M1 General Physics
Year 2021-2022
STATISTICAL PHYSICS
Partial exam 2022

Should be written on a separate paper.
Concise but explicative answers expected throughout. No bonus for verboseness.
Cell phones and calculator forbidden.

1 Basic questions

1) Why is the Ising model of particular interest in statistical physics?

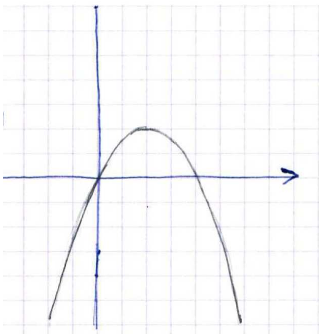
2) What is a liquid crystal?

3) Consider a thin membrane of liquid crystal, where the molecules can only lie within the plane of the membrane. Construct the relevant order parameter for studying the isotropic to nematic transition.

4) What are the competing effects at work in the isotropic to nematic transition of hard rod molecules, such as formed by certain solutions of viruses?

5) Sketch graphically the Legendre transform of the function in the graph below.

6) In order to check the above result, choose a suitable analytical form for a function $f(x)$, the graph of which should correspond with that below, and compute explicitly the Legendre transform. Is this compatible with the answer to the previous question?



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2 Thermodynamics of magnetic substances and Rushbrooke inequality

Using standard notations, we have for a magnetic substance

$$dU = TdS - MdB \tag{1}$$

1) Define the various quantities involved in (1). What name does this relation bear?

2) Define the other functions of state, by appropriate Legendre transformations.

3) Complete the following equalities

$$\left.\frac{\partial T}{\partial B}\right|_S = \left.\frac{\partial ...}{\partial ...}\right|_B \quad ; \quad \left.\frac{\partial S}{\partial B}\right|_T = \left.\frac{\partial M}{\partial T}\right|_B \quad ; \quad \left.\frac{\partial S}{\partial M}\right|_T = \left.\frac{\partial ...}{\partial ...}\right|_M \quad ; \quad \left.\frac{\partial T}{\partial M}\right|_S = \left.\frac{\partial ...}{\partial ...}\right|_M \tag{2}$$

4) Express the specific heats c_B and c_M, respectively at fixed magnetic field and fixed magnetization, as derivatives of the entropy S.

5) Write the definition of the isothermal susceptibility χ.

6) From the identity (to be made explicit)

$$\left.\frac{\partial S}{\partial T}\right|_M = \left.\frac{\partial S}{\partial T}\right|_B + \left.\frac{\partial S}{\partial B}\right|_T \left.\frac{\partial ...}{\partial ...}\right|_M, \tag{3}$$

deduce that

$$\chi(c_B - c_M) \;=\; -T\left.\frac{\partial M}{\partial B}\right|_T \left.\frac{\partial M}{\partial T}\right|_B \left.\frac{\partial ...}{\partial ...}\right|_M, \tag{4}$$

where the right hand side should be written explicitly.

7) Making use of the fact that

$$\left.\frac{\partial x}{\partial y}\right|_z \left.\frac{\partial y}{\partial z}\right|_x \left.\frac{\partial z}{\partial x}\right|_y \;=\; -1, \tag{5}$$

show that

$$\chi(c_B - c_M) \;=\; T\left(\left.\frac{\partial \ldots}{\partial T}\right|_B\right)^2. \tag{6}$$

8) For $T \to T_c$, $T < T_c$ and $B = 0$, express c_B, M, and χ as functions of $T_c - T$, and the critical exponents α, β and γ.

9) Using that $c_M > 0$, $\chi > 0$ and relation (6), show the Rushbrooke inequality $\alpha + a\beta + \gamma \geqslant 2$. What is a?

10) Onsager's solution for the 2d Ising model yields $\alpha = 0$, $\beta = 1/8$ and $\gamma = 7/4$. Comment. In 3 dimensions, we get $\alpha \simeq 0.1$, $\beta \simeq 0.33$ and $\gamma \simeq 1.24$. What do you notice?

11) Why do critical exponents differ in two and three dimensions?

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3 Back to Onsager's exact solution for Ising model

For the 2d Ising model, Onsager showed that the magnetization on the square lattice reads

$$M^8 \;=\; 1 - \sinh^{-4}\left(\frac{2J}{kT}\right) \tag{7}$$

at vanishing magnetic field.

1) What does J stand for here?

2) Compute the critical temperature T_c.

3) Which critical exponent(s) does this give access to? What is its/their value?