Should be written on a separate paper.

Electronic calculators, cell phones, and documents of all kinds are not allowed, except for lecture notes. Concise but explicative answers are expected throughout ; no bonus for verboseness. Write as neatly as possible and always specify clearly which question you are answering. No questions will be answered during the exam, for the sake of quietness and equity: if you detect what you believe to be an error or an inconsistency, explain so in your answer, and move on; you are also judged on your ability to understand the questions raised.

A Basic questions

1) In which sense is the study of specific heats relevant for phase transitions?

2) What are the mechanisms explaining phase transitions in the Ising model? Same question for hard rods exhibiting an isotropic to nematic transition.

3) We consider a Gaussian random variable with mean 2 and standard deviation 1. Write the probability density function, then compute $\langle X \rangle$, $\langle X^2 \rangle$, $\langle X^3 \rangle$ and $\langle e^X \rangle$.

B Transfer matrix formalism

Consider a modified Ising model in $d = 1$ dimension:

$$H(\{S_i\}) = -J \sum_{i=1}^{N} S_i S_{i+1} - h \sum_{i=1}^{N} S_i.$$

Assume periodic boundary conditions, $S_1 \equiv S_{N+1}$. Every spin can here take three possible values, $S_i = -1, 0, 1$. The partition function of the system reads:

$$Z_N = \sum_{\mathcal{C}} e^{-\beta H(\{S_i\})}$$

where $\mathcal{C}$ refers to all possible microscopic spin configurations.

1) On which parameters does the partition function depend?

2) How many terms are being summed in the partition sum above?

3) By analogy with the more familiar case where $S_i = -1, 1$ write the partition function of the system using the transfer matrix method (you can use the shorthand $K \equiv \beta J$ and $H \equiv \beta h$). Write explicitly the transfer matrix. To this end, you can recast

$$\sum_{i=1}^{N} S_i \qquad \text{as} \qquad \frac{1}{2} \sum_{i=1}^{N} (S_i + S_{i+1}).$$

Why is that so?

4) Find the expression of the free energy density in the thermodynamic limit:

$$f = \lim_{N \to \infty} \frac{1}{N} F_N$$

where F_N is the free energy of the N-spin system. Do not attempt at diagonalizing the transfer matrix.

5) We consider a variant of the problem with the same 3-state spins, but a modified Hamiltonian in which only like spins (ie spins of the same value) interact (with energy $-J$), while unlike spin interactions yield a 0 energy. Redo the analysis for $h = 0$, with explicit calculations throughout.