Electronic calculators, cell phones, and documents of all kinds are not allowed. Concise but explicative answers are expected throughout ; no bonus for verboseness. Write as neatly as possible and always specify clearly which question you are answering. No question will be answered during the exam, for the sake of quietness and equity: if you detect what you believe to be an error or an inconsistency in the assignment, explain so in your answer, and move on; you are also judged on your ability to understand the questions raised. The different sections are independent.

A Exercises

1) Minimizing a cumulant. We consider here a random variable X such that X and $-X$ are equally probable.

- **a)** Write the definitions of the moment and the cumulant generating functions for X.
- **b)** Obtain the expression of the fourth cumulant c_4 in terms of the relevant moments of X. One can use that for $|\epsilon| \ll 1$, $\log(1+\epsilon) = \epsilon - \epsilon^2/2 + \mathcal{O}(\epsilon^3)$.
- **c)** We define the rescaled cumulant as

$$\widetilde{c}_4 = \frac{c_4}{\left\langle (X - \langle X \rangle)^2 \right\rangle^2}. \tag{1}$$

  What is the minimal possible value for $\widetilde{c}_4$? Provide an explicit probability distribution realizing this minimum.

2) Let us consider the discrete time dynamics where a 1D random walker makes a jump η_i at step i. At time n, the walker's position is $x_n = \eta_1 + \eta_2 + \ldots + \eta_n$. Each jump η has a vanishing mean ($\langle \eta \rangle = 0$), and is broadly distributed with a Pareto index $\mu < 2$ at large $|\eta|$. Is the random walk sub-diffusive, diffusive, or super-diffusive? No lengthy calculation expected. What if $\langle \eta \rangle \neq 0$?

3) Discuss concisely an example of a sub-diffusive process.

B Itō-Doblin and Stratonovich for additive and multiplicative noises

1) For a Langevin equation with additive noise, such as

$$\frac{dx}{dt} = \mu F(x) + \sqrt{2D}\,\eta(t) \qquad \text{with} \qquad \langle \eta(t) \rangle = 0 \quad \text{and} \quad \langle \eta(t)\eta(t+\tau) \rangle = \delta(\tau), \tag{2}$$

and an arbitrary function $\varphi(x)$, check the consistency of Itō-Doblin and Stratonovich calculus for computing

$$\left\langle \frac{d\varphi(x(t))}{dt} \right\rangle. \tag{3}$$

2) What about when the diffusion coefficient D is x-dependent?

C Large deviations for a lattice random walk

Consider a one-dimensional random walk on $\mathbb{Z}$. At each discrete time step i, the walker's position changes by a jump η_i which can take three possible values: $x_i = x_{i-1} + \eta_i$, with

$$\eta_i = \begin{cases} -1 & \text{with probability} \quad 1/3 \\ \;\;\,0 & \text{with probability} \quad 1/3 \\ \;\;\,1 & \text{with probability} \quad 1/3 \end{cases}. \tag{4}$$

The walker starts at $x_0 = 0$ and the jumps are independent from step to step.

1) Compute the mean value $\langle x_n \rangle$ of the position and its variance at step n.

2) At large times n, how is x_n distributed (we restrict here to the typical fluctuations)?

3) How far does the previous "typical" behaviour extend, for the position distribution $P(x, n)$?

4) We are interested in the large deviations of $P(x, n)$, in the scaling limit where $n \to \infty$ and $x \to \infty$ with $z = x/n$ fixed. The corresponding rate function is denoted $\phi(z)$.

a) What is the connection between P and ϕ?

b) Without calculations, provide the values of $\phi(-1)$, $\phi(0)$, $\phi(1)$. Explain.

c) Calculate the rate function using the Gärtner-Ellis approach, where $\phi(z)$ is expressed as the Legendre transform of the large n limit of the scaled cumulant generating function

$$\kappa_n(t) = \frac{1}{n} \log \left\langle e^{t x_n} \right\rangle . \tag{5}$$

Does $\kappa_n(t)$ depend on n; why? Sketch the graph of $\phi(z)$. In which range does z vary? Does ϕ feature specific symmetry properties?

d) Show that the "typical" behaviour of question 2 is precisely recovered.

e) If one wants to check ϕ obtained using Sanov theorem, what are the main steps involved (no calculation asked)?

5) In which sense can the large deviation function $\phi(z)$ be related to the entropy of a physical system? Define explicitly the system in question.

D Constrained stochastic processes and effective Langevin equations

Among all possible trajectories of a Brownian motion that emerge from $X = 0$ at time $t = 0$, our interest goes to those that come back to $X = 0$ at time $t_f > 0$, where t_f is fixed; these are called *bridges*, with the position denoted by $B(t)$. A naive algorithm to create these bridge trajectories would be to generate a large number of Brownian paths for $0 \le t \le t_f$ and retain only those that do return to the origin at t_f. This would be both inefficient, and lead to incorrect statistics for any finite sample.

1) On intuitive grounds, sketch a graph of the variance of $B(t)$ versus time t for $0 \le t \le t_f$.

We propose here a more powerful approach, where $B(t)$ is the solution to an effective Langevin equation, that automatically takes into account the global constraint. Besides, the approach generalizes to the generation of other stochastic processes, such as *excursions* where X remains positive at all times, before going back to 0 at t_f, see Fig. 1.

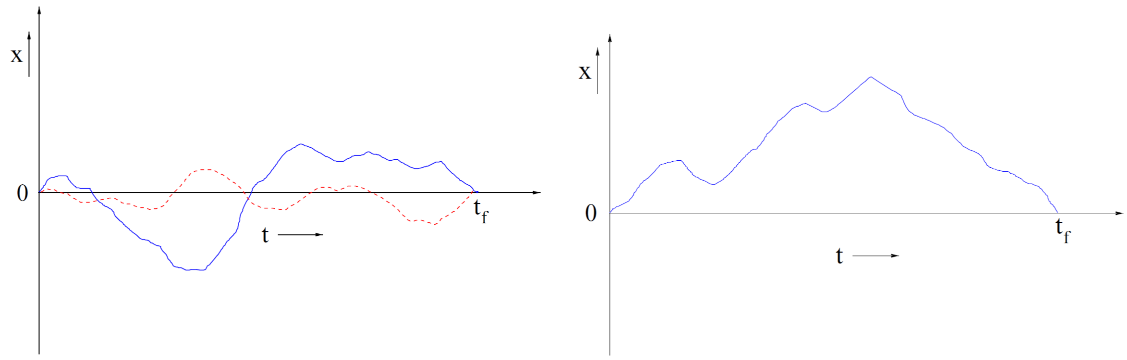

Figure 1: From [1]. Left: sketch of two Brownian bridges starting at the origin, in the time interval $0 \le t \le t_f$. Right: a typical excursion, with a positive position during the whole time window of interest.

The idea is illustrated in the one-dimensional setting of the overdamped Langevin equation

$$\frac{dx}{dt} = \frac{1}{m\gamma} F(x(t), t) + \sqrt{2D}\, \xi(t) \tag{6}$$

where x denotes the position of the Brownian object driven by an external force $F(x,t)$, m its mass, D the diffusion coefficient and $\xi(t)$ is the standard Gaussian white noise.

2) What is the physical dimension of the (positive) quantity γ? How is it related to the other quantities, including the temperature T of the fluid in which the Brownian particle (no demonstration asked)? How is the latter relation called?

3) What is the Fokker-Planck equation fulfilled by $P(x,t|x_0,t_0)$, the probability density that the particle be at point x at time t? We are here interested in the forward dynamics, involving derivatives with respect to x and t.

We introduce $\widetilde{\mathcal{P}}(x,t)$, the probability density over all possible paths starting at $x_0 = 0$ at time 0 and ending at position $x_\mathrm{f} = 0$ at time t_f, to find the particle at point x at time t.

4) Show that

$$\widetilde{\mathcal{P}}(x,t) \;=\; \mathcal{N}\, P(0,t_\mathrm{f}|x,t)\, P(x,t|0,0). \tag{7}$$

What is the expression of $\mathcal{N}$?

5) For the sake of convenience, we use the notation $Q(x,t) = P(0,t_\mathrm{f}|x,t)$. Write the Fokker-Planck equation obeyed by Q.

6) Knowing both the equations obeyed by P and Q, show that the equation ruling the dynamics of $\widetilde{\mathcal{P}}(x,t)$ reads

$$\partial_t \widetilde{\mathcal{P}}(x,t) \;=\; -D\,\partial_x \left[\left(\beta F(x,t) + \alpha \partial_x \log Q\right) \widetilde{\mathcal{P}} \right] \;+\; D\partial_x^2 \widetilde{\mathcal{P}}, \tag{8}$$

where α is a coefficient to be specified.

7) Show that the above Fokker-Planck equation corresponds to a Langevin dynamics with an additional potential proportional to $-\log Q(x,t)$, on top of the external force F. Discuss physically the minus sign in front of the log. We refer to this equation as the conditioned Langevin process.

It is the additional force in $\partial_x \log Q$ that guarantees proper sampling, i.e. that the trajectories starting at $(x=0, t=0)$ and ending at $(0, t_\mathrm{f})$ are statistically unbiased.

8) **Application to the free Brownian bridge**.

- **a)** When $F = 0$ (free Brownian motion), what is the explicit form of $Q(x,t)$?
- **b)** What is then the conditioned Langevin equation, for generating bridges? Discuss its content on physical grounds.

9) **Application to the free Brownian excursion**.
Here, the situation is more subtle and requires to consider first a final point $x_\mathrm{f} > 0$, with all trajectories remaining in the half-space $x > 0$. We thus take $Q(x,t) = P(x_\mathrm{f}, t_\mathrm{f}|x,t)$ and again $F = 0$. It can be shown that the propagator Q now obeys

$$Q(x,t) \;=\; \frac{1}{\sqrt{4\pi D(t_\mathrm{f}-t)}} \left[\exp\left(-\frac{(x_\mathrm{f}-x)^2}{4D(t_\mathrm{f}-t)} \right) - \exp\left(-\frac{(x_\mathrm{f}+x)^2}{4D(t_\mathrm{f}-t)} \right) \right]. \tag{9}$$

(Bonus question: why is that so?)

- **a)** Taking the limit $x_\mathrm{f} \to 0$ with the above expression for Q, write the constrained Langevin equation leading to excursions. What is the effective potential appearing? Discuss.
- **b)** Compare the two situations, bridge versus excursion.

10) **Application to the free Brownian meander**.
We introduce a last category of constraint, where it is demanded that the trajectory starting from the fixed position $x(0) = 0$ remain positive at all times, without a requirement on the end point $x(t_\mathrm{f})$. What is then Q for free motion (i.e. for $F = 0$)? Express the result in terms of the error function

$$\mathrm{erf}(x) \;=\; \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \tag{10}$$

Which constrained Langevin dynamics does this give rise to?

$\longrightarrow$ [1] Going further: S.N. Majumdar and H. Orland, J. Stat. Mech. P06039 (2015).