Ising correlation function in one dimension

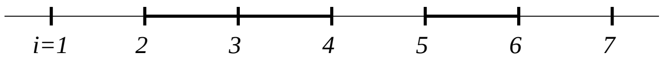
We aim at computing the correlation function $\langle S_i S_j \rangle$ for the one dimensional ferromagnetic Ising model ($J > 0$), without a magnetic field. We shall assume free boundary conditions. Noting that

$$e^{K S_i S_{i+1}} \;=\; \cosh(K) \;+\; S_i S_{i+1}\, \sinh(K), \tag{1}$$

the N spin partition function can be written

$$Z \;=\; (\cosh \beta J)^{N-1} \sum_{\{S_i\}_{1\leq i \leq N}} \prod_{i=1}^{N-1} \left[1 + (\tanh \beta J) S_i S_{i+1}\right]. \tag{2}$$

It is then convenient to associate a graph to each term in the expansion of the product. For instance, for the term $(\tanh \beta J)^3 (S_2 S_3)(S_3 S_4)(S_5 S_6)$:



where a thick segment joins nearest neighbors that are present in the term under consideration. Under which condition does a graph provide a non-vanishing contribution to the partition function? Compute Z. Using similar arguments, show that the correlation function reads

$$\langle S_i S_j \rangle \;=\; \exp(-|i-j|/\xi), \tag{3}$$

where ξ is the correlation length (provide its expression). Show that $\xi \to \infty$ in the limit $T \to 0$. Of which phenomenon is this the signature? What happens with periodic boundaries?

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