Show that $\widetilde{\xi}' = \widetilde{\xi}/2$. Taking advantage of the fact that $\tanh K' = (\tanh K)^2$, show that this yields $\widetilde{\xi} \propto 1/[-\log(\tanh K)]$.

2 The Niemeijer-Van Leeuwen decimation procedure

In the early days of the renormalization, Niemeijer and Van Leeuwen [Phys. Rev. Lett. **31**, 1411 (1973)] came up with an explicit, albeit approximate, procedure to integrate out a fraction of the degrees of freedom in a two-dimensional spin system. This is what we want to explore in this section. We consider a two-dimensional Ising model with N spins living on a triangular lattice with spacing a. The exchange energy J normalized by the temperature is again denoted by K.

1) The lattice is divided, as shown in figure 1, into triangular plaquettes. A spin variable $S_I = \pm 1$ is associated to each plaquette $I = \{i_1, i_2, i_3\}$ via a majority rule: $S_I = \mathrm{sign}(\sigma_{i_1} + \sigma_{i_2} + \sigma_{i_3})$. What is the number N' of plaquettes and what is the spacing a' of the triangular lattice the plaquettes make up?

2) The Hamiltonian $H = -J\sum_{\langle i,j\rangle}\sigma_i\sigma_j$ of course features interactions between spins σ_i belonging to the same plaquette I, but it also features interactions between spins σ_i and σ_j belonging to different nearest neighbor plaquettes I and J. We thus split the Hamiltonian into $H = H_1 + H_2$, with $H_1 = \sum_I h_1(I)$ and $H_2 = \sum_{\langle I,J\rangle} h_2(I, J)$. With these loose notations $h_1(I)$ actually denotes a function of the spins pertaining to plaquette I (same for $h_2(I, J)$). For a given triangular plaquette $I = \{i_1, i_2, i_3\}$ write the expression of $h_1(I)$ as a function of $\{\sigma_{i_1}, \sigma_{i_2}, \sigma_{i_3}\}$. Similarly, for two nearest neighbor plaquettes $I = \{i_1, i_2, i_3\}$ and $J = \{j_1, j_2, j_3\}$, write $h_2(I, J)$ as a function of $\{\sigma_{i_1}, \sigma_{i_2}, \sigma_{i_3}\}$ and $\{\sigma_{j_1}, \sigma_{j_2}, \sigma_{j_3}\}$.

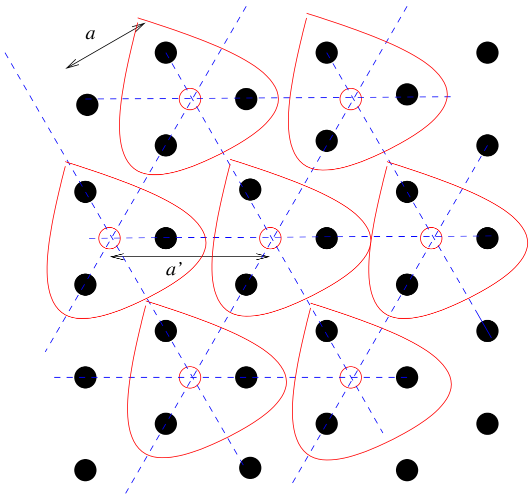

Figure 1: The original spins σ_i lie at the black bullets while the plaquette spins lie at the empty circles. They form a triangular lattice materialized with dashed lines.

3) We would like to rewrite the original partition function Z in terms of a summation over the $\{S_I\}$ configurations rather than over the $\{\sigma_i\}$ configurations, be it at the expense of modifying the Hamiltonian. As a step in that direction, we note that

$$Z(K,N,a) = \sum_{\{S_I\}} \sum_{\{\sigma_i\}}{}' \mathrm{e}^{-\beta H[\{\sigma_i\}]} \tag{2}$$

where $\sum'$ denotes a sumation over all $\{\sigma_i\}$ configurations at fixed plaquette configurations $S_I = \text{sign}(\sum_{i\in I}\sigma_i)$. Let $Z(\{S_I\}) = \sum'_{\{\sigma_i\}} \mathrm{e}^{-\beta H[\{\sigma_i\}]}$. Determine the approximate expression of $Z(\{S_I\})$, denoted by Z_1, when the plaquette-plaquette interactions are discarded.

4) Justify that

$$Z(K,N,a) = \sum_{\{S_I\}} Z_1 \langle \mathrm{e}^{-\beta H_2}\rangle_1 \tag{3}$$

where $\langle \ldots \rangle_1 = \frac{1}{Z_1}\sum'_{\{\sigma_i\}} \mathrm{e}^{-\beta H_1}\ldots$. How would you interpret the average brackets $\langle \ldots \rangle_1$?

5) In general, determining $\langle \mathrm{e}^{-\beta H_2}\rangle_1$ is a formidable task. Express the latter average in terms of the cumulants of H_2 with respect to the measure $\langle \ldots \rangle_1$.

6) We now implement the Niemeijer-Van Leeuwen approximation which consists in dropping all cumulants of order ≥ 2. What is the physical content, in terms of plaquette-plaquette interactions, of the second cumulant (which is neglected)?

7) Let σ_i be a spin belonging to a plaquette I. Show that

$$\langle \sigma_i \rangle_1 = S_I \frac{\mathrm{e}^{3K} + \mathrm{e}^{-K}}{\mathrm{e}^{3K} + 3\mathrm{e}^{-K}} \tag{4}$$

8) Within the proposed approximation, show that $Z(K,N,a) = (\mathrm{e}^{3K} + 3\mathrm{e}^{-K})^{N'} Z(K', N', a')$, where $K' = f(K)$ is to be identified.

9) Find the fixed points of f. Discuss their stability and their physical meaning. Find the critical temperature (within the proposed approximation); compare it with the mean-field value. The exact result is close to $3.6\, J/k$

10) Let ν be the exponent governing the divergence of the correlation length as criticality is approached. Find the value of ν predicted by the Niemeijer-Van Leeuwen approximation and compare it with both its mean-field counterpart and the exact value ($\nu_{\text{exact}} = 1$).

A few years later, Van Leeuwen and his collaborators [Phys. Rev. Lett. **40**, 1605 (1978)] came up with a decimation scheme exact in the limit of very large systems. While the specifics of the calculation itself are tedious, the idea was to begin with an N spin system and to eliminate, at each step of the decimation procedure, an infinitesimal fraction of spins.