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Condensed Matter, Quantum Physics and Soft Matter M2 programs

exam 2017-2018

Circling again with Lee and Yang - short correction

1) There are 8 microscopic states, with energies $-3J-3B$ (all spins aligned with B), $J-B$ (1 spin anti-aligned with B, 3-fold degenerate), $J+B$ (also 3-fold), and $3J+3B$.

a)

$$Z_3(T,B) \;=\; e^{3K+3\beta B} + 3\,e^{-K+\beta B} + 3\,e^{-K-\beta B} + e^{3K-3\beta B}$$

b)

$$\exp(-3\beta B)\,Z_3(T,B) \;=\; e^{3K} + 3\,e^{-K-2\beta B} + 3\,e^{-K-4\beta B} + e^{3K-6\beta B}$$

$$\Longrightarrow \quad \boxed{\exp(-3\beta B)\,Z_3(T,B) \;=\; e^{3K} \,+\, 3\,e^{-K}\,z \,+\, 3\,e^{-K}\,z^2 + e^{3K}\,z^3} \tag{1}$$

c) There is one obvious root, $z_1=-1$. The polynomial has real coefficients, and the two other roots, z_2 and z_3 have to be complex conjugate. The product of the three roots is -1, so that $z_2z_3=1$. Hence $|z_1|^2=|z_2|^2=|z_3|^2=1$ and the three roots are on the unit circle.

2) Since $\boxed{z=e^{-2\beta B}}$, the smallest power corresponds to all spins aligned with B, with a contribution in $\exp(N\beta B)=z^{-N/2}$. Likewise, the largest power is $\exp(-N\beta B)=z^{N/2}$. Hence $P_N(z)=Z_N\exp(-N\beta B)=z^{N/2}Z_N$ is a polynomial of degree N in z.

3) Up to an unknown function of $u(T)$, P_N reads

$$\boxed{P_N(z) \;=\; u(T)\prod_{j=1}^{N}(z-z_j) \;=\; u(T)\prod_{j=1}^{N}(z-e^{-2i\beta b_j})}. \tag{2}$$

4) We have

$$Z_N \;=\; e^{N\beta B}\,u(T)\,\prod_{j=1}^{N}(e^{-2\beta B}-e^{-2i\beta b_j}) \tag{3}$$

from which the magnetization follows as

$$\begin{aligned} m(T,B) \;&=\; \frac{1}{N}\frac{\partial\log Z}{\partial\beta B} \;=\; 1+\frac{1}{N}\sum_{j=1}^{N}\frac{-2\,e^{-2\beta B}}{e^{-2\beta B}-e^{-2i\beta b_j}} \;=\; 1-\frac{2\,z}{N}\sum_{j=1}^{N}\frac{1}{z-e^{-2i\beta b_j}} \\ &=\; -\frac{1}{N}\sum_{j=1}^{N}\frac{e^{-2\beta B}+e^{-2i\beta b_j}}{e^{-2\beta B}-e^{-2i\beta b_j}} \;=\; \frac{1}{N}\sum_{j=1}^{N}\frac{e^{-\beta B+i\beta b_j}\,+\,e^{\beta B-i\beta b_j}}{-e^{-\beta B+i\beta b_j}\,+\,e^{\beta B-i\beta b_j}} \end{aligned} \tag{4}$$

5) Taking the large N limit, we get

$$\boxed{m(T,B) \;=\; \int\rho(b)\,\frac{1}{\tanh[\beta(B-ib)]}\,db} \tag{5}$$

6) The fluctuation-response connection states that $\chi\,kT \;=\; \sum_{k,l}\Gamma(\mathbf{r}_k-\mathbf{r}_l) \;=\; N\int\Gamma(r)\,d\mathbf{r}$ in a homogeneous system. The compressibility behaves like $\delta B^{1/\delta'-1}$ where $\delta B=B-i\,b_{\min}$, and we have to pay more attention for the scaling of $\int\Gamma$. While it is wrong to write that the asymptotic behaviour of Γ is given by $\exp(-r/\xi)/r^{d-2+\eta'}$, we can nevertheless state that

$$\Gamma(\mathbf{r}) \sim \frac{1}{\xi^{d-2+\eta'}}\,F\left(\frac{r}{\xi}\right) \qquad \text{for } r\to\infty \tag{6}$$

where F is some scaling function, decaying exponentially fast to zero at infinity. Hence,

$$\int \Gamma(r)\, d\mathbf{r} \;=\; \int \frac{1}{\xi^{\eta'-2}}\, F\left(\frac{r}{\xi}\right) \frac{d\mathbf{r}}{\xi^d} \;\propto\; \xi^{2-\eta'} \;\propto\; \delta B^{-\nu'(2-\eta')} \quad\Longrightarrow\quad \boxed{1-\frac{1}{\delta'} \;=\; \nu'(2-\eta')} \tag{7}$$

7) We are interested in the magnetization $\mathcal{M}$ inside a coherence volume $\mathcal{C}$, say a ball of radius ξ. For the sake of counting the corresponding number of spins, we introduce the lattice spacing a, and write

$$\mathcal{M} \;=\; \frac{\sum_{k\in\mathcal{C}} S_k}{\#\text{ spins in }\mathcal{C}} \;=\; \frac{a^d}{\xi^d} \sum_{k\,\in\,\mathcal{C}} S_k. \tag{8}$$

The statement is that scaling-wise, $\langle\mathcal{M}^2\rangle - \langle\mathcal{M}\rangle^2 \propto (\delta m)^2 \;\propto\; (\delta B)^{2/\delta'}$ and we are left with finding the behaviour of the left hand side. This is possible with the help of Γ. We start from

$$\Gamma_{kl} \;\equiv\; \langle S_k S_l\rangle - \langle S_k\rangle\langle S_l\rangle \;=\; \langle S_k S_l\rangle - \langle S\rangle^2 \tag{9}$$

and $$\left\langle(\mathcal{M}-\langle\mathcal{M}\rangle)^2\right\rangle \;=\; \frac{a^{2d}}{\xi^{2d}} \left\langle \sum_{k,l\,\in\,\mathcal{C}} (S_k-\langle S\rangle)(S_l-\langle S\rangle)\right\rangle \;=\; \frac{a^{2d}}{\xi^{2d}} \sum_{k,l\,\in\,\mathcal{C}} \Gamma_{kl} \tag{10}$$

which yields, remembering the analysis of the previous question for the last relation :

$$\left\langle(\mathcal{M}-\langle\mathcal{M}\rangle)^2\right\rangle \;\propto\; \frac{a^d}{\xi^d} \int_{\mathcal{C}} \Gamma(r)\, d\mathbf{r} \;\propto\; \xi^{-d}\,\xi^{2-\eta'}. \tag{11}$$

Finally

$$\boxed{\frac{1}{\delta'} \;=\; \frac{\nu'}{2}\,\left(d-2+\eta'\right)}. \tag{12}$$

Unlike the previous scaling relation, this one only holds below the upper critical dimension. When mean-field exponents become exact, the fluctuations are dominated :

$$\langle\mathcal{M}^2\rangle - \langle\mathcal{M}\rangle^2 \ll (\delta m)^2. \tag{13}$$

8) From

$$m(T,i\widetilde{b}) - m(T,ib_{\min}) \;=\; \int \rho(b)\left\{\frac{1}{\tanh[\beta(i\widetilde{b}-ib)]} \;-\; \frac{1}{\tanh[\beta(ib_{\min}-ib)]}\right\} db, \tag{14}$$

we perform the suggested expansion :

$$m(T,i\widetilde{b}) - m(T,ib_{\min}) \;\sim\; kT \int \rho(b)\left\{\frac{1}{i\widetilde{b}-ib} \;-\; \frac{1}{ib_{\min}-ib}\right\} db \;\sim\; -ikT \int \rho(b) \frac{b_{\min}-\widetilde{b}}{(\widetilde{b}-b)(b_{\min}-b)} db$$

We anticipate that $\delta m \gg b_{\min}-\widetilde{b}$, i.e. $\sigma < 1$, and we are interested in the singular behaviour, ruled by the vicinity of $b_{\min}$. With $x = (b-b_{\min})/\delta b$ where $\delta b = b_{\min}-\widetilde{b} > 0$, and $\rho(b) \propto (b-b_{\min})^\sigma$, the integral[1] becomes

$$m(T,i\widetilde{b}) - m(T,ib_{\min}) \;\sim\; -ikT \int_0^{\cdots} (x\,\delta b)^\sigma \frac{\delta b}{\delta b(x+1)(x\delta b)}\, \delta b\; dx \;\propto\; (\delta b)^\sigma \quad\Rightarrow\quad \boxed{\delta' = \frac{1}{\sigma}}. \tag{15}$$

1. the integral is convergent for $\sigma > 0$. Yet, in $d = 1$ and $d = 2$, σ takes values -1/2 and -1/6 respectively. We would have to work a bit more to take due account of these cases. Note that the integral with the tanh, defining $m(T, ib_{\min})$ also diverges for $\sigma < 0$.

9) Gathering results :

$$\boxed{\sigma = \frac{1}{\delta'} = \frac{d-2+\eta'}{d+2-\eta'}} \qquad \text{and} \qquad \boxed{\nu' = \frac{2}{d+2-\eta'}}. \tag{16}$$

This is indeed compatible with the Figure. For instance, we read for $d=1$ that $\eta' \simeq -1$, $\delta' \simeq -2$, which fulfil the scaling relation above. Same remark for $d=6$ (the upper critical dimension), where $\eta' \simeq 0$, $\delta' \simeq 2$.

10) The standard mean-field argument seen in class applies :

$$\boxed{m = \tanh\left[\beta\left(B + cJm\right)\right]} \qquad \text{and} \qquad \boxed{kT_c = cJ}. \tag{17}$$

11) Differentiating Eq. (17) with respect to B :

$$dm = (1-\tanh^2)\left[\beta\left(dB + cJdm\right)\right] \qquad \Longrightarrow \qquad \boxed{\chi kT = \frac{1-m^2}{1-\beta J\, c\,(1-m^2)}}. \tag{18}$$

12) At the edge, the susceptibility diverges, so that

$$\beta J\, c\,(1-m^2) = 1 \qquad \Longrightarrow \qquad \boxed{1 - m(T, ib_{\min})^2 = \frac{T}{T_c}}. \tag{19}$$

Let $t = (T-T_c)/T_c > 0$. We have $m(T, ib_{\min})^2 = -t$, i.e. $m(T, ib_{\min}) = i\sqrt{-t}$. We then use (17), assuming $b_{\min}$ small,

$$m \sim \beta\left(ib_{\min} + cJm\right) - \frac{\beta^3}{3}\left(ib_{\min} + cJm\right)^3 \sim \beta\left(ib_{\min} + cJm\right) - \frac{1}{3}(\beta cJm)^3 \tag{20}$$

$$\Longrightarrow \qquad \beta i b_{\min} \sim m(1-\beta cJ) + \frac{1}{3}\left(\frac{T_c}{T}\right)^3 m^3 \sim m\left(1 - \frac{T_c}{T}\right) - \frac{1}{3}mt \sim \frac{2}{3}m\,t \tag{21}$$

Hence, $\boxed{b_{\min} \propto t^{3/2}}$ for small $t > 0$.

13) To obtain δ', computing first order variations like in Eq. (18) is not sufficient (since the susceptibility diverges at $B = ib_{\min}$). This divergence of χ is indicative of the fact the variations of m and B no longer are of the same order, but such that δB becomes infinitesimally smaller than δm. This hint will be useful below. We thus Taylor expand the equation of state (17) one order higher, to get

$$\delta m = (1-\tanh^2)\,\delta\left(\beta B + \beta cJm\right) + \frac{1}{2}(-2)(\tanh)(1-\tanh^2)\,\left[\delta\left(\beta B + \beta cJm\right)\right]^2 \tag{22}$$

Keeping (19) in mind, $\delta B \ll \delta m$, and simplifying by $1 - \tanh^2 = 1 - m^2$, we arrive at

$$\beta\delta B \sim m\left[\delta\left(\beta cJm\right)\right]^2 \qquad \Longrightarrow \qquad \boxed{\delta' = 2}. \tag{23}$$

It is noteworthy that this is indeed what the graph provided in the text is telling us, for $d \geqslant 6$ (6 being the upper critical dimension).

14) The approach makes sense for $T > T_c$.

15) We would need to account for a space-dependent magnetic field B_j and start from

$$m_k = \tanh\left[\beta B_k + J\sum_{\ell} c_{k\ell}\, m_\ell\right] \tag{24}$$

where $c_{j\ell}$ is the adjacency matrix, such that $c_{j\ell} = 1$ if sites j and ℓ are nearest neighbors. Its Fourier transform

$$\widehat{c}(\mathbf{k}) \;=\; \sum_{\ell} c_{j\ell}\, e^{i\mathbf{k}\cdot(\mathbf{r}_j - \mathbf{r}_\ell)} \tag{25}$$

is independent from j due to translational invariance, and defined for the discrete $\mathbf{k}$-vectors compatible with the periodicity of the lattice. The correlation function follows from

$$\Gamma_{k\ell} \;\propto\; -\frac{\partial^2 F}{\partial B_k\,\partial B_\ell} \;=\; \frac{\partial m_k}{\partial B_\ell}, \tag{26}$$

a facet of the fluctuation-response connection. For technical reasons, it is more convenient to compute $\frac{\partial B_k}{\partial m_\ell}$, which yields the inverse matrix element $\Gamma^{-1}_{k\ell}$. From (24) :

$$\delta_{kj} \;=\; (1 - m_k^2)\left[\beta\frac{\partial B_k}{\partial m_j} + \beta J c_{kj}\right] \qquad \Rightarrow \qquad \Gamma^{-1}_{kj} \;\propto\; \frac{1}{1-m^2}\,\delta_{kj} - \beta J c_{kj}, \tag{27}$$

where after having taken the derivative, we have restricted to a homogeneous system with order parameter m. Hence, in Fourier-space :

$$\frac{1}{\widehat{\Gamma}(\mathbf{k})} \;=\; \frac{1}{1-m^2} \;-\; \beta J\;\widehat{c}(\mathbf{k}). \tag{28}$$

To get the large distance behaviour of Γ, we expand $\widehat{c}(\mathbf{k}) = c(1 - k^2a^2) + \mathcal{O}(k^4)$, where a is some microscopic length, and we finally arrive at

$$\widehat{\Gamma}(\mathbf{k}) \;\propto\; \frac{1}{\xi^{-2} + k^2} \qquad \text{with} \qquad \xi^{-2} \;\propto\; \frac{kT}{1-m^2} \;-\; Jc \;\propto\; \frac{T}{1-m^2} \;-\; T_c, \tag{29}$$

a quantity that vanishes when $m = m(T, ib_{\min})$. The exponent ν' is defined as $\xi \propto (\delta B)^{-\nu'}$ and we thus use $m^2 - m^2(T, ib_{\min}) \propto \delta m \propto (\delta B)^{1/2}$ to conclude that $\xi^{-2} \propto (\delta B)^{1/2}$. Hence, $\boxed{\nu' = 1/4}$.

16) The transfer matrix technique does the job, see the course.

17) The formula tells us that $\boxed{\sin(\beta b_{\min}) = \exp(-2K)}$. $b_{\min}$ cannot vanish (unless $K \to \infty \Leftrightarrow T \to 0$), and we recover that there is no phase transition in one dimension, with short range interactions.

18) In the vicinity of the edge : $\rho(b) \propto 1/\sqrt{b - b_{\min}}$, meaning that $\boxed{\sigma = -1/2}$.

19) We have by definition one zero in each interval of length Δb : $\boxed{\rho(b) \propto 1/\Delta b}$.

20) The slope in the graph is $2/4 = 1/2$. Thus, $\Delta b \propto (b - b_{\min})^{1/2}$ and we recover $\sigma = -1/2$.

21) One should approach the limit either from above, or from below :

$$\lim_{B\to 0^+} m(T,B) \;=\; m_s(T) \;=\; -\lim_{B\to 0^-} m(T,B) \tag{30}$$

22) From

$$m(T,B) \;=\; \int_0^{\pi/2} \rho(b)\left\{\frac{1}{\tanh[\beta(B - ib)]} \;+\; \frac{1}{\tanh[\beta(B + ib)]}\right\} db, \tag{31}$$

we see that for $B \to 0$, tanh being an odd function, the integral is dominated by the vicinity of $b = 0$. It is then legitimate to Taylor expand the two tanh to leading order in their (small) arguments :

$$m(T,B) \;\sim\; \int_0^{\pi/2} \rho(b)\left\{\frac{1}{\beta(B - ib)} \;+\; \frac{1}{\beta(B + ib)}\right\} db \;\sim\; kT\,2B \int_0^{\pi/2} \rho(b)\,\frac{1}{B^2 + b^2}\,db. \tag{32}$$

At this point, we recover that the sign of B is essential. Indeed, we take $x = b/B$:

$$m(T,B) \;\sim\; 2kT \int_0^{\pi/(2B)} \frac{\rho(Bx)}{1+x^2}\,dx \;\sim\; 2kT\,\rho(0) \int_0^{\pi/(2B)} \frac{1}{1+x^2}\,dx \;\sim\; kT\,\rho(0)\,\mathrm{sign}(B)\pi \tag{33}$$

so that

$$\boxed{m_s(T) \;=\; \pi\, kT\,\rho(0)}. \tag{34}$$