Renormalization à la Migdal-Kadanoff - short correction

A- The 1d case

1) We have $\exp(KS_iS_j) = \cosh K + S_iS_j \sinh K = (\cosh K)(1 + S_iS_j \tanh K)$. This stems from the fact that $S_iS_j = \pm 1$. If this is not seen immediately, one way to proceed is to write

$$e^{KS_iS_j} = \cosh(KS_iS_j) + \sinh(KS_iS_j) = \cosh(K) + \sinh(KS_iS_j) = \cosh K + S_iS_j \sinh K, \quad (1)$$

from parity.

2) S_2 will later on be a spin to be decimated :

$$\begin{aligned} \sum_{\{S_2\}} e^{KS_1S_2+KS_2S_3} &= (\cosh K)^2 \sum_{\{S_2\}} (1 + S_1S_2 \tanh K)(1 + S_2S_3 \tanh K) \\ &= 2(\cosh K)^2 \left[1 + S_1S_3 (\tanh K)^2\right]. \end{aligned} \quad (2)$$

The terms in S_2^0 and S_2^2 only do survive upon integrating out S_2.

3) The partition function can be written

$$Z(K,N,a) = \sum_{\{S_i\}} \prod_i \exp(KS_iS_{i+1}) = (\cosh K)^N \sum_{\{S_i\}} \prod_i (1 + S_iS_{i+1} \tanh K). \quad (3)$$

Using repeatedly relation (2), for all spins marked with a cross, we arrive at

$$Z(K,N,a) = (\cosh K)^N Z(K', N/b, ba) \quad (4)$$

where since there are $b-1$ spins integrated out between successive retained spins, it appears that the renormalized model has lattice constant K' such that

$$\boxed{\tanh K' = (\tanh K)^b}. \quad (5)$$

One may note that for $b = 2$, $\tanh K' = (\tanh K)^2 \iff K' = \frac{1}{2} \log\cosh(2K)$, a form that we already met (see the tutorials).

B- The two-dimensional model / take 1

4) When computing the partition function, integrating out spins marked with a square in Fig. C1 couples the 4 neighboring spins. Couplings proliferate upon decimation, which is not sustainable.

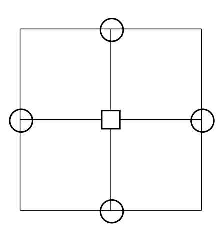

FIGURE C1 – Proliferation of couplings under naive renormalization. In one iteration, integrating out the □ spin, the spins marked with circles become coupled by a 4-body term.

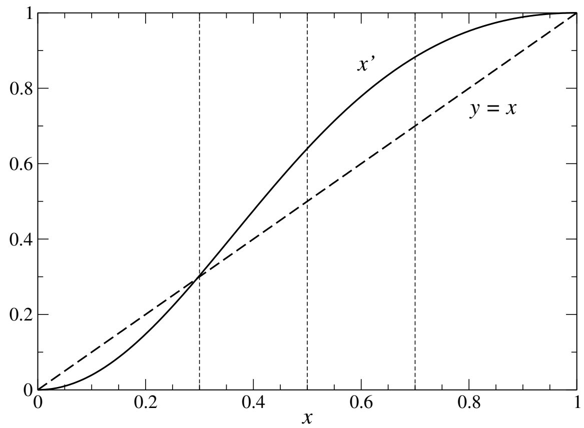

FIGURE C2 – Finding the fixed point of the recursion relation (7). The graph is for the $x \mapsto x'$ mapping, with $x = \tanh K$. The non-trivial fixed point is at $x_c \simeq 0.3$.

5) Moving the bonds makes the problem locally unidimensional, so that we can make use of the result shown in section A. Therefore, the recursion relation is

$$\boxed{\tanh K' = \left[\tanh(2K)\right]^2}. \tag{6}$$

In terms of $x = \tanh K$, this means

$$x' = \left[\tanh(2K)\right]^2 = \left[\frac{2\tanh K}{1+\tanh^2 K}\right]^2 = \left[\frac{2x}{1+x^2}\right]^2. \tag{7}$$

6) Fixed points for relation (7). There are two trivial fixed points : a large-temperature one for $K = 0$, and a low-temperature one for $K \to \infty$. With $x = \tanh K$, they correspond respectively to $x = 0$ and $x \to 1$. It can be seen on Fig. C2 that they are both stable, since the derivative is smaller than unity in their vicinity [1]. The figure also displays a third (and non-trivial) fixed point, $x_c \simeq 0.3$. For such a value, the $t \mapsto \tanh t$ graph provided in the main text shows that $\tanh t \simeq t$ is a very fair approximation (remember the next Taylor term in the expansion, which is $t^3/3$). Thus, $\boxed{K_c \simeq 0.3}$. A more precise calculation shows that $K_c \simeq 0.305$. This fixed point is unstable, as it should.

7) The mean-field prediction is $K_c^{\mathrm{mf}} = 1/4$, since each site has 4 neighbors on the lattice. As always, mean-field overestimates the critical temperature, and correspondingly underestimates the critical K, since it discards fluctuations that destroy order : $\boxed{K_c^{\mathrm{mf}} < K_c^{\mathrm{exact}}}$. Note that Migdal and Kadanoff do a better job here than mean-field.

8) Lars Onsager solved the $d = 2$ Ising model in the 1940s. Rudolf Peierls had previously rigorously shown the existence of a phase transition for the $d = 2$ Ising model, in the 1930s.

9) Since the large scale features are preserved by renormalization, $\xi' = \xi$, meaning that $\widetilde{\xi}'/\widetilde{\xi} = 1/b$.

10) We know that when $K \to K'$, $\widetilde{\xi} \to \widetilde{\xi}/b$. To loop the loop, we need $\widetilde{\xi} \propto |K - K_c|^{-\nu}$. Denoting $K = K_c + \delta K$, $K' = K_c + \delta K'$, this means

$$\frac{(\delta K')^{-\nu}}{(\delta K)^{-\nu}} = \frac{\widetilde{\xi}'}{\widetilde{\xi}} = \frac{1}{b} \implies \boxed{\left.\frac{\partial K'}{\partial K}\right|_{K_c} = b^{1/\nu}}. \tag{8}$$

---

1. We are supposed to study the $K \mapsto K'$ mapping, rather than $x \mapsto x'$. Both are equivalent, and if a fixed point is (un)stable in one variable, then so is it for the other. Indeed, let us call f the function behind the mapping $K \to K'$: $K' = f(K)$, and we are interested in some K^* with $K^* = f(K^*)$. We change variable to $x = \varphi(K)$, and we take φ to be a bijection (here, a tanh). Then, $x' = \varphi(K') = \varphi(f(K)) = \varphi(f(\varphi^{-1}(x)))$. Elementary calculus shows that

$$\frac{dx'}{dx} = \varphi'\big(f(\varphi^{-1}(x))\big)\,\frac{f'\big(\varphi^{-1}(x)\big)}{\varphi'(\varphi^{-1}(x))} = \varphi'\big(f(K)\big)\,\frac{f'(K)}{\varphi'(K)} \implies \left.\frac{dx'}{dx}\right|_{x^*} = \varphi'\big(f(K^*)\big)\,\frac{f'(K^*)}{\varphi'(K^*)} = f'(K^*).$$

This proves that a fixed point exhibits the same stability features in both representations, K or x.

Differentiating (6), we get

$$(1 - \underbrace{\tanh^2 K'}_{\tanh^4(2K)}) \frac{\partial K'}{\partial K} = 4\tanh(2K)(1-\tanh^2 2K) \implies \left.\frac{\partial K'}{\partial K}\right|_{K_c} = \frac{4\tanh 2K_c}{1+\tanh^2 2K_c} = 2\tanh(4K_c), \tag{9}$$

using one more time $\tanh(2t) = \dfrac{2\tanh t}{1+\tanh^2 t}$. Since $K_c \simeq 0.3$, we can read for the graph given in the main text that $\tanh(1.2) \simeq 0.84$. We need to find ν satisfying $2^{1/\nu} \simeq 2 \times 0.84 \simeq 1.68 \simeq 5/3$:

$$\nu \simeq \frac{\log 2}{\log 5/3} \simeq \frac{0.69}{0.51} \simeq \frac{0.69}{0.5(1+2\,10^{-2})} \simeq \frac{0.69}{0.5}(1-2\,10^{-2}) \simeq 1.38 - 2*0.14 \simeq 1.35. \tag{10}$$

To conclude, we have found

$$\boxed{\nu \simeq 1.35 \qquad \text{while} \qquad \nu^{\text{mf}} = \frac{1}{2} \quad \text{and} \quad \nu^{\text{exact}} = 1}. \tag{11}$$

On this count as well, we improve over mean-field.

C- The two-dimensional model / take 2

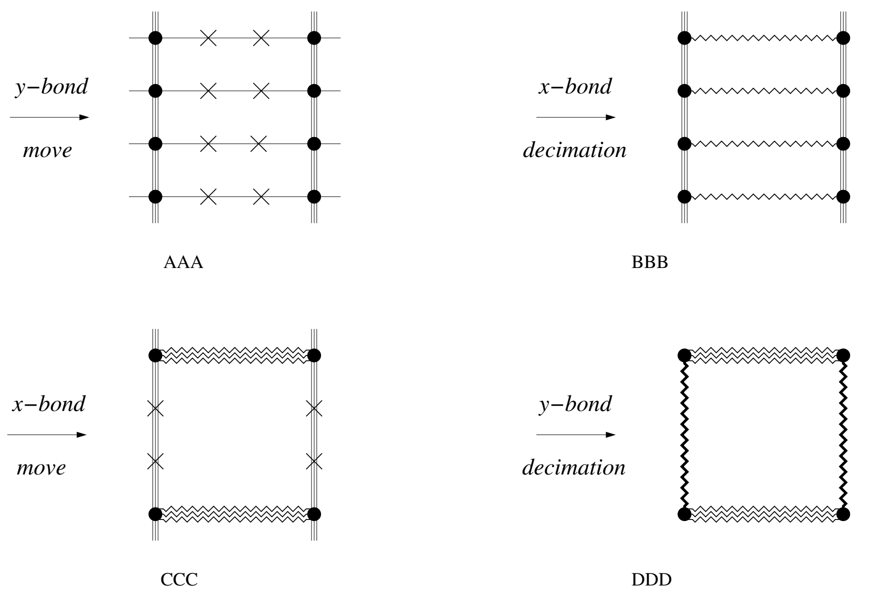

FIGURE C3 – The 4 steps involved in the procedure, with translation in terms of coupling strengths.

11) We decimate $b-1$ bonds. According to the result of section A, this means that

$$\boxed{\tanh \widetilde{K}_x = (\tanh K_x)^b}, \tag{12}$$

while $\widetilde{K}_y = bK_y$.

12) The x-bonds are broken and moved as shown in the figure, which then opens the possibility to decimate the y-bonds. Then, the x-bonds undergo a b-fold increase and in the perpendicular direction, $\tanh K'_y = [\tanh(\widetilde{K}_y)]^b$. The chain of four moves/decimation is addressed in Fig. C3, where the corresponding change affecting the couplings is sequenced. To summarize :

$$\boxed{K'_x = b\ \tanh^{-1}\Big(\tanh^b K_x\Big)\ ; \qquad K'_y = \tanh^{-1}\Big(\tanh(bK_y)\Big)^b}. \tag{13}$$

13) We take $b = 1+\epsilon$; we will need $\dfrac{d\tanh^{-1}(x)}{dx} = \dfrac{1}{1-x^2}$ and $\dfrac{1}{1-\tanh^2} = \cosh^2$.

$$\begin{aligned} K'_x &= (1+\epsilon)\tanh^{-1}\big[(\tanh K_x)^{1+\epsilon}\big] && (14)\\ &= (1+\epsilon)\tanh^{-1}\big[(\tanh K_x)(1+\epsilon\log\tanh K_x)\big] \;+\; \mathcal{O}(\epsilon^2) && (15)\\ &= (1+\epsilon)\left\{\tanh^{-1}(\tanh K_x) \;+\; \epsilon\,\frac{(\tanh K_x)\log\tanh K_x}{1-\tanh^2 K_x}\right\} + \mathcal{O}(\epsilon^2) && (16)\\ &= K_x \;+\; \epsilon\left\{K_x \;+\; \log\tanh K_x\,\frac{\tanh K_x}{1-\tanh^2 K_x}\right\} + \mathcal{O}(\epsilon^2). && (17)\end{aligned}$$

and thus we take, to linear order in ϵ,

$$\begin{aligned} K'_x &= K_x \;+\; \epsilon\big\{K_x \;+\; (\cosh K_x)(\sinh K_x)\,\log\tanh K_x\big\} && (18)\\ K'_x &= \boxed{K_x \;+\; \epsilon\left\{K_x \;+\; \frac{\sinh(2K_x)}{2}\,\log\tanh K_x\right\}}. && (19)\end{aligned}$$

We treat the recursion relation for K'_y similarly,

$$\begin{aligned} K'_y &= \tanh^{-1}\Big\{\big[\;\underbrace{\tanh((K_y+\epsilon K_y)}_{\tanh K_y+\epsilon K_y(1-\tanh^2 K_y)}\;\big]^{1+\epsilon}\Big\} && (20)\\ &= \tanh^{-1}\big\{\tanh K_y + \epsilon K_y(1-\tanh^2 K_y) \;+\; \epsilon\,(\tanh K_y)\,\log\tanh K_y\big\} \;+\; \mathcal{O}(\epsilon^2) && (21)\\ &= K_y \;+\; \epsilon\,\frac{K_y(1-\tanh^2 K_y) \;+\; (\tanh K_y)\,\log\tanh K_y}{1-\tanh^2 K_y}. && (22)\end{aligned}$$

Finally, we arrive at the same relation as (19)

$$\boxed{K'_y \;=\; K_y \;+\; \epsilon\left\{K_y \;+\; \frac{\sinh(2K_y)}{2}\,\log\tanh K_y\right\}}. \qquad (23)$$

14) With the exact critical coupling $K_c^{\text{exact}} = \dfrac{1}{2}\log(1+\sqrt{2})$, one has

$$\exp(2K_c^{\text{exact}}) = 1+\sqrt{2}, \qquad \sinh\big(2K_c^{\text{exact}}\big) = 1, \qquad \tanh K_c^{\text{exact}} \;=\; \frac{1}{1+\sqrt{2}} \qquad (24)$$

which proves the statement : K_c^{exact} is a fixed point of the recursion relation (19).

15) We start with Eq. (8), and we denote $F(K)$ the term in the curly brackets in Eq. (19) :

$$(1+\epsilon)^{1/\nu} \;=\; 1+\frac{\epsilon}{\nu}+\mathcal{O}(\epsilon^2) \;=\; \left.\frac{\partial K'}{\partial K}\right|_{K_c^{\text{exact}}} \;=\; 1+\epsilon\,F'(K_c^{\text{exact}}) \quad\Longrightarrow\quad \frac{1}{\nu} \;=\; F'(K_c^{\text{exact}}). \qquad (25)$$

Then,

$$\begin{aligned} F'(K) &= 1+\cosh(2K)\log\tanh K \;+\; \frac{\sinh(2K)}{2}\,\frac{1-\tanh^2 K}{\tanh K} \;=\; 1+\cosh(2K)\log\tanh K \;+\; 1.\\ \Longrightarrow\quad F'(K_c^{\text{exact}}) &= 2-\sqrt{2}\log(1+\sqrt{2}. && (26)\end{aligned}$$

To leading order in ϵ, the critical exponent ν for the correlation length thus reads

$$\boxed{\nu \;=\; \frac{1}{2-\sqrt{2}\log(1+\sqrt{2})}}. \qquad (27)$$

The numerical value of ν, 1.327, is closer to the exact result than its "take 1" counterpart, but not by an impressive amount. Here, the effort did not spectacularly pay off.

D- Generalization to arbitrary dimension (take 1 route)

16) On a three-dimensional cubic lattice with $b = 2$, an iteration increases length by a factor of 2, so that $N \to N/8$. In between retained sites, and before decimation, there are two bonds. To preserve the total "strength" of the lattice, the bond moving step strengthens each bond by a factor 4 (see Fig. C4, where the 4-fold increase is recovered by a different argument). The "take 1" recursion relation between K and K' is thus

$$\boxed{\tanh K' \;=\; \big[\tanh(4K)\big]^2}. \tag{28}$$

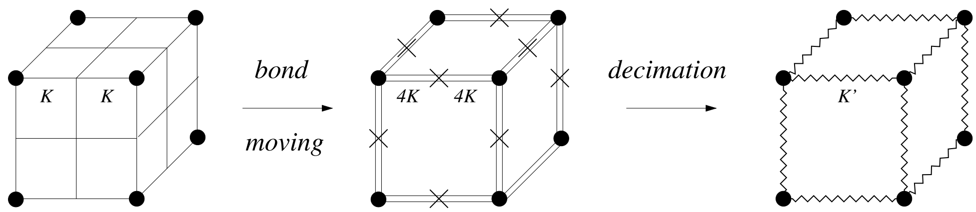

FIGURE C4 – Sketch of the $d = 3$, $b = 2$ procedure. To get the strengthening factor of the bonds, one can focus on "vertical bonds", and count how many there are, before and after bond moving. Before bond-moving, we have $4 \times 1/4 + 4/2 + 1 = 4$ pairs of bonds, where the weights 1/4, 1/2 and 1 come from the fact that these bonds are shared 4 times (along the edges of the cube), 2 times (faces of the cube) or not shared (the central pair of bonds). After bond moving, we have just $4/4 =$ one pair left. This 4-fold decrease in the number of bonds is compensated by a 4-fold increase in the strength. A similar argument applies to bonds in the two other directions (not "vertical").

17) Similarly, for a d-dimensional cubic lattice, N decreases to $N/2^d$ and keeping in mind the factor 2 alluded to in the previous question, we have

$$\boxed{\tanh K' \;=\; \big[\tanh(2^{d-1}K)\big]^2}. \tag{29}$$

18) We start from $K' \;=\; \tanh^{-1}\big[\tanh(2^{d-1}K)\big]^2$ and use that

$$\tanh^{-1}(x) \;=\; \frac{1}{2}\,\log\left(\frac{1+x}{1-x}\right) \qquad \text{together with} \qquad 1 - \tanh^2 = \frac{1}{\cosh^2}. \tag{30}$$

In the large K regime, with $x = \tanh(2^{d-1}K) \to 1$, this yields

$$K' \sim \frac{\log 2}{2} - \frac{1}{2}\,\log(1-x^2) \sim \frac{1}{2}\,\log\big[\cosh^2(2^{d-1}K)\big] \sim 2^{d-1}\,K. \tag{31}$$

Therefore, the small temperature sink is stable for $d > 1$ (and then $K' > K$). This is consistent with the lower critical dimension equal to unity.

19) For arbitrary b and d, we start by noting that $N \to N/b^d$. In between two retained sites, there are b bonds, so that the bond moving step strengthens each bond by a factor b^{d-1}. Decimation then leads to the generalized form

$$\boxed{\tanh K' \;=\; \big[\tanh(b^{d-1}K)\big]^b}. \tag{32}$$

Migdal-Kadanoff scheme gets worse, when compared to known results, when d increases. One can see in particular that for large d, we get $K_c \sim 2^{2-2d}$, which translates into a d-dependent critical exponent ν for all d. There is no sign of an upper critical dimension. Besides, it can be shown that ν approaches 1 for $d \to \infty$, which is not the mean-field result.