The Kosterlitz-Thouless transition

The XY model which follows does not display a *bona fide* critical point but it can nevertheless be approached with renormalization methods. Among classical references, we direct the interested students to Leo P. Kadanoff [Statistical Physics: Statics, Dynamics, and Renormalization, World Scientific, Singapore, (2000)].

1 Introduction

The Nobel Prize in Physics 2016 was divided, one half awarded to David J. Thouless, the other half jointly to F. Duncan M. Haldane and J. Michael Kosterlitz "for theoretical discoveries of topological phase transitions and topological phases of matter".

Figure 1: Kosterlitz and Thouless

The Kosterlitz-Thouless transition that we want to investigate here was discovered back in 1972. This transition both differs and resembles other phase transitions that you may have encountered. The summary provided by the Nobel committee says: "In 1972 J. Michael Kosterlitz and David J. Thouless identified a completely new type of phase transition in two-dimensional systems where topological defects play a crucial role. Their theory applied to certain kinds of magnets and to superconducting and superfluid films, and has also been very important for understanding the quantum theory of one-dimensional systems at very low temperatures". Interested readers are directed to the Nobel Prize in Physics website. The goal of this problem is to guide you through the pecularities of the Kosterlitz-Thouless transition. We will adopt the XY model of interacting spins language.

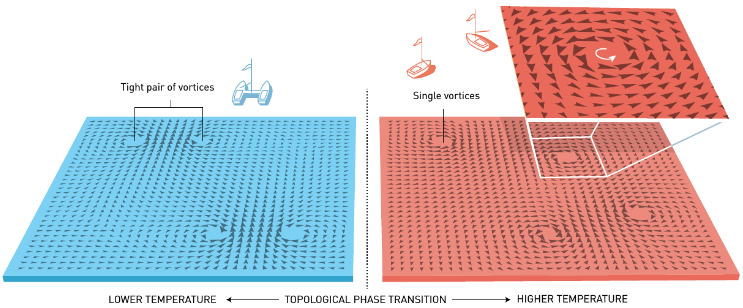

Figure 2: What do interacting spins have to do with vortices? Why are there little boats in the cartoon? Answering this question is the topic of the present problem. The little arrows depicted here do not represent spins, but the gradient of the angle θ introduced below.

An XY model consists of two-dimensional vectorial and classical spins $\mathbf{S}_i$ localized at the vertices $\mathbf{x}$ of a regular lattice with N sites ($N = (L/a)^2$ where a is the lattice spacing) and interacting via a ferromagnetic interaction $H = -J\sum_{\langle \mathbf{x},\mathbf{y}\rangle} \mathbf{S}_{\mathbf{x}} \cdot \mathbf{S}_{\mathbf{y}}$. The XY model is relevant to the description of superfluid helium or hexatic liquid crystals. In the specific two-dimensional case, correlations decay as a power law at low temperature, while above a certain critical temperature, they become short-range. While the nature of correlations changes drastically according to the temperature regime, there is nevertheless no ordering transition. Our purpose here is to convince ourselves of the existence of qualitatively different correlation regimes, and then to analyze the predictions of the renormalization group in the scale invariant regime. Throughout the text, we will make extensive use of the properties of the Green's function G of the Laplacian in two dimensions. These are gathered at the end. Some of the derivations are rather technical; these have been made explicit in the grey box. The focus will be on physical interpretation.

2 Correlations at low and high temperatures

1) Each spin $\mathbf{S}_{\mathbf{x}}$ being characterized by an orientation $\theta_{\mathbf{x}}$, what are the symmetries of the Hamiltonian?

2) What is the ground state of H?

3) Why is $H = \frac{J}{2}\sum_{\langle \mathbf{x},\mathbf{y}\rangle}(\theta_{\mathbf{x}} - \theta_{\mathbf{y}})^2$ a good approximation for H in the low temperature limit?

4) Show that in the **low temperature regime** $\langle \theta_{\mathbf{x}}\theta_{\mathbf{y}}\rangle = \frac{1}{K}G(\mathbf{x}-\mathbf{y})$ where $G(\mathbf{r})$ is defined in the appendix. We shall call Z_{sw} the partition function in this approximate so-called spin-wave regime. We will use the notation $K = \beta J$.

5) How does the spin-spin correlation $C(\mathbf{x},\mathbf{y}) = \langle \mathbf{S}_{\mathbf{x}} \cdot \mathbf{S}_{\mathbf{y}}\rangle$ behave in this low temperature regime? Is there any spontaneous magnetization? The properties of $G(\mathbf{r})$ are given in the appendix.

6) Prove that $\int \frac{\mathrm{d}\theta}{2\pi}\cos(\theta_1 - \theta)\cos(\theta - \theta_2) = \frac{1}{2}\cos(\theta_1 - \theta_2)$.

7) Let $\mathcal{N}(\mathbf{0},\mathbf{r})$ be the number of shortest paths connecting $\mathbf{0}$ to an arbitrary point $\mathbf{r} = (x,y)$. Justify that $\mathcal{N}(\mathbf{0},\mathbf{r}) = \binom{|x|+|y|}{|x|}$. The combination $|x|+|y|$ is sometimes labeled $||\mathbf{r}||_1$. This is the Manhattan distance between the origin and $\mathbf{r}$ (which is also called the 1-norm). Argue that $\mathcal{N}(\mathbf{0},\mathbf{r})$ has upper bound $2^{||\mathbf{r}||_1}$.

8) We now sit in the **high-temperature limit**. After justifying that $Z \simeq \int \frac{\prod_{\mathbf{x}} \mathrm{d}\theta_{\mathbf{x}}}{(2\pi)^N}\prod_{\langle \mathbf{x},\mathbf{y}\rangle}(1 + K\cos(\theta_{\mathbf{x}} - \theta_{\mathbf{y}}))$, show that to leading order in an expansion in powers of K as $K \to 0$

$$C(\mathbf{x},\mathbf{y}) \sim \mathcal{N}(\mathbf{x},\mathbf{y})(K/2)^{||\mathbf{x}-\mathbf{y}||_1} \quad (1)$$

where $||\mathbf{r}||_1$ denotes the Manhattan distance between spins $\mathbf{x}$ and $\mathbf{y}$. We admit that we can restrict to those non-vanishing contributions in (1) that are of lowest order in K. Define and express the correlation length ξ in terms of K. We are interested in the K dependence only.

3 Towards a Coulomb gas within the Villain approximation

Our goal is to establish a connection between the original XY model and a system of charges interacting via a Coulomb potential in two space dimensions. This section is mostly technical at first sight. However, the effective electric charges that appear in reformulating the partition function can be seen as vortices of the local magnetization field that we start from. The first two questions have to do with the Villain approximation, while the remainder is a series of steps and mappings connecting the Villain model with a Coulomb gas. You are urged to read through the various technical steps to get a feel of how the Coulomb gas emerges in technical terms.

1) The Bessel function of imaginary argument $I_n(x) = \int_0^{2\pi}\frac{\mathrm{d}\theta}{2\pi}\mathrm{e}^{x\cos\theta + in\theta}$ allows us to expand $\mathrm{e}^{K\cos u} = \sum_{n=-\infty}^{+\infty}\mathrm{e}^{inu}I_n(K)$. What is the K regime in which one can approximate $I_n(K) \simeq \frac{\mathrm{e}^{K-n^2/(2K)}}{\sqrt{2\pi K}}$?

2) We will henceforth use that $\mathrm{e}^{K\cos u} \simeq \frac{\mathrm{e}^K}{\sqrt{2\pi K}}\sum_{n=-\infty}^{\infty}\mathrm{e}^{inu - \frac{n^2}{2K}}$. With physical symmetries in sight, what is the advantage of the latter approximation with respect to the simpler approximation $\mathrm{e}^{K\cos u} \simeq \mathrm{e}^{K - Ku^2/2}$?

Our starting point is the partition function, written in the form

$$Z = \int \frac{\prod_{\mathbf{x}} \mathrm{d}\theta_{\mathbf{x}}}{(2\pi)^N}\prod_{\langle \mathbf{x},\mathbf{y}\rangle} e^{K\cos(\theta_{\mathbf{x}} - \theta_{\mathbf{y}})}. \quad (2)$$

Sometimes the notation $\mathcal{D}\theta$ instead of the heavier $\frac{\prod_{\mathbf{x}}\mathrm{d}\theta_{\mathbf{x}}}{(2\pi)^N}$ is used. We note that $\prod_{\langle\mathbf{x},\mathbf{y}\rangle}$ is equivalent to $\prod_{\mathbf{x},\mu=x,y}$ with $\mu = x$ or y refering to the bond relating $\mathbf{x}$ to its nearest neighbor. We also denote $\mathbf{e}_x$ and $\mathbf{e}_y$ the units vectors along x and y respectively. We now introduce a two-dimensional vector field $\mathbf{n}(\mathbf{x})$ with integer components. Let's argue why the partition function of the XY model, up to an overall multiplicative constant, can be written in the form

$$Z = \int \frac{\prod_{\mathbf{x}}\mathrm{d}\theta_{\mathbf{x}}}{(2\pi)^N} \prod_{\mathbf{x},\mu=x,y} \sum_{n_\mu(\mathbf{x})=-\infty}^{\infty} \mathrm{e}^{in_\mu(\mathbf{x})\partial_\mu\theta_{\mathbf{x}}}\mathrm{e}^{-n_\mu(\mathbf{x})^2/2K} \tag{3}$$

where ∂_μ refers to a discrete derivative along the space direction $\mu = x$ or y (also denoted by 1 or 2): $\partial_\mu\theta(\mathbf{x}) = \theta_{\mathbf{x}+\mathbf{e}_\mu} - \theta_{\mathbf{x}}$. To do so, we first realize that for each bond of nearest neighbor lattice sites $\mathbf{x}$ and $\mathbf{x}+\mathbf{e}_\mu$, we have

$$\mathrm{e}^{K\cos(\theta_{\mathbf{x}+\mathbf{e}_\mu}-\theta_{\mathbf{x}})} = \sum_{n_\mu(\mathbf{x})=-\infty}^{+\infty} \frac{\mathrm{e}^K}{\sqrt{2\pi K}}\mathrm{e}^{in_\mu(\mathbf{x})(\theta_{\mathbf{x}+\mathbf{e}_\mu}-\theta_{\mathbf{x}})-n_\mu^2/2K} \tag{4}$$

where $\theta_{\mathbf{x}+\mathbf{e}_\mu} - \theta_{\mathbf{x}}$ will henceforth be denoted by $\partial_\mu\theta$. Gathering the above representation for all possible bonds leads to Eq. (3), which can be recast in the following form:

$$Z = \sum_{\{n_\mu(\mathbf{x})\}} \int \frac{\prod_{\mathbf{x}}\mathrm{d}\theta_{\mathbf{x}}}{(2\pi)^N} \left(\frac{\mathrm{e}^K}{\sqrt{2\pi K}}\right)^{2N} \mathrm{e}^{i\sum_{\mathbf{x},\mu} n_\mu(\mathbf{x})(\theta_{\mathbf{x}+\mathbf{e}_\mu}-\theta_{\mathbf{x}})-\frac{n_\mu^2}{2K}} \tag{5}$$

where the main summation with $\{n_\mu(\mathbf{x})\}$ amounts to $\{n_x(\mathbf{x}_1), n_y(\mathbf{x}_1), n_x(\mathbf{x}_2), n_y(\mathbf{x}_2), \ldots\}$, assigning two sets of integers to each lattice site. The K-dependent prefactors will hereafter be omitted (they do contribute the free energy, but they do not affect θ or $\mathbf{n}$ dependent calculations).

We now integrate out each $\theta(\mathbf{x})$ degree of freedom. This leads to a constraint over the $\mathbf{n}(\mathbf{x})$ degrees of freedom that can be expressed in terms of the discrete version of the divergence operator. Indeed, using that $\sum_{\mathbf{x},\mu} n_\mu(\mathbf{x})\partial_\mu\theta(\mathbf{x}) = -\sum_{\mathbf{x},\mu}\partial_\mu n_\mu(\mathbf{x})\theta(\mathbf{x})$ (ignoring possible boundary contributions), and using that for any integer z, we have $\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta}{2\pi}\mathrm{e}^{-i\theta z} = \delta_{z,0}$, we arrive at the constraint that for each $\mathbf{x}$, we have $\sum_\mu \partial_\mu n_\mu(\mathbf{x}) = \boldsymbol{\nabla}\cdot\mathbf{n} = 0$. A more pedestrian route goes as follows. In Eq. (5), before integrating over a given $\theta_{\mathbf{x}}$, we note that it appears in the following combination

$$n_x(\mathbf{x})[\theta_{\mathbf{x}+\mathbf{e}_x} - \theta_{\mathbf{x}}] + n_y(\mathbf{x})[\theta_{\mathbf{x}+\mathbf{e}_y} - \theta_{\mathbf{x}}] + n_x(\mathbf{x}-\mathbf{e}_x)[\theta_{\mathbf{x}} - \theta_{\mathbf{x}-\mathbf{e}_x}] + n_y(\mathbf{x}-\mathbf{e}_y)[\theta_{\mathbf{x}} - \theta_{\mathbf{x}-\mathbf{e}_y}]. \tag{6}$$

Thus, $\theta_{\mathbf{x}}$ goes together with

$$-n_x(\mathbf{x}) - n_y(\mathbf{x}) + n_x(\mathbf{x}-\mathbf{e}_x) + n_y(\mathbf{x}-\mathbf{e}_y). \tag{7}$$

This quantity should vanish, otherwise the $\theta_{\mathbf{x}}$ integration is cancelled out. We are back to the 'divergence' condition $\sum_\mu \partial_\mu n_\mu(\mathbf{x}) = 0$.

While it may be surprising that the scalar field θ was traded for a dual vector field $\mathbf{n}$, this is actually just a mathematical illusion. As is true in continuum space, if $\boldsymbol{\nabla}\cdot\mathbf{n} = 0$ then $\mathbf{n}$ can be written in the form of a curl, $\mathbf{n} = \boldsymbol{\nabla}\times\mathbf{A}$, where $\mathbf{A} = p(\mathbf{x})\mathbf{e}_z$. Here $p(\mathbf{x})$ is a scalar integer-valued field related to $\mathbf{n}$ via $n_1(\mathbf{x}) = p(\mathbf{x}+\mathbf{e}_2) - p(\mathbf{x}) = \partial_2 p$ and $n_2(\mathbf{x}) = p(\mathbf{x}) - p(\mathbf{x}+\mathbf{e}_1) = -\partial_1 p$. Given the linear relation between $\mathbf{n}$ and p, any possible Jacobian would be a constant. This allows us to cast the partition function in the form of a summation over configurations of $p(\mathbf{x})$:

$$Z = \sum_{\{p(\mathbf{x})\in\mathbb{Z}\}} \mathrm{e}^{-\frac{1}{2K}\sum_{\mathbf{x}}(\boldsymbol{\nabla}p)^2} \tag{8}$$

An interesting interpretation of the exponential weight in the above partition involves a fictitious temperature K and a Hamiltonian $H = \frac{1}{2}\sum_{\mathbf{x}}(\boldsymbol{\nabla}p)^2$. More can be found in the section dealing with the roughening transition, which at first sight looks to be a physical problem remote from the XY model of spins.

A couple more steps remain before we see the Coulomb gas picture emerge. The first one rests on the use of the Poisson formula, which states that for an arbitrary function f we have that

$$\sum_{p=-\infty}^{\infty} f(p) = \sum_{m=-\infty}^{\infty}\int_{-\infty}^{+\infty}\mathrm{d}\phi f(\phi)\mathrm{e}^{i2\pi m\phi} \tag{9}$$

We apply that formula for each summation over $p(\mathbf{x})$ (namely, for each $\mathbf{x}$), with the result that

$$Z = \left(\prod_{\mathbf{x}} \int_{-\infty}^{\infty} \mathrm{d}\phi(\mathbf{x})\right) \sum_{\{m(\mathbf{x})\in\mathbb{Z}\}} \mathrm{e}^{-\frac{1}{2K}\sum_{\mathbf{x}} \boldsymbol{\nabla}\phi^2 + 2i\pi\sum_{\mathbf{x}} m(\mathbf{x})\phi(\mathbf{x})} \tag{10}$$

The last step consists in integrating out the Gaussian field $\phi(\mathbf{x})$ explicitly. This is done using the methods seen in homework 1. The useful formula is that if ϕ is a Gaussian field with correlation $\langle\phi(\mathbf{x})\phi(\mathbf{y})\rangle = G(\mathbf{x},\mathbf{y})$ then for any arbitrary $j(\mathbf{x})$ we have that $\langle \mathrm{e}^{\int \mathrm{d}^2 x j(\mathbf{x})\phi(\mathbf{x})}\rangle = \mathrm{e}^{\frac{1}{2}\int \mathrm{d}^2x \mathrm{d}^2x' j(\mathbf{x})G(\mathbf{x},\mathbf{x}')j(\mathbf{x}')}$. We apply this formula to $G = (-\Delta/K)^{-1}$, namely to the Green's function of the Laplacian, with $j(\mathbf{x}) = 2i\pi m(\mathbf{x})$. Denoting by $Z'_{\mathrm{sw}} = \int \prod_{\mathbf{x}} \mathrm{d}\phi(\mathbf{x})\mathrm{e}^{-\frac{1}{2K}\sum_{\mathbf{x}}\boldsymbol{\nabla}\phi^2}$ we thus arrive at

$$Z = Z'_{\mathrm{sw}} \sum_{\{m(\mathbf{x})\}} \mathrm{e}^{-2\pi^2 K \sum_{\mathbf{x},\mathbf{y}} m(\mathbf{x})G(\mathbf{x}-\mathbf{y})m(\mathbf{y})} \tag{11}$$

where G is the Green's function of the discrete Laplacian. We find it convenient to rewrite Z with the help of a more regular function $\overline{G}(\mathbf{r}) = G(\mathbf{r}) - G(\mathbf{0})$. After substitution we get

$$\begin{aligned} Z &= Z'_{\mathrm{sw}} \sum_{\{m(\mathbf{x})\}} \mathrm{e}^{-2\pi^2 K G(\mathbf{0})(\sum_{\mathbf{x}} m(\mathbf{x}))^2} \mathrm{e}^{-2\pi^2 K \sum_{\mathbf{x},\mathbf{y}} m(\mathbf{x})\overline{G}(\mathbf{x}-\mathbf{y})m(\mathbf{y})} \\ Z &= Z'_{\mathrm{sw}} \sum_{\{m(\mathbf{x})\}} \mathrm{e}^{-2\pi^2 K G(\mathbf{0})(\sum_{\mathbf{x}} m(\mathbf{x}))^2} \mathrm{e}^{-2\pi^2 K \sum_{\mathbf{x}\neq\mathbf{y}} m(\mathbf{x})\overline{G}(\mathbf{x}-\mathbf{y})m(\mathbf{y})} \end{aligned} \tag{12}$$

At this stage, nothing constrains the configurations of the integer field $m(\mathbf{x})$. However, given that $G(\mathbf{0}) \simeq \frac{1}{2\pi}\ln\frac{L}{a}$ we see that in the large L limit configurations which have $\sum_{\mathbf{x}} m(\mathbf{x}) \neq 0$ are killed. Hence our conclusion that **only** *neutral* **configurations of m enter the partition function**, namely those which verify $\sum_{\mathbf{x}} m(\mathbf{x}) = 0$. An explicit expression for $\overline{G}$ is not available, but its large distance behavior is well-known, while it is regular at short distances. Without any loss for the description of large scale phenomena, we extrapolate the large distance asymptotics of $\overline{G}$ downto the lattice scale by using the approximate expression $\overline{G}(\mathbf{r}) \simeq -\frac{1}{2\pi}\ln\frac{||\mathbf{r}||}{a} - \frac{1}{4}$ (see the Appendix). This is useful in the second line of Eq.(12).

At last, we are ready for the Coulomb gas identification: the quantity $Z_{\mathrm{v}} = Z/Z'_{\mathrm{sw}}$ is the partition function of a two-dimensional Coulomb gas with charges $2\pi\sqrt{J}m(\mathbf{x})$ sitting at the lattice sites whose density is governed by $y = \mathrm{e}^{-\pi^2 K/2}$ that plays the role of a fugacity in that it controls the density of charges. This partition function

$$Z = Z'_{\mathrm{sw}} \sum_{\{m(\mathbf{x})\}} y^{\sum_{\mathbf{x}} m(\mathbf{x})^2} \mathrm{e}^{\pi K \sum_{\mathbf{x}\neq\mathbf{y}} m(\mathbf{x})\ln(||\mathbf{x}-\mathbf{y}||/a)m(\mathbf{y})} \tag{13}$$

is close enough to the partition function one would write out of the box for a two component Coulomb gas in the grand-canonical ensemble. The specifics of the fugacity term differ, though, especially at higher densities of charges.

It is high time we endow the field $m(\mathbf{x})$ with a physical meaning that connects to the original problem of interacting spins. The quantity $m(\mathbf{x})$ can be viewed as the circulation of the local magnetization field around some location $\mathbf{x}$ and it can thus be interpreted as a vorticity field. This interpretation can be traced back in the series of technical steps we have just gone through ($\theta \to \mathbf{n} \to p \to \{\phi, m\} \to m$). It would of course deserve a bit more work to be fully clarified, but this is at least consistent with the knowledge one has from hydrodynamics where vortices a distance r apart interact via a $\ln r$ interaction. As y increases, more and more charges appear while $y \to 0$ has a vanishing number of such charges (or vortices) which eventually stop interacting with each other. The former regime in which vortices proliferate is found in at high temperatures: correlations decay exponentially fast. At low temperatures, by contrast, quasi long range order sets in, characterized by power law correlations.

4 Real-space renormalization

We will now focus on the Z_{v} partition function that cannot be evaluated exactly. In the low fugacity limit, the behavior of the system is well understood, which suggests to attempt a $y \to 0$ expansion of Z_{v}. Below, one should not confuse the fugacity y with a Cartesian coordinate.

1) We begin with the correlation function $C(\mathbf{x},\mathbf{y}) = \langle \mathbf{S}_{\mathbf{x}} \cdot \mathbf{S}_{\mathbf{y}} \rangle$ and with one more accepted result. It is possible to show (see the beautiful 1977 paper [Phys. Rev. B **16**, 1217 (1977)] by José, Kadanoff, Kirkpatrick and Nelson, equation (5.1)) that a blunt expansion of C in powers of y leads to the expression

$$C(\mathbf{x},\mathbf{y}) \propto ||\mathbf{x}-\mathbf{y}||^{-\frac{1}{2\pi K_{\text{eff}}}} \tag{14}$$

with

$$\frac{1}{K_{\text{eff}}} = \frac{1}{K} + 4\pi^3 y^2 \int_a^L \frac{\mathrm{d}r}{a}\left(\frac{r}{a}\right)^{3-2\pi K}. \tag{15}$$

For what K regime is the perturbation expansion in powers of y well-defined (for lack of a better word)?

2) We introduce $b > 1$ and split the integral in the right hand side of (15) into $\int_a^L \mathrm{d}r \ldots = \int_a^{ba} \mathrm{d}r \ldots + \int_{ba}^L \mathrm{d}r \ldots$. We define K' by $K'^{-1} = K^{-1} + 4\pi^3 y^2 \int_a^{ba} \frac{\mathrm{d}r}{a}\left(\frac{r}{a}\right)^{3-2\pi K}$. We thus arrive at

$$K_{\text{eff}}^{-1} = K'^{-1} + 4\pi^3 y^2 \int_{ba}^L \frac{\mathrm{d}r}{a}\left(\frac{r}{a}\right)^{3-2\pi K}. \tag{16}$$

Upon rescaling ba into a, show that the relationship between K_{eff}, K' and y' in (16) is strongly reminiscent of (15) between K_{eff}, K and y, from which one can define the renormalized fugacity $y' = b^{2-\pi K} y$.

3) Assuming the elimination of short scale fluctuations between a and ba is infinitesimal, with $b = e^{\ell}$, show that the running couplings $K(\ell)$ and $y(\ell)$ evolve according to

$$\frac{\mathrm{d}K}{\mathrm{d}\ell} = -4\pi^3 y^2 K^2, \qquad \frac{\mathrm{d}y}{\mathrm{d}\ell} = (2-\pi K)y. \tag{17}$$

4) As a consistency check, the flow equation on y can be recovered in a simpler way. First show that as $y \to 0$, $Z_{\text{v}} = Z/Z'_{\text{sw}} = 1 + \frac{y^2}{a^4}\int_0^L \mathrm{d}^2x\mathrm{d}^2y \left(\frac{a}{||\mathbf{x}-\mathbf{y}||}\right)^{2\pi K}$. The double space integral avoids the $||\mathbf{x}-\mathbf{y}|| < a$ region.

5) Show now that, after splitting in the double integral into two regions ($||\mathbf{x}-\mathbf{y}|| < ba$ and $||\mathbf{x}-\mathbf{y}|| > ba$) it is possible to rewrite Z_{v} in the form

$$Z_{\text{v}} = (1+y^2 I)\left[1 + \frac{y'^2}{a^4}\int_{a<|||\mathbf{x}-\mathbf{y}|||<L/b} \mathrm{d}^2x\mathrm{d}^2y\left(\frac{a}{||\mathbf{x}-\mathbf{y}||}\right)^{2\pi K}\right] \tag{18}$$

where $I = a^{-4}\int_{a||\mathbf{x}-\mathbf{y}||<ba} \mathrm{d}^2x\mathrm{d}^2y\left(\frac{a}{||\mathbf{x}-\mathbf{y}||}\right)^{2\pi K}$. The $(1+y^2 I)$ prefactor is not renormalizing any of the y or K couplings. In your opinion, what does it renormalize? Recover from this analysis that $y' = y\, b^{2-\pi K}$.

6) Recall the relationship between y and K at the microscopic level (before any sort of renormalization). Plot the $y(K)$ function in the (K, y) plane. This is the so-called line of initial conditions. Explain the latter terminology.

7) The RG flow is made up of the two equations (17). What are the fixed points of the RG flow? Position the fixed points in the same (K, y) plane as in the previous question.

8) Show that if K remains in the vicinity of $2/\pi$ we must have

$$16\pi^2 y^2 - (2-\pi K)^2 = C,$$

namely that the flow lines are hyperboles in the (K, y) plane. Draw the asymptotes in the (K, y) plane along with a few possible trajectories.

9) Recall that our approach is based on a small y (virial) expansion. Discuss the stability of the fixed points and provide their physical interpretation.

10) What can you say about the nature of spin-spin correlations at each of these fixed points?

5 Correlation length from the high temperature region

We want to exploit the RG flow to predict the temperature dependence of the correlation length in the high-temperature phase.

1) What is the correlation length in the low temperature phase?

2) How would you define the critical temperature T_c? Carry out a graphical check of your definition by plotting the fixed points reached by the flow depending on whether $T > T_c$ or $T < T_c$.

3) Justify that as $T \to T_c$, we must have $C \simeq a(T - T_c)$, where a is a constant whose sign will be given.

4) In this regime close to the critical point, find $K(\ell)$ by direct integration of the flow between $\ell = 0$ and ℓ.

5) How would you define the correlation length ξ? Prove that as $T \to T_c^+$,

$$\xi \sim \exp\left(\frac{\text{cst}}{\sqrt{T - T_c}}\right). \tag{19}$$

Appendix

Let $G(\mathbf{r})$ be the Green's function of the two-dimensional Laplacian on a square lattice with $N = (L/a)^2$ sites, defined by

$$-\sum_{\mu=x,y} \left(G(\mathbf{r} + a\mathbf{e}_\mu) + G(\mathbf{r} - a\mathbf{e}_\mu) - 2G(\mathbf{r})\right) = \delta_{\mathbf{r},\mathbf{0}} \tag{20}$$

where a is the lattice spacing, and a $1/N$ contribution to the right hand side (a uniform background) has been neglected. Introducing the Fourier transform $\widetilde{G}(\mathbf{q})$ defined by

$$\widetilde{G}(\mathbf{q}) = \sum_{\mathbf{r}} \mathrm{e}^{i\mathbf{q}\cdot\mathbf{r}} G(\mathbf{r}) \tag{21}$$

we have that for $\mathbf{q} \neq \mathbf{0}$,

$$\widetilde{G}(\mathbf{q}) = \frac{1}{4 - 2\cos aq_x - 2\cos aq_y} \tag{22}$$

The Fourier modes are indexed by $\mathbf{q} = \frac{2\pi}{L}(n_x, n_y)$, where n_x and $_y$ are integers between $-L/(2a)$ and $L/(2a)$. The direct space Green's function is

$$G(\mathbf{r}) = \frac{1}{N} \sum_{\mathbf{q}\neq\mathbf{0}} \frac{\mathrm{e}^{i\mathbf{q}\cdot\mathbf{r}}}{4 - 2\cos aq_x - 2\cos aq_y}. \tag{23}$$

When injecting this form into (20), one indeed checks that up to the aforementioned (and not specified) $1/N$ contribution to the right hand side, we have the Green's function sought for. We shall not prove any of the properties below, but we freely use them

$$G(\mathbf{0}) \simeq \frac{1}{2\pi} \ln \frac{L}{a}, \qquad \overline{G}(\mathbf{r} \gg a) \simeq -\frac{1}{2\pi} \ln \frac{||\mathbf{r}||}{a} - c + o(1) \tag{24}$$

where $\overline{G}(\mathbf{r}) = G(\mathbf{r}) - G(\mathbf{0})$ and where $c = \frac{1}{2\pi}\left(\gamma + \frac{3}{2}\ln 2\right) \simeq \frac{1}{4}$.