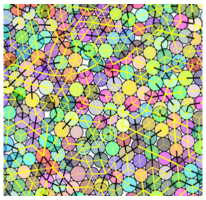

IPa school on disorder in complex systems
Introduction to phase transitions and critical phenomena
Tutorial 5

prepared with M. Lenz and F. van Wijland

Renormalization à la Migdal-Kadanoff

We consider a d-dimensional Ising model with spins $S_i = \pm 1$ and dimensionless coupling constant $K > 0$, having partition function

$$Z = \sum_{\{S_i\}} \exp\left[K \sum_{\langle i,j\rangle} S_i S_j\right], \tag{1}$$

where $\langle i, j\rangle$ denotes a pair of nearest neighbors on the lattice.

1 The 1d case

We start with the $d = 1$ situation, where real-space decimation can be carried out exactly. We will repeatedly need in the remainder the corresponding recursion relation of this explicit renormalization : decimating $b - 1$ consecutive sites so that the renormalized model has lattice constant multiplied by b, the coupling strength transforms into

$$\boxed{\tanh K' = (\tanh K)^b}, \tag{2}$$

see Fig. 1 illustrating the case $b = 3$.

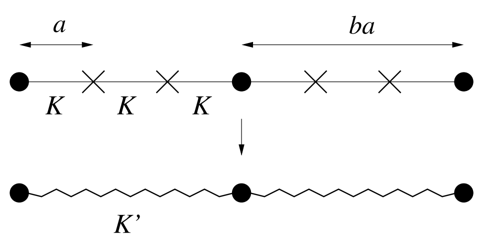

Figure 1 – In this Ising chain, spins shown with a cross are integrated out while those with a circle are kept. Here, the renormalization is performed via a decimation by a factor $b = 3$; the zigzag lines indicate a change of coupling upon decimation, from K to K' (see Eq. 2).

1) Prove that one can always write $\exp(KS_iS_j) = c\,(1 + d\,S_iS_j)$, where the constants c and d depend on K.

2) Spin values S_1 and S_3 being fixed, compute

$$\sum_{\{S_2\}} \exp\left(KS_1S_2 + KS_2S_3\right). \tag{3}$$

3) Proceed then to show relation (2). To this end, one can express the partition function $Z(K, N, a)$ for an N-spin system and lattice constant a in terms of $(\ldots)\, Z(K', N', a')$, where $(\ldots)$ refers to a K-dependent prefactor. A discussion of boundary conditions is not asked ; we aim to remain here at the simplest possible level.

2 The two-dimensional model / take 1

Beyond dimension 1 unfortunately, a simple decimation of the Ising model is problematic : starting from nearest neighbor interactions, longer-range couplings are generated, which quickly become out of control.

4) Briefly justify the previous statement.

To circumvent this difficulty, approximations have to be made. We will follow here an idea put forward by Migdal (1975) and Kadanoff (1976), which yields reasonable phase diagrams in a number of situations, such as absorbed noble gases on surfaces like graphite. We consider a $b = 2$ decimation on the square lattice. The idea, sketched in Fig. 2 is to ultimately remove the bonds that are not connected to the retained spins. The retained spins, shown with a circle in Fig. 2, are then connected by a bond of strength $2K$ rather than K (see the middle panel). In doing so, we obtain a system of spins that although not rigorously equivalent to the original one, is convenient to treat and yields insightful results.

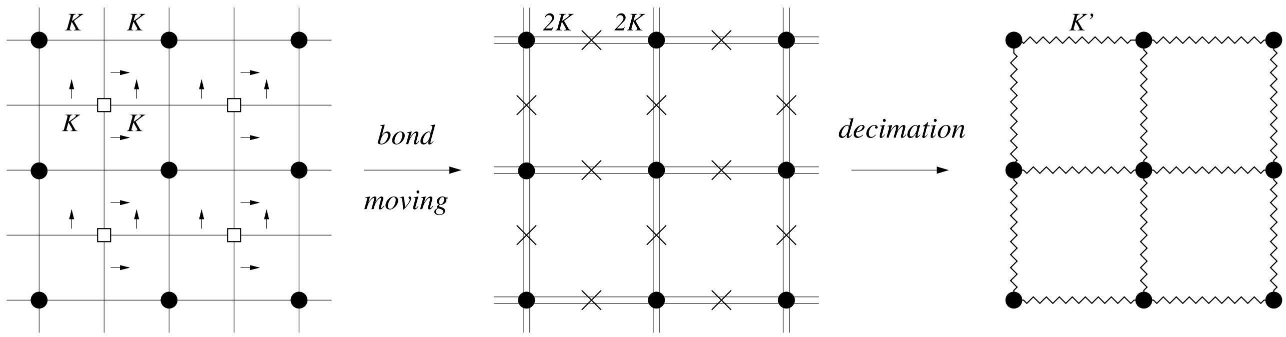

FIGURE 2 – The bond moving scheme for $d = 2$ and $b = 2$. Spins to be decimated are again shown with a cross. The small arrows in the left panel indicate in which direction the bonds are moved. Their precise direction in not essential, provided that one ends up with the configuration shown in the central panel, where all bonds have been doubled in strength. Note that the "central" spins, shown with squares in the left panel, simply disappear after bond-moving. The right panel is for the system resulting from integrating out the spins with a cross. This system is endowed with a new coupling constant, K', embodied by the zigzag lines.

5) What is the new coupling strength K' after decimation ? Put this recursion relation in the form $\tanh K' = \ldots$

6) Discuss the fixed points for the recursion relation. One can here define $x = \tanh K$ and use

$$\tanh(2t) = \frac{2\tanh t}{1+\tanh^2 t}, \tag{4}$$

together with Fig. 3. Provide an approximate value of the critical coupling K_c.

7) What is the mean-field prediction, K_c^{mf}, for the above quantity (no heavy calculation asked) ? Compare to the exact value worked out by Onsager

$$K_c^{\mathrm{exact}} = \frac{1}{2}\log(1+\sqrt{2}) \simeq 0.44, \tag{5}$$

and comment.

8) When did Onsager solve the $d = 2$ Ising model ? Can you name a predecessor of his who obtained interesting exact results pertaining to the study of phase transitions ?

We measure the correlation length ξ in units of the relevant lattice spacing : $\widetilde{\xi} = \xi/a$, $\widetilde{\xi}' = \xi'/(ba)$. We seek next the critical exponent associated to the divergence of $\widetilde{\xi}$ in the vicinity of K_c : $\widetilde{\xi} \propto |K - K_c|^{-\nu}$.

9) How is $\widetilde{\xi}'/\widetilde{\xi}$ related to the scaling factor b ?

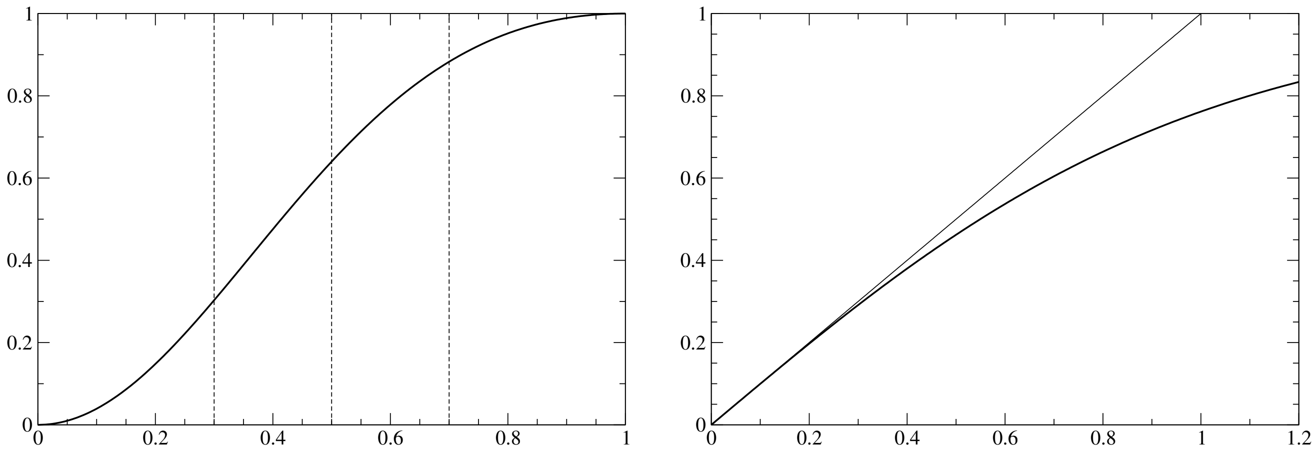

FIGURE 3 – Left : graph of the function $x \mapsto 4x^2/(1+x^2)^2$. Right : graph of the function $t \mapsto \tanh t$; the first bisector is also shown.

10) Show that $\left.\dfrac{\partial K'}{\partial K}\right|_{K_c}$ allows to compute ν. What is ν ? Compare to the mean-field prediction ν^{mf} (if you remember it, no calculation asked), and to the exact result $\nu^{\text{exact}} = 1$. It can be useful here to use $\log 2 \simeq 0.69$ and $\log(5/3) \simeq 0.51$.

3 The two-dimensional model / take 2

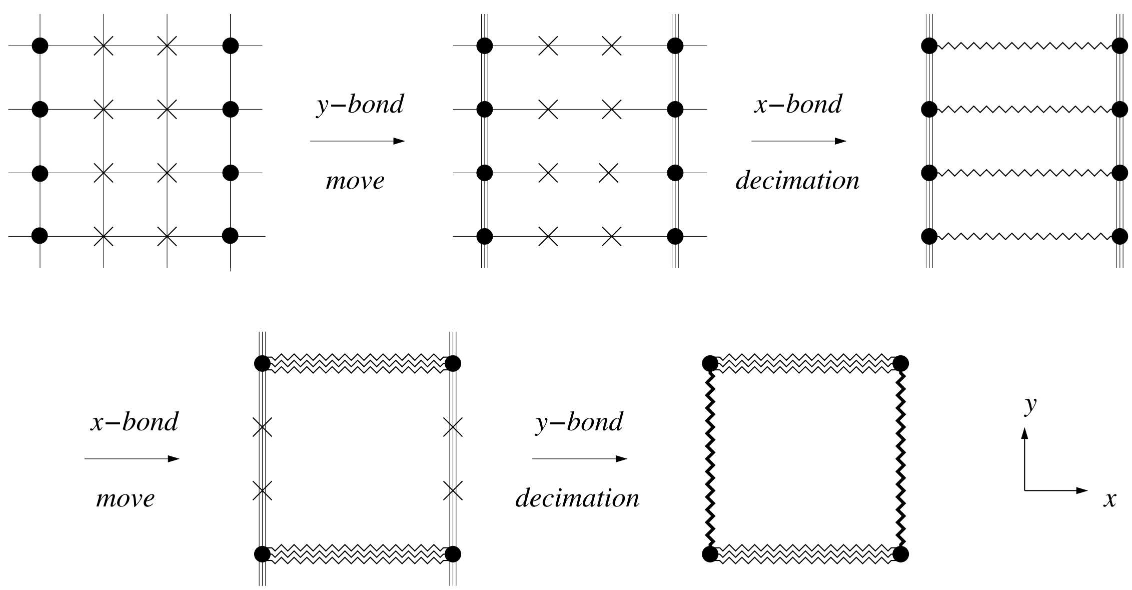

FIGURE 4 – The "take 2" Migdal-Kadanoff procedure on the square lattice, with $b = 3$. After breaking and moving the y-bonds along x, we obtain sites that are no longer connected to their neighbors along the y-axis. These sites (shown with a cross) are integrated out by a one dimensional decimation akin to that discussed in section 1. The process is repeated by interchanging x and y directions, to restore the symmetry between x and y directions. At the end of the procedure, one obtains couplings K'_x and K'_y.

A refined procedure may be proposed, which starts by allowing coupling constants to differ along the x and y directions, and proceeds in four steps (see Fig. 4). We first break and move $b-1$ bonds along x to create new bonds of strength $\widetilde{K}_y = bK_y$, at every b-th line. The spins that are not connected by y-bonds are marked with a cross in Fig. 4 ; they are subsequently decimated.

11) At this stage, what is the value of the resulting coupling constant along x, $\widetilde{K}_x$?

The ensuing lattice is no longer square, but rectangular. To circumvent this shortcoming, we move x-bonds as indicated in Fig. 4 and we finally decimate the sites that are not connected by x-bonds.

12) Express K'_x and K'_y as a function of K_x and K_y.

In spite of our efforts, the resulting model features an undesirable anisotropy. Indeed, starting from $K_x = K_y = K$, we end up with $K'_x \neq K'_y$. An interesting trick to get round this drawback is to generalize our recursion relations to non-integer b-values, and consider the limit $b \to 1$.

13) Denoting $b = 1 + \epsilon$, with $\epsilon \ll 1$, show that K'_x and K'_y now coincide to linear order in ϵ, with

$$K'_x \;=\; K_x \,+\, \epsilon \left\{ K_x + f(K_x)\, \log \tanh K_x \right\} \tag{6}$$

where f is some function, to be specified.

14) Check that the exact critical coupling K_c^{exact} given by Eq. (5) is a fixed point of the new recursion relation (6).

15) To leading order in ϵ, show that the critical exponent ν for the correlation length reads

$$\nu \;=\; \frac{1}{2 - \sqrt{2}\log(1+\sqrt{2})} \simeq 1.327. \tag{7}$$

Conclude.

4 Generalization to arbitrary dimension (take 1 route)

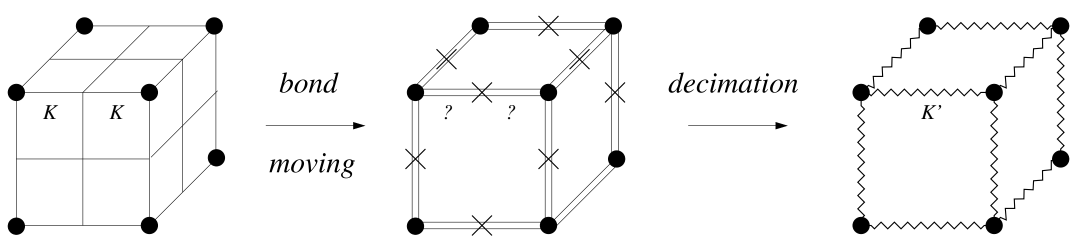

FIGURE 5 – The "take 1" procedure in three dimensions. Here, $b = 2$.

16) On a three-dimensional cubic lattice with $b = 2$ (see Fig. 5), what would the "take 1" recursion relation between K and K' be?

17) How does this result generalize to a d-dimensional cubic lattice?

18) Discuss the stability of the low temperature fixed point (large K regime) where it turns out that K' and K are proportional. Connect this result to the lower critical dimension of the present discrete model.

19) Write the recursion relation for arbitrary b and d.

References :

→ *Recursion relations in gauge field theories*, A.A. Migdal, Sov. Phys.-JETP **42**, 413 (1975).

→ *Notes on Migdal's recursion formulas*, L.P. Kadanoff, Ann. Phys. **100**, 359 (1976).