Introduction to Mean Field Games (and applications)

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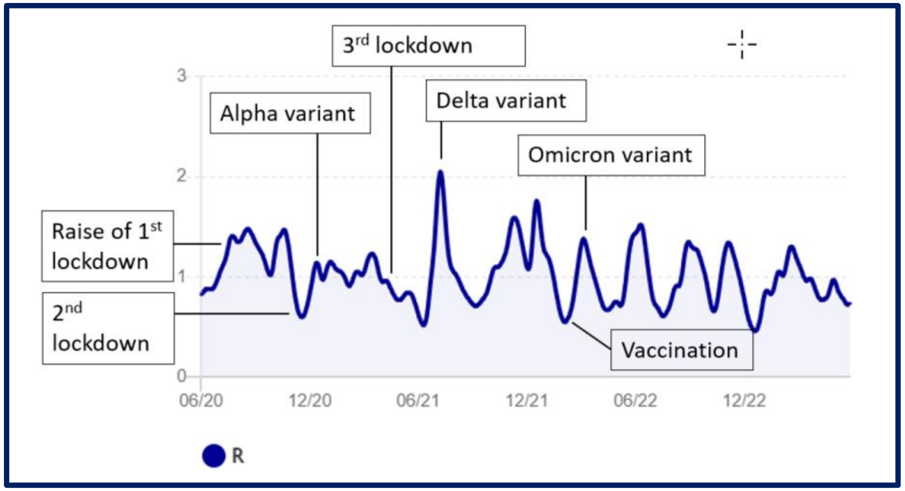
[Nicolas et al. Scientific Reports **9**, 105 (2019)]



[Data from "Santé Publique France", author: Guillaume Rozier [https://covidtracker.fr].)

Introduction

The three levels of pedestrian dynamics

*[Hoogendoorn and Bovy, Transportation Research Part B: **38**, 169 (2004)]*

- 1: Departure time choice, and activity pattern choice (strategic level)
- 2: Activity scheduling, activity area choice, and route-choice to reach activity areas (tactical level);
- 3: Walking behavior (operational level)

Operational level : how they will move along that route in response to interactions with other people

[Abhishek Atre · Jun 29, 2015 (https://www.youtube.com/watch?v=95wrgAvV474)]

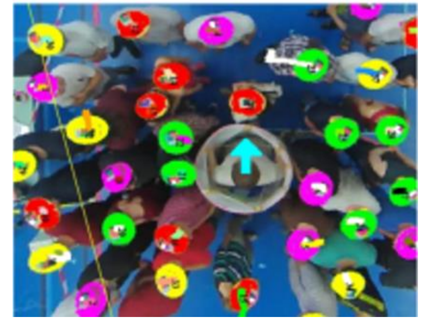
Crossing a static crowd

[Nicolas et al. Scientific Reports ***9****, 105 (2019)]*

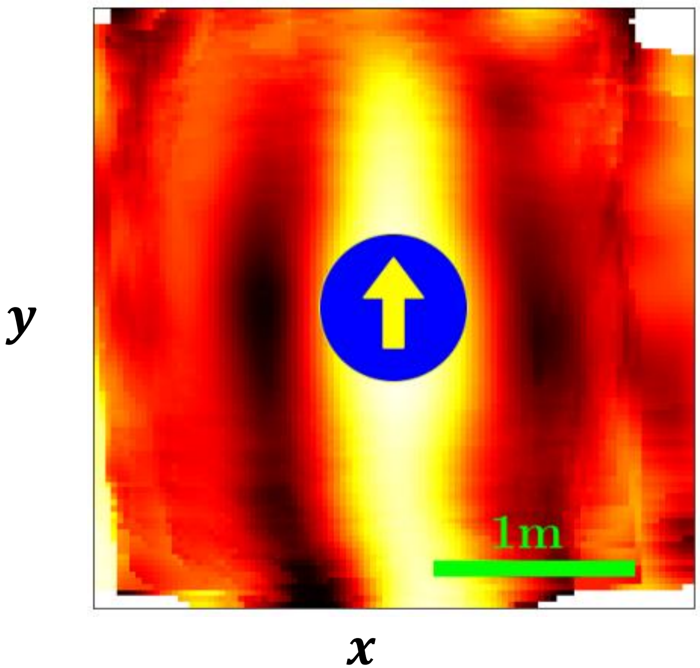
Experimental data

$m_0 = 2.5$ pedestrian/m²

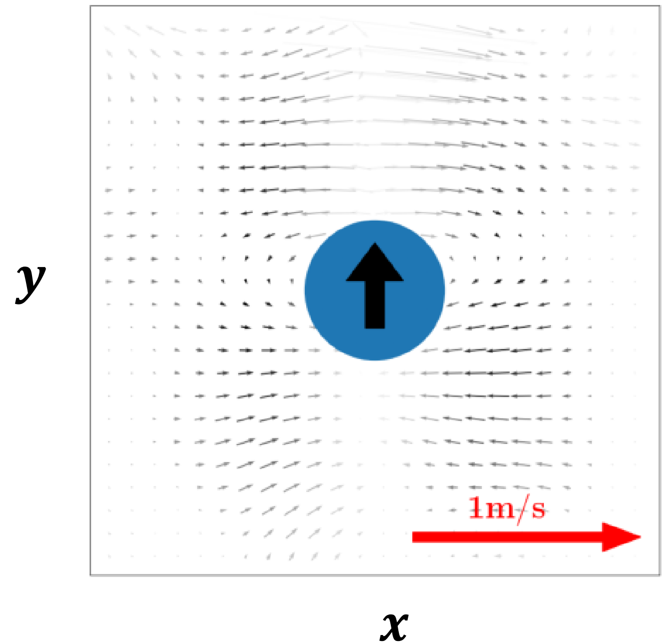
$v_y = 0.6$ m/s $R = 0.32$m



density plot



velocity plot



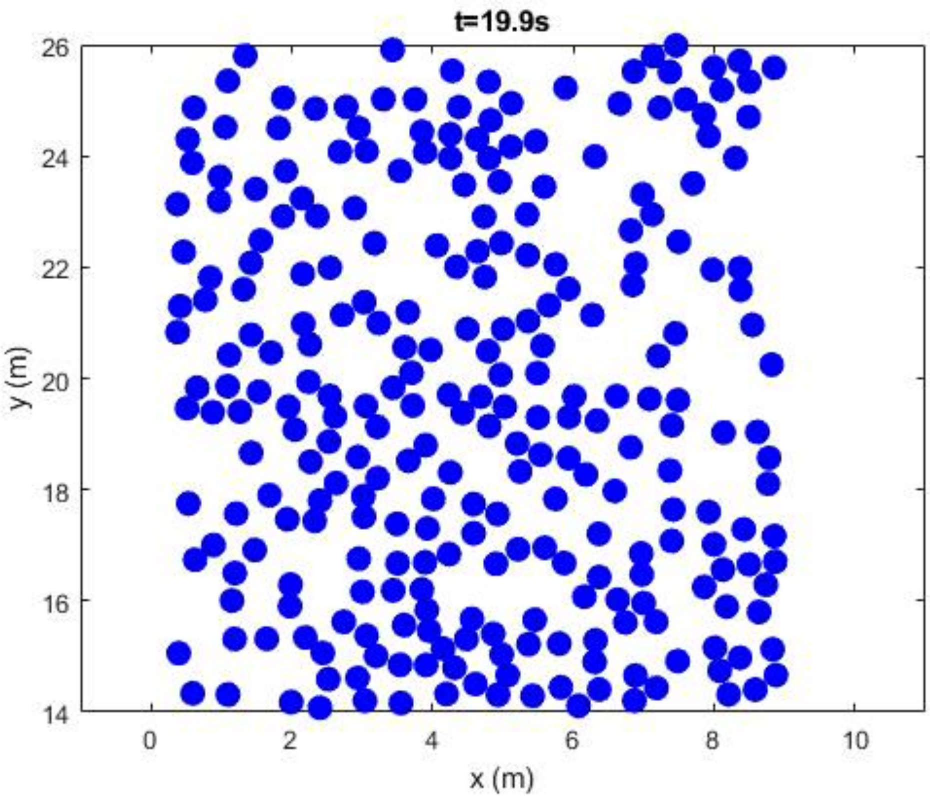
- Rather symmetric density plot :
  - Lower density in front an behind the cylinder
  - Higher density on the wing

- Motion is lateral

*Adapted from [Nicolas et al. Scientific Reports **9**, 105 (2019)]*

Question : Can we interpret these data with "dynamical" models ?

First try : Granular model

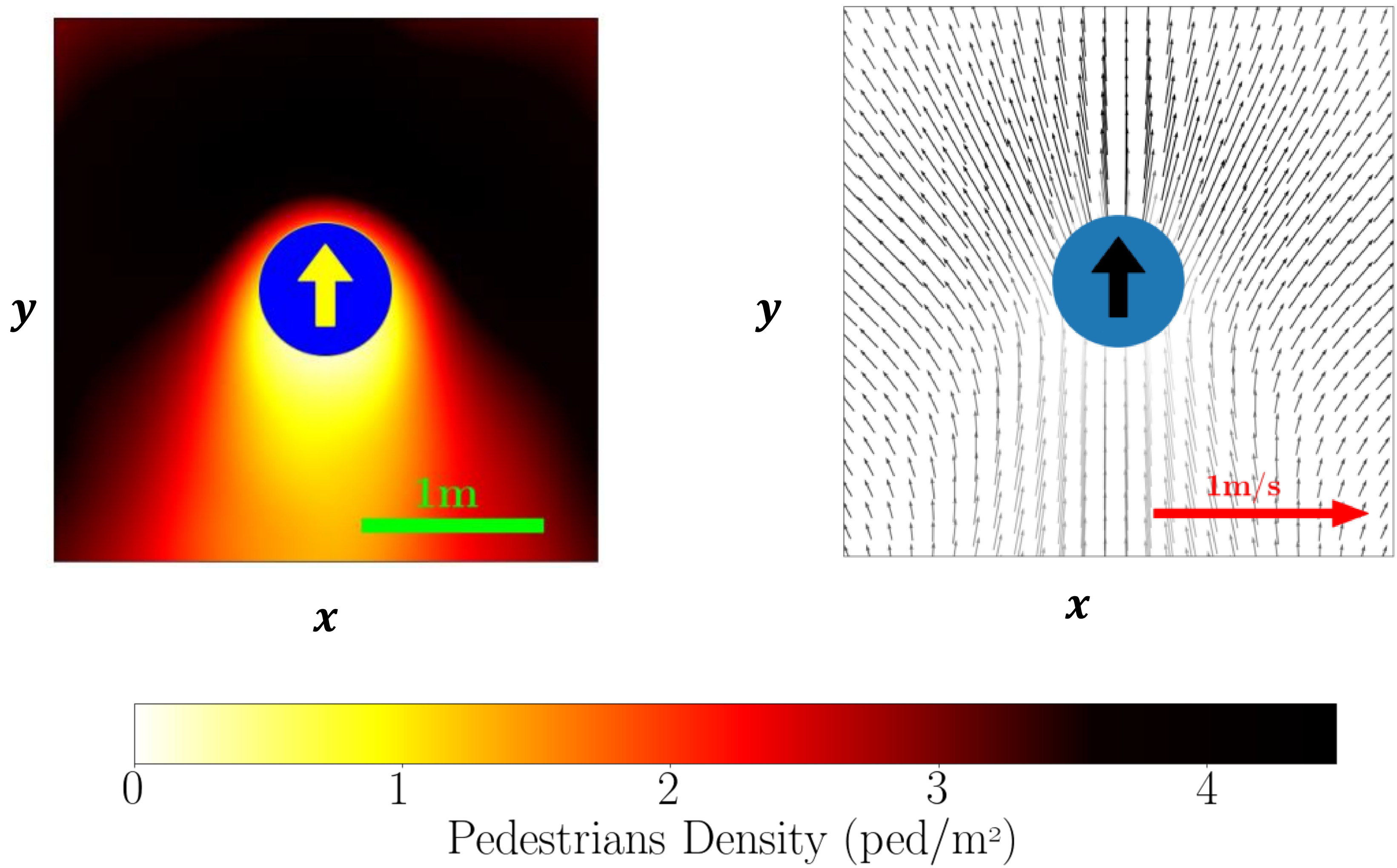


[Details of the calculations : Seguin et al. EPL (2009)]

$m_0 = 2.5$ ●/m²
$v_y = 0.6$ m/s
$R = 0.32$m
Gaussian noise + Inelastic collisions
(restitution coef: $e = 0.5$)

Density and velocity fields for the granular model

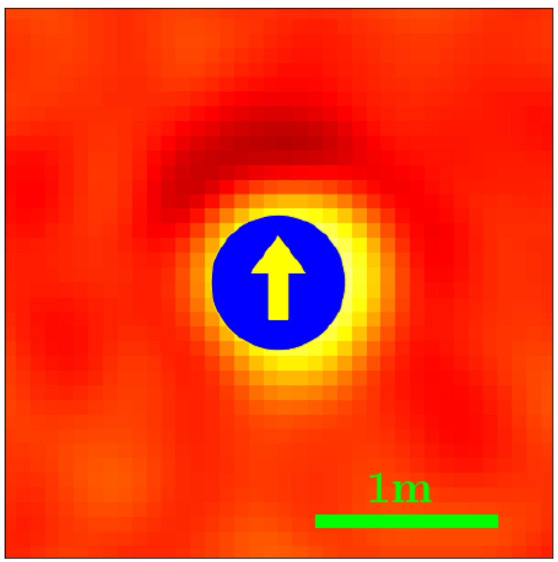
density plot **velocity plot**



Second try : “social force model

density plot

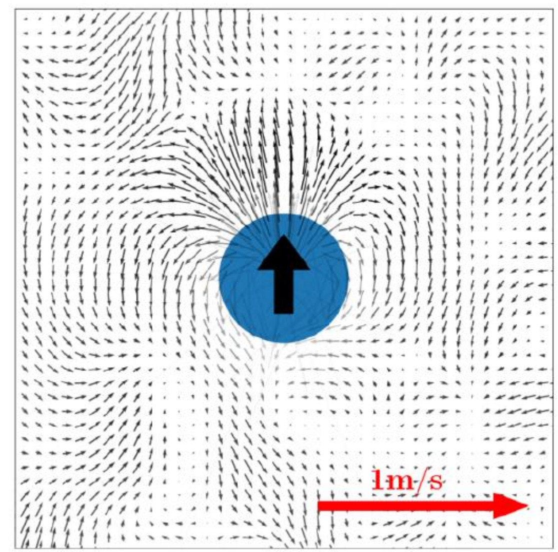
y



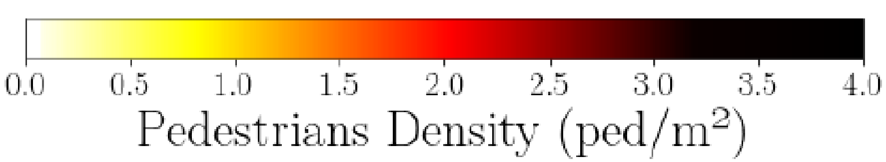
x

velocity plot

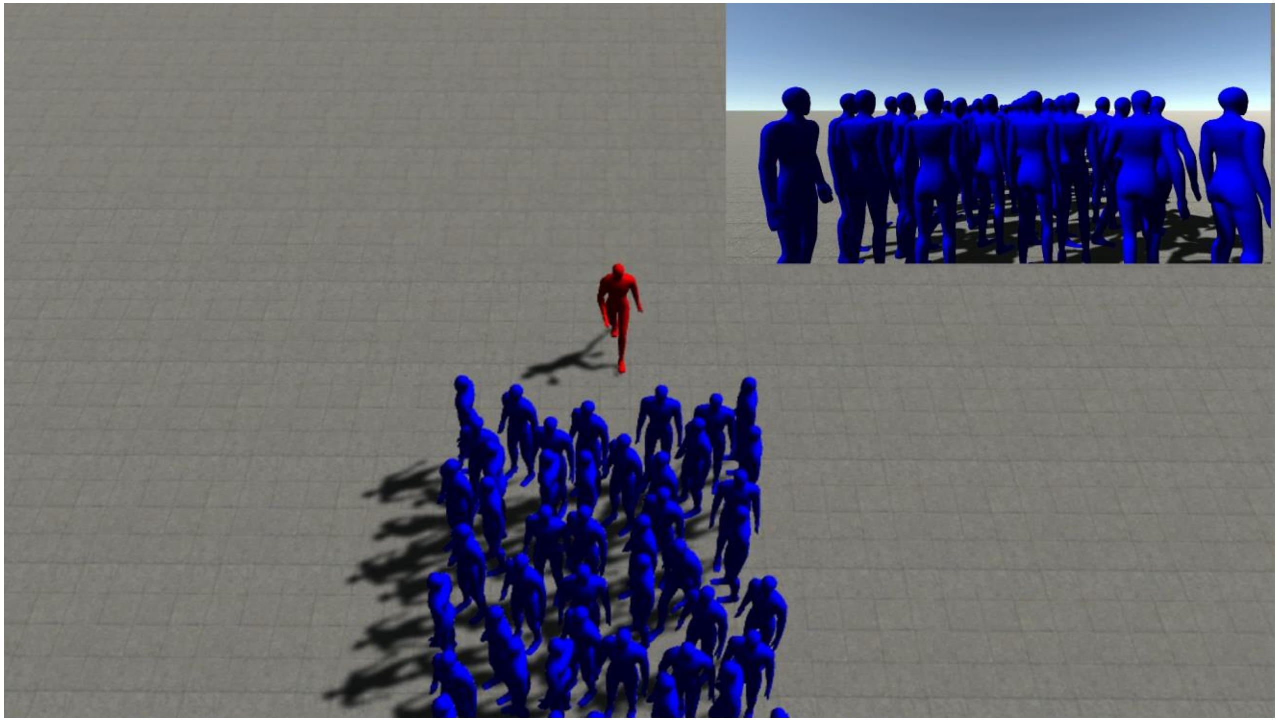
y



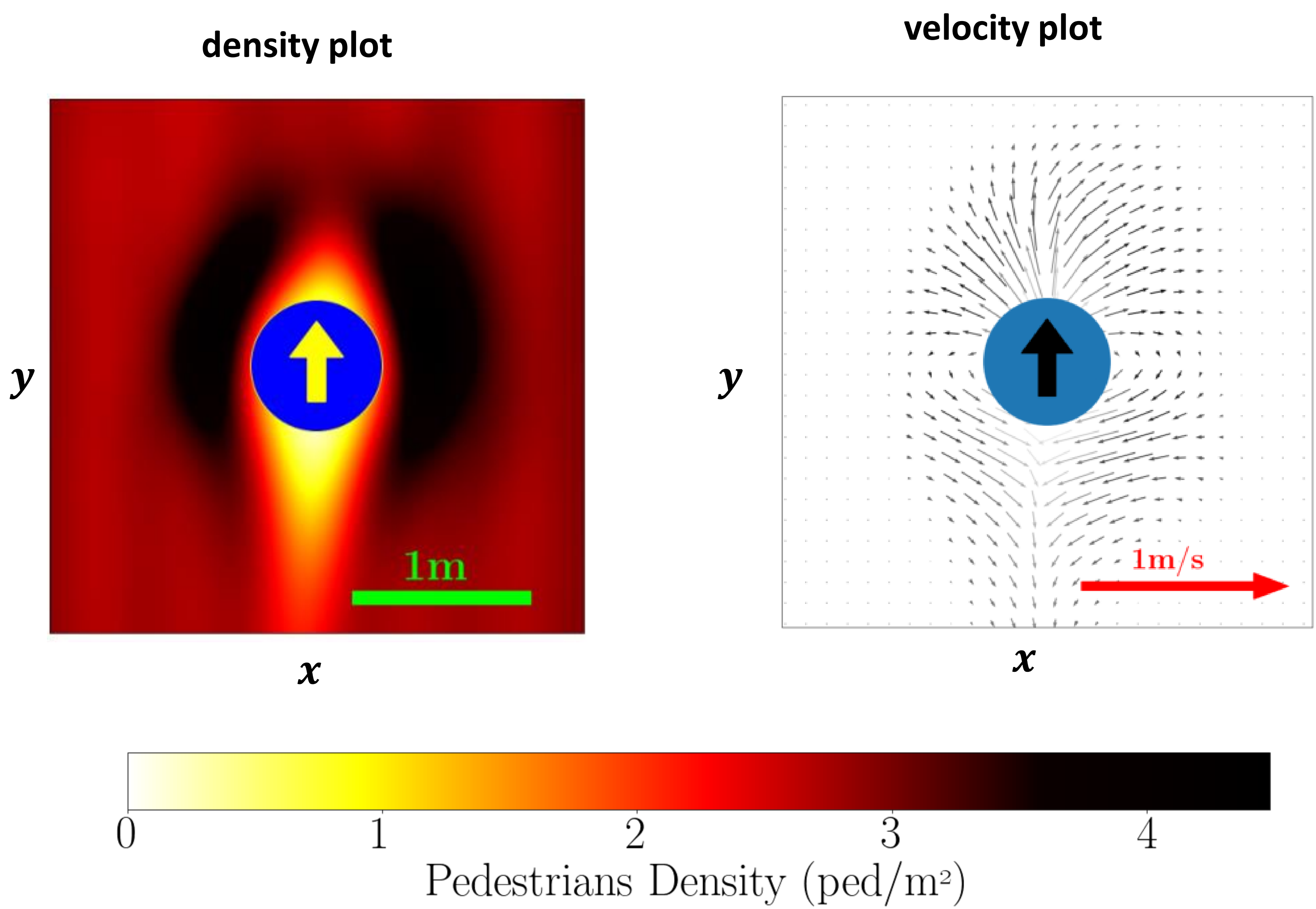
x



Third try : “anticipated-time-to-first collision model” inspired from Karamouzas et al. (2017)



Density and velocity fields for the "Time to Collision" model



- Simple dynamical models (granular, “social force”) fail drastically to reproduce the qualitative features of the experiments
- Even the more modern version of these models (“time to collision”) will struggle to do so.
- On the other hand, the experimental observations are rather intuitive: pedestrians **anticipate** that it will cost them less effort to step aside and then resume their positions, even if it entails enduring high densities for some time, than to endlessly run away from an intruder that will not deviate from its course.

This requires a change of paradigm :

- anticipation ➔ competitive optimization ➔ Game theory
- Large crowd ➔ Many Body Problem ➔ Mean Field

Mean-Field Games

Content

- I – Basics of Game theory
  - Strategic form of a game
  - Lotteries and mixed dynamics
  - Nash equilibrium
- II – Introduction to Mean-Field Games
  - Optimal control / Bellman equation
  - Coupling with Fokker-Planck equation
- III – The fish-school (toy) model
  - Connection with the non-linear Schrödinger equation
  - Heisenberg picture and action
- IV – Pedestrian dynamics
  - Permanent regime
  - Scaled equations
  - Parameter space
- V- epidemics dynamics
  - The SIR dynamics
  - MFG equations for discrete state variable
  - Discussion on the “cost of anarchy”

I – Game Theory 101

[Game theory, Maschler, Solan and Zamir (2013)]

Game theory

- methodology of using mathematical tools to model and analyse situations of interactive decision making.
- These are situations involving several decision makers (called players) with different goals, in which the decision of each affects the outcome for all the decision makers.

Canonical / strategic form of a game

Eg : rock-paper-scissor

		Player II		
		Rock	Paper	Scissors
Player I	Rock	0, 0	1, –1	1, –1
	Paper	1, –1	0, 0	– 1, 1
	Scissors	–1, 1	–1, 1	0, 0

NB1 : what we call a “strategy” may be a bit more complicated than this

Eg : Tic-Tac-Toe

Example, of a tic-tac-toe game where the first player (X) wins the game in seven steps [fromhttps://en.wikipedia.org/wiki/Tic-tac-toe]

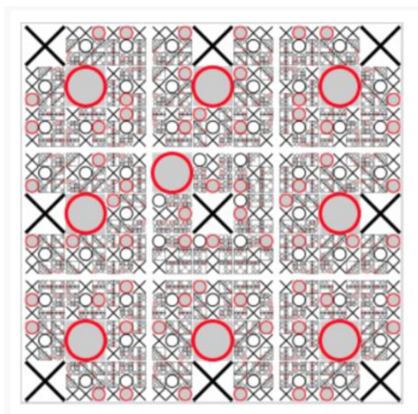


Illustration of **one strategy** for player I

NB1 : here outcome = win → 1, or loose → -1, but more generally we expect some **utility** of a given outcome.

Players : $i = 1, \ldots, N$

$S_i = \{\text{strategy of S}\}$ $\quad S = (S_1, \ldots, S_N)$

$\{-i\} = \{j \neq i\}$

$$\begin{aligned} u_i : S &\longrightarrow \mathbb{R} \\ s = (s_i, s_{-i}) &\longmapsto u_i(s) \ . \end{aligned}$$

utility

Dominates strategies & Nash equilibrium

Dominates strategy

A strategy s_i of a player i is strictly dominated by another strategy t_i of the same player if

$$\forall s_{-i} \in S_{-i},\ u(s_i, s_{-i}) < u(t_i, s_{-i})\ .$$

⇒ useful to simplify a game by eliminating dominated strategies

Nash equilibrium

A strategy vecor $s^* = (s_1^*, \ldots, s_N^*)$ is a **Nash equilibrium**

$$\Leftrightarrow \forall i \in N, \forall s_i \in S_i \quad u(s_i, s^*_{-i}) \leq u(s^*)$$

⇒ nobody can expect a gain by changing strategy *alone*

Nash Theorem

Mixed strategy

A *mixed strategy* is simply a probability distribution over the set of *pure strategies* S_i

$$\Sigma_i \equiv \left\{ \sigma_i : S_i \mapsto [0,1] / \sum_{s_i \in S_i} \sigma_i(s_i) = 1 \right\}.$$

Nash Theorem (1950)

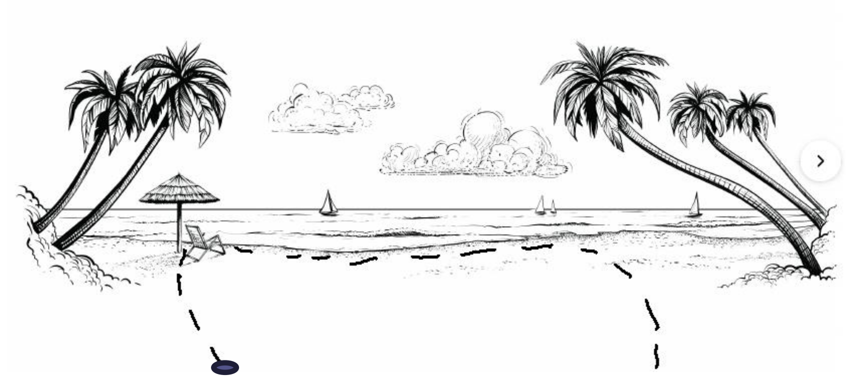
Every games in strategic form (with a finite number of players, each of them having a finite number of strategies) has (at least) one Nash equilibrium in mixed strategies

II - Introduction to Mean Field Game

[Lasry & Lions (2006), Huang et al (2006)]

Optimal control

- Agent characterized by a *state variable* $\boldsymbol{X}$ (for pedestrians → spatial position)



- Dynamics : Langevin

$$\mathrm{d}\boldsymbol{X}_t = \underbrace{\boldsymbol{v}_t}_{\text{control}}\, dt + \sigma\, \underbrace{d\xi_t}_{\text{white noise}}$$

- Cost function

$$c[\mathbf{v}(\cdot)](t,\mathbf{x}) = \left\langle \int_t^T \underbrace{\left[\frac{m\mathbf{v}^2(\tau)}{2} - V(\mathbf{X}_\tau)\right]_{\mathbf{X}_t=\mathbf{x}}}_{\text{running cost}} d\tau \;+\; \underbrace{C_T(\mathbf{X}_T)}_{\text{terminal cost}} \right\rangle_{\text{noise}}$$

Solving the optimization problem when $\sigma = 0$ (noiseless case)

- Dynamics : $\dot{X}_t = v_t$

- Cost function

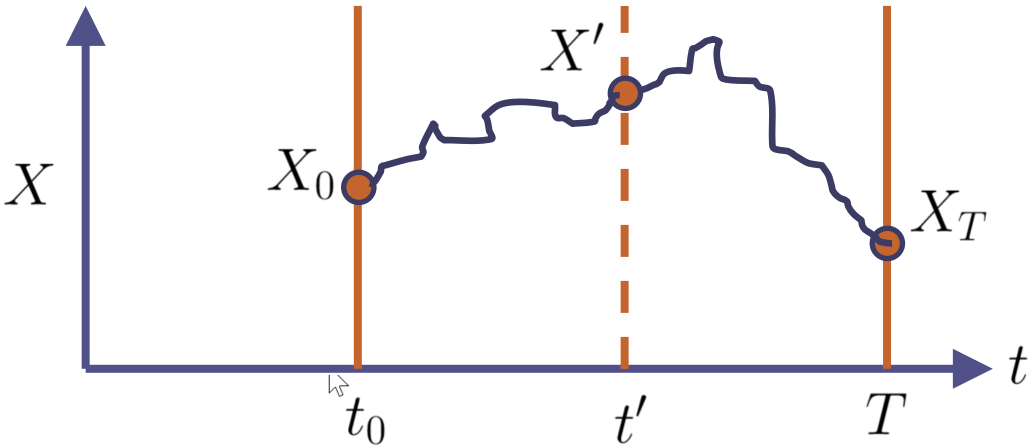
$$c[\dot{\mathbf{X}}(\cdot)](t,\mathbf{x}) \equiv \int_t^T \underbrace{\left[\frac{m}{2}\dot{\mathbf{X}}_\tau^2 - V(\mathbf{X}_\tau)\right]}_{\text{Lagrangian } L(X,\dot{X})} d\tau$$

- Classical dynamics → Lagrange equation

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{v}}\right) = \frac{\partial L}{\partial \mathbf{X}}} \Rightarrow \boxed{m\ddot{\mathbf{X}} = -\partial_{\mathbf{X}} V}$$

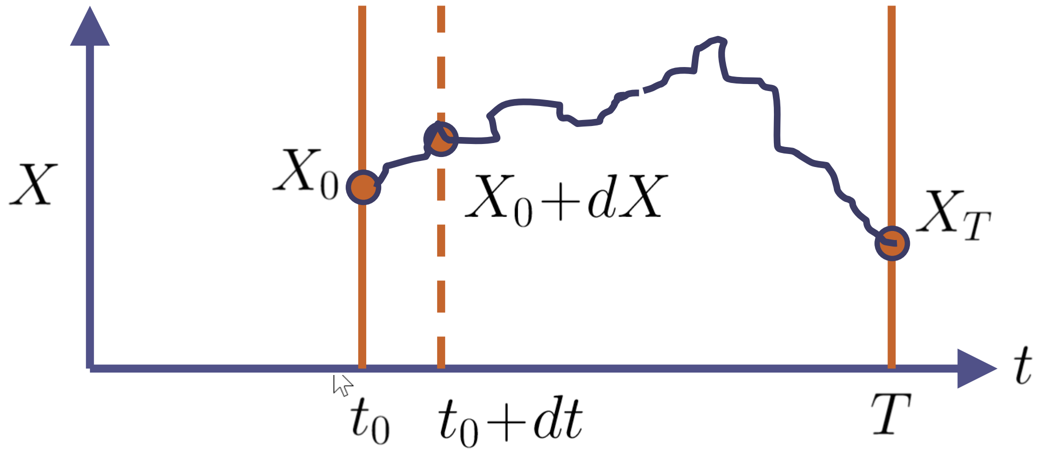
"Optimal control" $\boldsymbol{v}^*(\cdot)$ ≡ velocity solution of the classical equations of motion

Solving the optimization problem when $\sigma \neq 0$: Linear programming and Bellman Equation



Optimization ($t_0 \rightarrow T$) = Optimization ($t_0 \rightarrow t'$) + Optimization ($t' \rightarrow T$)
+ Optimization at t'

Solving the optimization problem when $\sigma \neq 0$: Linear programming and Bellman Equation



Optimization (t_0 → T) = Optimization (t_0 → t') + Optimization (t' → T)
+ Optimization at t'

t' = t_0 + dt ➔ differential equation

More formally :

- Introduce de value function $u(\mathbf{x},t) = \min_{\mathbf{v}(\cdot)} c[\mathbf{v}(\cdot)](\mathbf{x},t)$

- Apply Bellman

$$\partial_t u(\mathbf{x},t) - \frac{(\nabla_{\mathbf{x}} u)^2}{2m} + \frac{\sigma^2}{2}\Delta u(\mathbf{x},t) = V(\mathbf{x})$$

Hamilton-Jacobi-Bellman equation

- Boundary condition : $u(\cdot,T) \equiv C_T(\cdot)$ (backward)

- Optimal control : $\boldsymbol{v}_t^*(\boldsymbol{x}) = -\nabla u(\boldsymbol{x},t)/m$

- As $\sigma \to 0$: HJB → Hamilton-Jacobi equation (solution ≡ Cassical action S)

$$\frac{\partial S}{\partial \mathbf{x}} = \mathbf{p} \equiv m\mathbf{v}$$

Mean Field Games

- N agents $i = 1, 2, \ldots, N$
- Agents : *state variable* $\boldsymbol{X}_t^i$
  control variable $\boldsymbol{v}_t^i$
- Dynamics : Langevin $\quad \mathrm{d}\boldsymbol{X}_t^i = \boldsymbol{v}_t^i dt + \sigma\, d\xi_t^i$
- Interaction between agents $\quad V(\mathbf{X}) \to V(\mathbf{X}^i, \mathbf{X}^{-i}) \to \tilde{V}[\rho(\cdot)](\mathbf{X}^i, t)$

[density of agents] $\quad \boxed{\rho(\mathbf{x}) \equiv \frac{1}{N} \sum_{j \neq i} \delta(\mathbf{X}^j - \mathbf{x})}$

- Cost function

$$c^i[\mathbf{v}^i(\cdot)](t, \mathbf{x}) = \left\langle \int_t^T \left[\frac{m}{2} (\mathbf{v}_\tau^i)^2 - \tilde{V}[\rho](\mathbf{X}_\tau^i) \right]_{\mathbf{X}_t^i = \mathbf{x}} d\tau \; + \; C_T(\mathbf{X}_T^i) \right\rangle_{\text{noise}}$$

- Apply Bellman for each agent i

$$\partial_t u_i(t,\mathbf{x}) - \frac{1}{2m}[\nabla u_i(t,\mathbf{x})]^2 + \frac{\sigma^2}{2}\Delta u_i(t,\mathbf{x}) = \tilde{V}[\rho](x)$$

➔ N coupled differential equations = **Many Body Game Theory**

- Mean-field approximation :

$$\rho(\mathbf{x}) \equiv \underbrace{\frac{1}{N}\sum_{j\neq i}\delta(\mathbf{X}^j - \mathbf{x})}_{\text{stochastic}} \longrightarrow \underbrace{\frac{1}{N}\left\langle\sum_{j}\delta(\mathbf{X}^j - \mathbf{x})\right\rangle_{\text{noise}}}_{\text{deterministic}}$$

$$\begin{cases} \partial_t\rho + \nabla_{\mathbf{x}}(\mathbf{v}\,\rho) - \dfrac{\sigma^2}{2}\Delta_{\mathbf{x}}\rho = 0 \\ \rho(\mathbf{x}, t=0) = \rho^0(\mathbf{x}) \end{cases} \qquad \text{(Kolmogorov)}$$

Mean Field Game [Lasry & Lions (2006), Huang et al (2006)]= coupling between a (collective) stochastic motion and an (individual) optimization problem through a mean field

- Langevin dynamics : $\mathrm{d}\boldsymbol{X}_t = \boldsymbol{v}_t\, dt + \sigma\, d\xi_t$ leads to a *(forward)* diffusion equation for the mean density $\rho(\boldsymbol{x}, t)$

$$\begin{cases} \partial_t \rho + \nabla_{\mathbf{x}} (\mathbf{v}\, \rho) - \dfrac{\sigma^2}{2} \Delta_{\mathbf{x}} \rho = 0 \\ \rho(\mathbf{x}, t=0) = \rho^0(\mathbf{x}) \end{cases} \qquad \text{(Kolmogorov)}$$

- Optimization problem, through linear programming, leads to a *(backward)* Hamilton-Jacobi-Bellman equation for the value function $\mathrm{u}(\boldsymbol{x}, t)$

$$\begin{cases} \partial_t u - \dfrac{1}{2m} (\nabla_{\mathbf{x}} u)^2 + \dfrac{\sigma^2}{2} \Delta_{\mathbf{x}} u = \tilde{V}[\rho](\mathbf{x}) \\ u(T, \mathbf{x}) = C_T(\mathbf{x}) \end{cases} \qquad \text{(HJB)}$$

- Kolmogorov coupled to HJB through the drift velocity $\boldsymbol{v}(\boldsymbol{x}, t) = -\nabla u(\boldsymbol{x}, t)/m$

- HJB coupled to Kolmogorov through the mean field $\tilde{V}[\rho](\boldsymbol{x}, t)$

III – The fish-school (toy) model

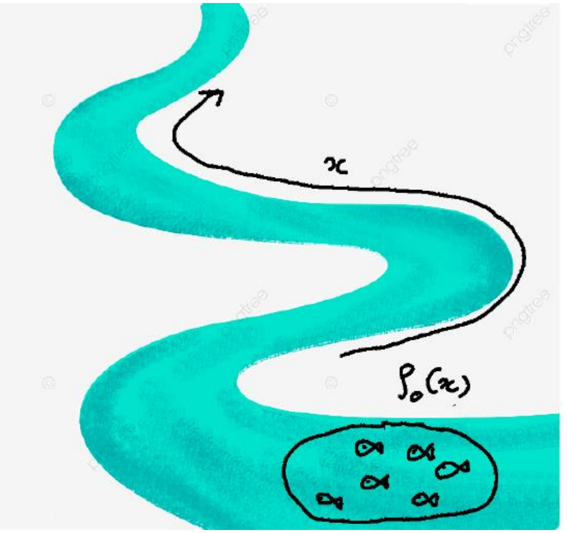
Position of the problem:

School of fish in a (1d) river :

- Start in the morning with an initial distribution $\rho_o(\boldsymbol{x})$.
- During the day → to gather food, while staying close together to protect themselves from predators (and the latter concern take priority on the former).

$$\tilde{V}[\rho(\cdot)](\boldsymbol{x}) = V_o(\boldsymbol{x}) + g\rho_o(\boldsymbol{x}) \qquad g \gg 0$$

- At the end of the day: find shelter in the best possible place → $C_T(\boldsymbol{x})$.



$$\partial_t \rho_t + \partial_x(v_t \rho_t) - \frac{\sigma^2}{2}\partial^2_{xx}\rho_t = 0$$

$$\partial_t u_t - \frac{(\partial_x u_t)^2}{2m} + \frac{\sigma^2}{2}\partial^2_{xx}u_t = \tilde{V}[\rho_t(.)](x) = V_0(x) + g\rho(x) ,$$

$$\rho_{t=0}(\cdot) = \rho_0(\cdot) \qquad u_T(\cdot) = C_T(\cdot) \qquad v_t(x) = -\frac{1}{m}\,\partial_x u_t$$

Transformation to NLS

- Cole-Hopf transform:

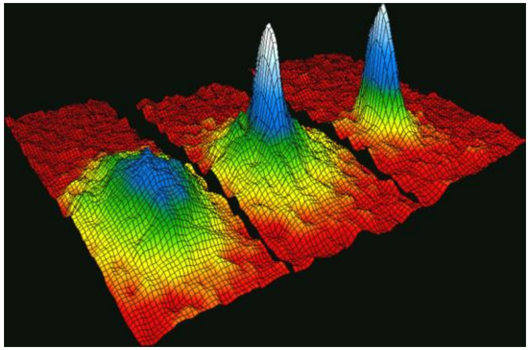
$$\Phi(\mathbf{x},t) = \exp\left(-\frac{1}{m\sigma^2}u(\mathbf{x},t)\right)$$

- "Hermitization" of Kolmogorov:

$$\Gamma(\mathbf{x},t) = \rho(\mathbf{x},t)\exp\left(\frac{1}{m\sigma^2}u(\mathbf{x},t)\right) \quad \Rightarrow \quad \rho_{\mathrm{er}}(\mathbf{x},t) = \Gamma(\mathbf{x},t)\Phi(\mathbf{x},t)$$

$$\Rightarrow \quad \begin{cases} -m\sigma^2\partial_t\Phi = \dfrac{m\sigma^4}{2}\Delta\Phi + [V_0(\mathbf{x}) + g\rho]\Phi \\ +m\sigma^2\partial_t\Gamma = \dfrac{m\sigma^4}{2}\Delta\Gamma + [V_0(\mathbf{x}) + g\rho]\Gamma \end{cases}$$

$$\boxed{\Phi_T(\cdot) = \exp(-C_T(\cdot)/m\sigma^2)} \qquad \boxed{\Gamma_{t=0}(\cdot) = \rho^0(\cdot)/\Phi_{t=0}(\cdot)}$$



$$i\hbar\partial_t\Psi = -\frac{\hbar^2}{2m}\Delta\Psi + [V_0(\mathbf{x}) + g|\Psi|^2]\Psi$$

Non-Linear Schrödinger

MFG equations, specifying to $\tilde{V}[\rho(\cdot)](\boldsymbol{x}) = V_o(\boldsymbol{x}) + g\rho_o(\boldsymbol{x})$

$$\begin{cases} -m\sigma^2\partial_t\Phi = \dfrac{m\sigma^4}{2}\Delta\Phi + [V_0(\mathbf{x}) + g\rho]\Phi \\ +m\sigma^2\partial_t\Gamma = \dfrac{m\sigma^4}{2}\Delta\Gamma + [V_0(\mathbf{x}) + g\rho]\Gamma \end{cases}$$

Formal change $(\Psi, \Psi^*, i\hbar) \to (\Phi, \Psi, m\sigma^2)$ maps NLS to MFG !!!

Intermezzo: few different ways to use the connection with NLS

Tool #1 : Heisenberg representation & Ehrenfest relations

Quantum mechanics

- State of the system $\equiv$ wave function $\Psi(x,t)$
- Observables $\equiv$ operators: $\hat{O} = f(\hat{p}, \hat{x})$
- Average $\langle \hat{O} \rangle \equiv \int dx \Psi^*(x) \hat{O} \Psi(x)$
- Hamiltonian $\equiv$ $\hat{H} = \frac{\hat{p}^2}{2\mu} + V(x) = -\frac{\hbar^2}{2\mu}\Delta_x + V(x)$

$$\hat{x} \equiv x\times$$
$$\hat{p} \equiv i\hbar\partial_x$$

$$i\hbar\partial_t \Psi = \hat{H}\Psi \qquad \Rightarrow \qquad i\hbar\frac{d}{dt}\langle \hat{O}\rangle = \langle [\hat{H}, \hat{O}]\rangle$$

$$\Rightarrow \quad \begin{cases} \frac{d}{dt}\langle \hat{x}\rangle = \frac{1}{\mu}\langle \hat{p}\rangle \\ \frac{d}{dt}\langle \hat{p}\rangle = -\langle \nabla_x V(\hat{x})\rangle \end{cases} \qquad \text{(Ehrenfest)}$$

Quadratic Mean Field Games $(\hbar_{\text{eff}} \equiv m\,\sigma^2)$

- Operators: $\hat{X}\,(\cdot) \equiv x \times (\cdot)$ $\quad \hat{\Pi}\,(\cdot) \equiv -\hbar_{\text{eff}}\,\partial_x\,(\cdot)$ $\quad \hat{O} = f(\hat{X}, \hat{\Pi})$

- Average: $\langle \hat{O} \rangle(t) \equiv \langle \Phi_t | \hat{O} | \Gamma_t \rangle = \int dx\, \Phi_t(x)\, \hat{O}\, \Gamma_t(x)$ $\qquad \boxed{\rho = \Phi\Gamma}$

  If $\hat{O} = f(\hat{X}, \cancel{\hat{\Pi}})$ $\qquad \langle \hat{O} \rangle(t) = \int dx\, \rho_t(x) O(x)$

$$\left(\langle \hat{1} \rangle = \int d\mathbf{x} \rho_t(\mathbf{x}) = 1 \qquad \langle \hat{X} \rangle = \int d\mathbf{x}\, \mathbf{x}\, \rho_t(\mathbf{x}) \right)$$

- Hamiltonian: $\hat{H} = \frac{\hat{\Pi}^2}{2m} + \tilde{V}[\rho](\hat{X})$

$$\begin{cases} -m\sigma^2 \partial_t \Phi = \hat{H}\Phi \\ +m\sigma^2 \partial_t \Gamma = \hat{H}\Gamma \end{cases} \quad \Rightarrow \quad \frac{d}{dt}\langle \hat{O} \rangle = -\frac{1}{m\sigma^2} \langle [\hat{H}, \hat{O}] \rangle$$

Force operator : $\hat{F}[m_t] \equiv -\nabla_x V[m_t](\hat{X})$

Exact relations

- Force operators: $\hat{F}[\rho_t] = -\nabla\tilde{V}[\rho_t](\hat{X})$

$$\Sigma^2 \equiv \langle(\hat{X}^2\rangle - \langle\hat{X}\rangle^2) \qquad \Lambda \equiv (\langle\hat{X}\hat{\Pi} + \hat{\Pi}\hat{X}\rangle - 2\langle\hat{\Pi}\rangle\langle\hat{X}\rangle)$$

$$\begin{cases} \dfrac{d}{dt}\langle\hat{X}\rangle = \dfrac{1}{m}\langle\hat{\Pi}\rangle \\ \dfrac{d}{dt}\langle\hat{\Pi}\rangle = \langle\hat{F}[\rho_t]\rangle \end{cases} \qquad \begin{cases} \dfrac{d}{dt}\Sigma^2 = \dfrac{1}{m}\Big(\langle\hat{X}\hat{\Pi} + \hat{\Pi}\hat{X}\rangle - 2\langle\hat{\Pi}\rangle\langle\hat{X}\rangle\Big) \\ \dfrac{d}{dt}\Lambda = -2\langle\hat{X}\hat{F}[\rho_t]\rangle + 2\langle\hat{\Pi}^2\rangle \end{cases}$$

- Local interactions: $\tilde{V}[\rho_t](\boldsymbol{x}) = V_{\boldsymbol{o}}(\boldsymbol{x}) + f\big(\rho_t(\boldsymbol{x})\big)$

$\rightarrow \hat{F}[\rho_t] = \hat{F}_o - g\,(\nabla\rho_t)\,f'(\rho_t) \qquad (\hat{F}_o = -\nabla V_{\boldsymbol{o}}(\hat{\boldsymbol{X}}))$

$$\boxed{\langle\hat{F}\rangle = \langle\hat{F}_o\rangle} \qquad \boxed{\langle\hat{X}\hat{F}\rangle = \langle\hat{X}\hat{F}_o\rangle - \int dx\, x(\nabla\rho(x))f'(\rho(x))}$$

Tool #2 : action and variational approach

Action

$$S[\Gamma(x,t),\Phi(x,t)] \equiv \int dt\,dx\,\left[\frac{\mu\sigma^2}{2}(\partial_t\Phi\,\Gamma - \Phi\partial_t\Gamma) - \frac{\mu\sigma^4}{2}\nabla\Phi.\nabla\Gamma + U_0(x)\Phi\Gamma + \frac{g}{2}\Phi^2\Gamma^2\right]$$

$$\left[\frac{\delta S}{\delta\Gamma}=0\right] \Leftrightarrow -\mu\sigma^2\partial_t\Phi = \frac{\mu\sigma^4}{2}\Delta_{\mathbf{x}}\Phi + V[\mathbf{x},m]\Phi$$

$$\left[\frac{\delta S}{\delta\Phi}=0\right] \Leftrightarrow +\mu\sigma^2\partial_t\Gamma = \frac{\mu\sigma^4}{2}\Delta_{\mathbf{x}}\Gamma + V[\mathbf{x},m]\Gamma$$

- Conserved quantity: $\mathcal{E}_{\text{tot}}\cancel{(t)} \equiv \frac{1}{2\mu}\langle\hat{\Pi}^2\rangle + \langle U_0(\hat{X})\rangle + \langle\hat{H}_{\text{int}}\rangle$
- Variational anzatz $\Longrightarrow$ Ordinary Differential Equations

$$\langle\hat{H}_{\text{int}}\rangle \equiv \frac{g}{2}\int dx\, m_t(x)^2$$

Tool #3 : solitons and integrability

For *d=1* and *V[m](x) = g m + U₀ (x)* ➔ NLS, and thus MFG, integrable

➔ Infinite number of conserved quantities *[Bonnemain et al. 2021]*

$$Q_1 = \frac{1}{2}\int_{\mathbb{R}} (w_1\Phi + \tilde{w}_1\Gamma)dx = \frac{\mu\sigma^4}{2g}\int_{\mathbb{R}} (\Gamma\partial_x\Phi - \Phi\partial_x\Gamma)dx \propto P$$

$$Q_2 = \frac{1}{2}\int_{\mathbb{R}} (w_2\Phi + \tilde{w}_2\Gamma)dx = \frac{\mu\sigma^4}{2g^2}\int_{\mathbb{R}} dx\left[-\frac{\mu\sigma^4}{2}\nabla\Gamma.\nabla\Phi + \frac{g}{2}(\Gamma\Phi)^2\right] \propto E_{\text{tot}}$$

$$Q_3 = \frac{1}{2}\int_{\mathbb{R}} (w_3\Phi + \tilde{w}_3\Gamma)dx = \frac{\mu^2\sigma^8}{2g^2}\int_{\mathbb{R}} dx\left[\frac{\mu\sigma^4}{g}(\Gamma\partial_{xxx}\Phi - \Phi\partial_{xxx}\Gamma) + \frac{3}{2}\left(\Gamma^2\partial_x\Phi^2 - \Phi^2\partial_x\Gamma^2\right)\right]$$

$$Q_4 = \frac{1}{2}\int_{\mathbb{R}} (w_4\Phi + \tilde{w}_4\Gamma)dx = \frac{\mu^2\sigma^8}{2g^2}\int_{\mathbb{R}} dx\left[\frac{\mu^2\sigma^8}{g^2}\partial_{xx}\Phi\partial_{xx}\Gamma + \frac{3\mu\sigma^4}{g}\partial_x\Phi^2\partial_x\Gamma^2 + (\Phi\Gamma)^3\right]$$

➔ Scaling solutions (Thomas-Fermi regime) *[Bonnemain et al. 2020]*

$$\begin{cases} m(t,x) = \dfrac{3\left(z(t)^2 - x^2\right)}{4z(t)^3} \\ v(t,x) = -\dfrac{z'(t)}{z(t)}x \end{cases}$$

$$z(t) = 3\left(\frac{|g|}{4\mu}\right)^{1/3} t^{2/3}$$

Limiting case $U_0(x) \equiv 0$ (NB: $g > 0$)

In that case solution of stationary NLS known (bright soliton)

$$\Psi_{\mathrm{e}}(x) = \frac{\sqrt{\eta}}{2} \frac{1}{\cosh\left(\frac{x}{2\eta}\right)}$$

$$\eta \equiv 2\mu\sigma^4/g$$

caracteristic length scale

"Strong coordination" regime

- meaning : variations of $U_0(x)$ on the scale η are small
- ergodic state

$$m_{\mathrm{e}}(x) \simeq \frac{\eta}{4} \frac{1}{\cosh^2\left(\frac{x - x_{\max}}{2\eta}\right)}$$

$$x_{\max} = \mathrm{argmax}[U_0]$$

Resulting Generic scenario [for strong positive coordination]

1) Herd formation: extension η, mean position $x_0 = \langle x \rangle_{m_0}$ (very short time process)

2) Propagation of the herd :

   - as a classical particle of mass μ in pot $U_0(x)$
   - initial position: $X(0) = x_0$
   - final momentum: $P(T) = -\partial_x c_T(X(T))$

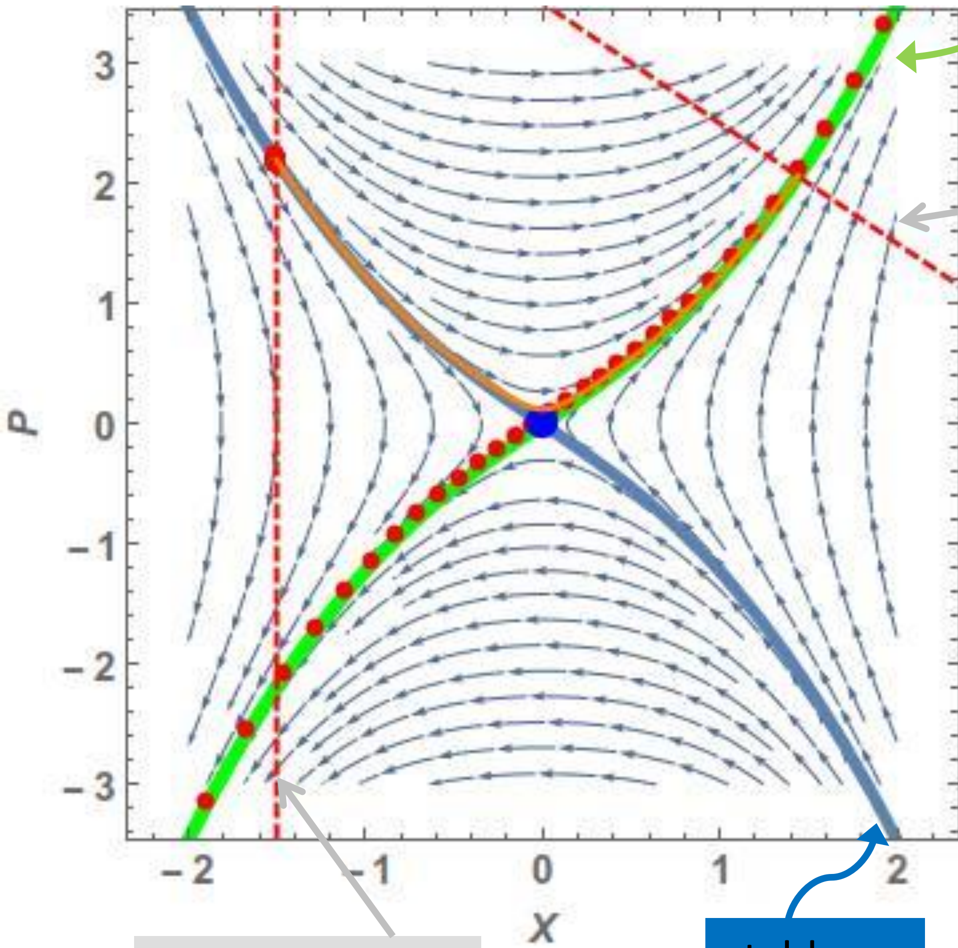
3) Herd dislocation near $t = T$ (again very short process)

NB: Boundary pb rather than initial valuer pb

- **possibly more than one solution**
- $[T \to \infty]$ **motion governed by unstable fixed points**

Propagation phase in the long time limit : role of the unstable fix points

unstable manifold



Final condition

$$C_T(x) = -\frac{7}{2}x + \frac{x^2}{2}$$

$$\Downarrow$$

$$P_T = \frac{7}{2} - X_T$$

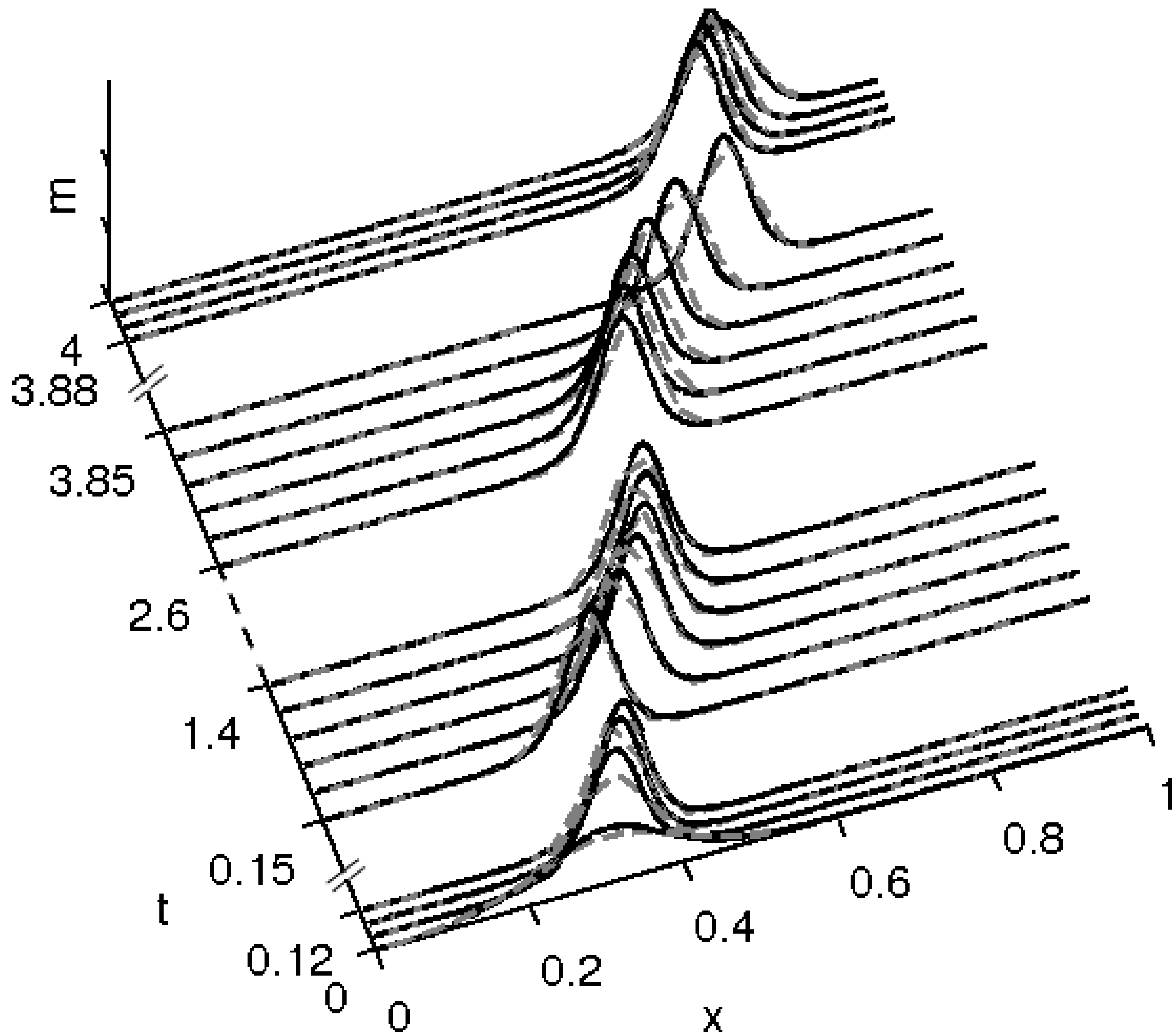
Initial condition

$$[X_0 = -1.5]$$

stable manifold

$$U_0(x) = -\frac{x^4}{4} - \frac{x^2}{2}$$

Comparison with numerical simulation (with variational ansatz for the herd formation)



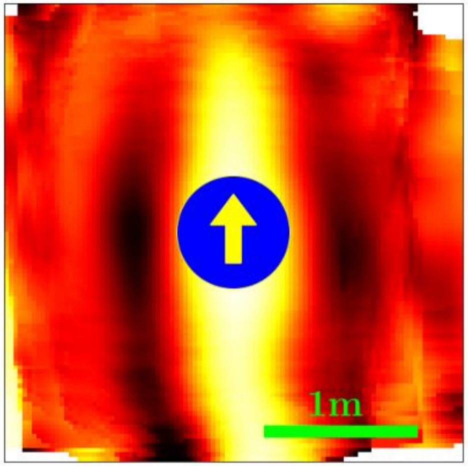
Conclusion for this toy model

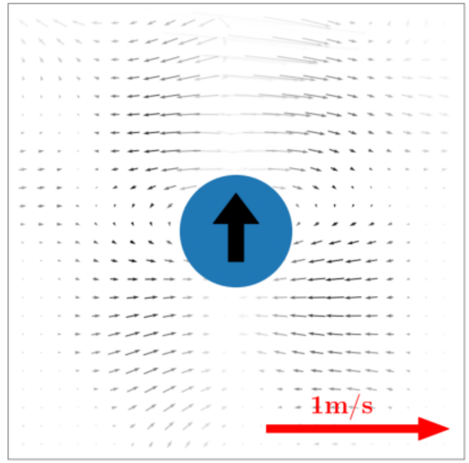
- Formal, but deep, relation between a class of mean field games and the Non-Linear Schrödinger equation dear to the heart of physicists
- Classical tools developed in that context (Ehrenfest relations, solitons, variational methods, etc ..) can be used to analyze mean field games
- Here: application to a simple population dynamics model

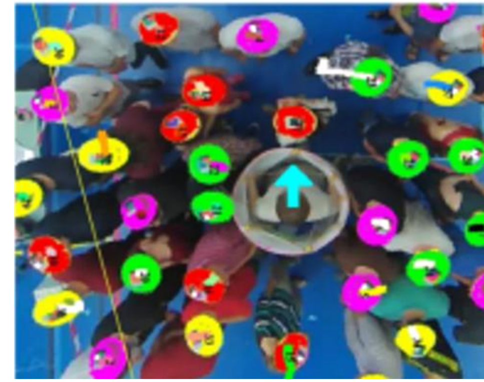
  → rather thorough understanding of this model

  (including more structured initial conditions, collapse of the soliton,..)

IV – Back to pedestrian dynamics







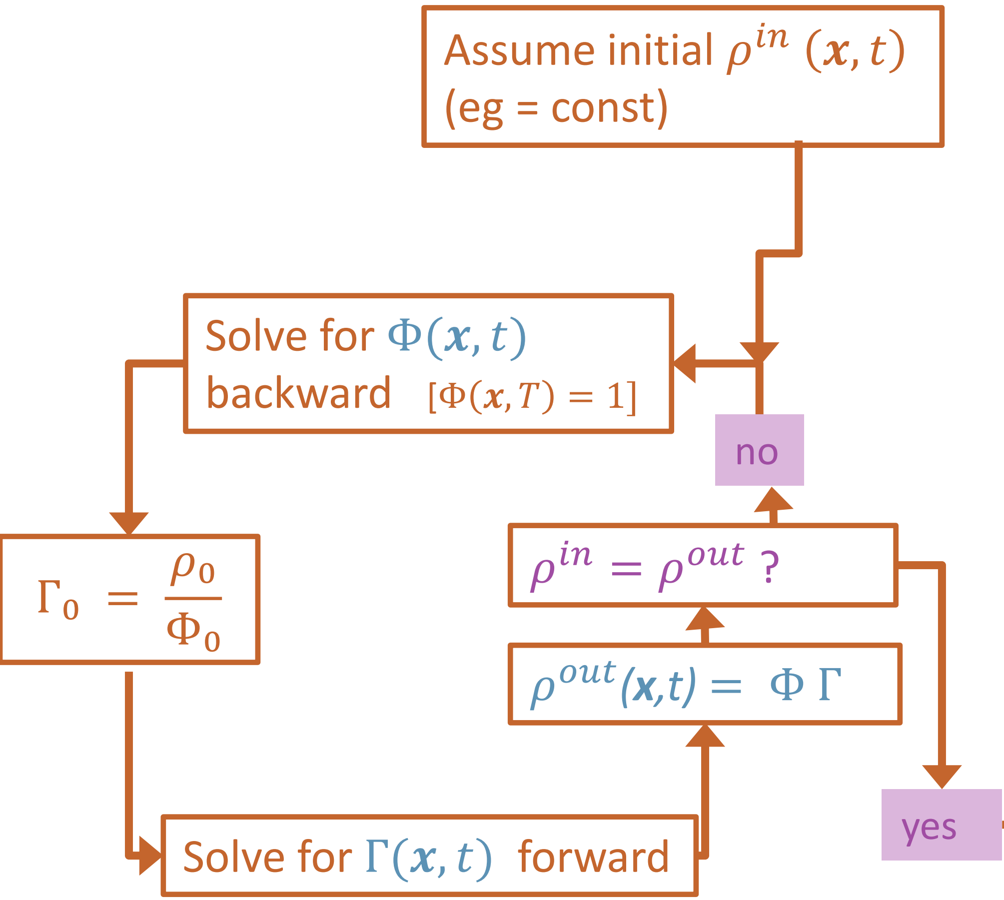
$$
\begin{cases}
\partial_t \rho_t(\mathbf{x}) - \dfrac{1}{m}\boldsymbol{\nabla}.(\rho_t(\mathbf{x})\,\boldsymbol{\nabla}(\mathbf{x})) - \dfrac{\sigma^2}{2}\Delta\rho_t(\mathbf{x}) = 0 \\[2ex]
\partial_t u_t(\mathbf{x}) - \dfrac{(\nabla u_t(\mathbf{x}))^2}{2m} + \dfrac{\sigma^2}{2}\Delta u_t(\mathbf{x}) = V_0(|\mathbf{x} - \mathbf{v}_c t|) + g\rho_t(bx)
\end{cases}
$$

$$
\rho_{t=0}(\cdot) = \rho_0(\cdot) \qquad u(T,\cdot) = 0
$$

$$
\begin{cases}
+\hbar_{\text{eff}}\partial_t\Gamma_t(\mathbf{x}) = +\dfrac{\hbar_{\text{eff}}^2}{2m}\Delta\Gamma_t(\mathbf{x}) + (V_0(|\mathbf{x} - \mathbf{v}_c t|) + g\rho_t(\mathbf{x}))\Gamma_t(\mathbf{x}) \\[2ex]
-\hbar_{\text{eff}}\partial_t\Phi_t(\mathbf{x}) = +\dfrac{\hbar_{\text{eff}}^2}{2m}\Delta\Phi_t(\mathbf{x}) + (V_0(|\mathbf{x} - \mathbf{v}_c t|) + g\rho_t(\mathbf{x}))\Phi_t(\mathbf{x})
\end{cases}
$$

$$
\Phi_T(\cdot) \equiv 1,\ \Gamma_{t=0}(\mathbf{x}) = \rho^0/\Phi_{t=0}(\mathbf{x})
$$

Numerical implementation



Propagation :

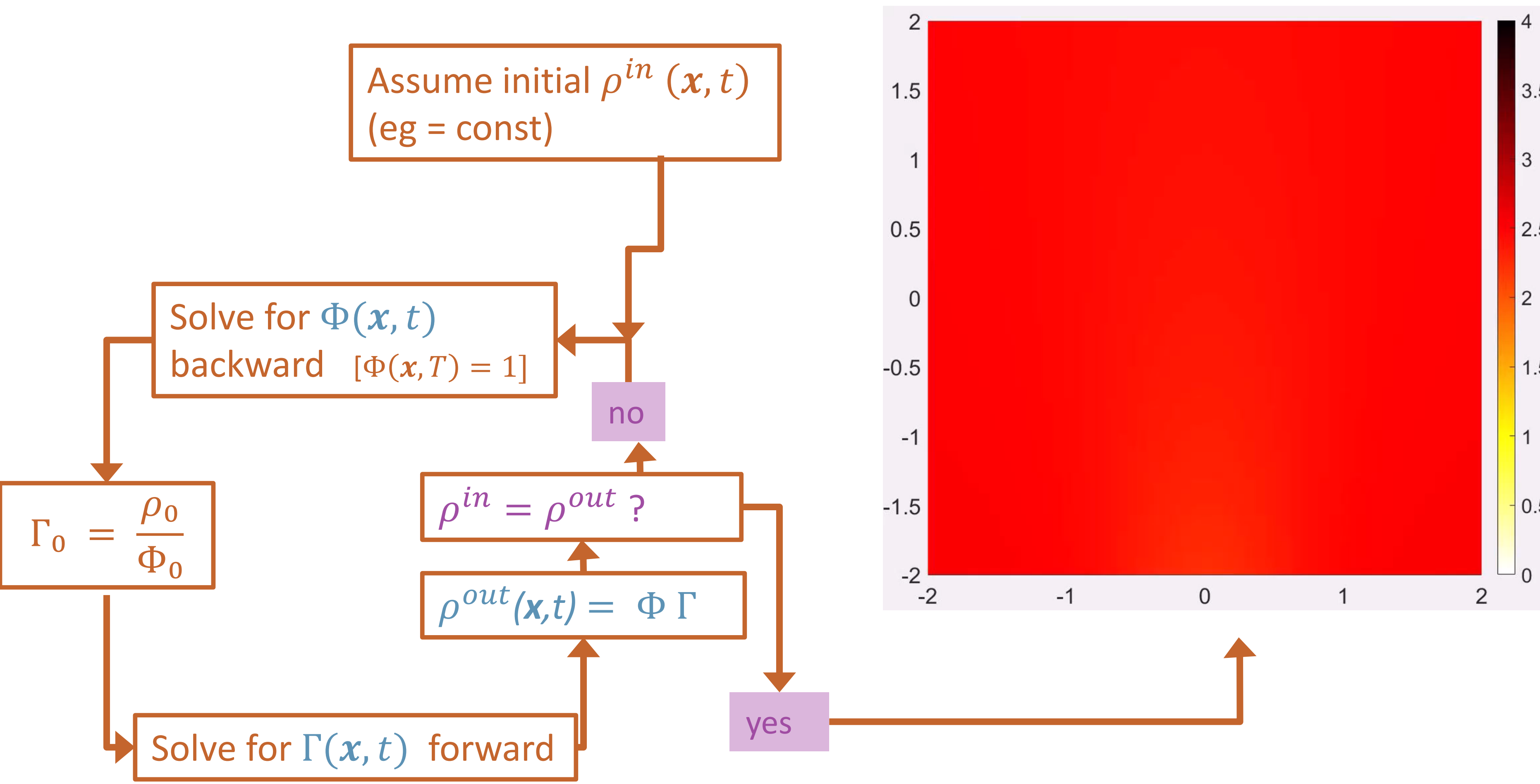
$$\begin{cases} +\hbar_{\text{eff}}\partial_t\Gamma_t = +\frac{\hbar_{\text{eff}}^2}{2m}\Delta\Gamma_t + (V_0 + g\rho_t^{\text{in}})\Gamma_t \\ -\hbar_{\text{eff}}\partial_t\Phi_t = +\frac{\hbar_{\text{eff}}^2}{2m}\Delta\Phi_t + (V_0 + g\rho_t^{\text{in}}))\Phi_t \end{cases}$$

$$\Phi(T,\cdot) = 1 \qquad \Gamma(t=0,\cdot) = \rho_0(\cdot)/\Phi(T,\cdot)$$

Self consistent equation :

$$\rho^{\text{out}}(\mathbf{x},t) = \Phi(\mathbf{x},t)\Gamma(\mathbf{x},t) = \rho^{\text{in}}(\mathbf{x},t)$$

Numerical implementation



However :

- We are not really interested in the full dynamics (transient regime, etc…)
- Self consistence is expensive.

⇒ Permanent regime (a.k.a ergodic) *[cf Cardialaguet et al. (2013)]*

NB $\Gamma_t(\mathbf{x}) = \exp[-\lambda t/\hbar_{\text{eff}}]\Gamma_{\text{er}}$ $\Phi_t(\mathbf{x}) = \exp[\lambda t/\hbar_{\text{eff}}]\Phi_{\text{er}}$ (time dependent)

$m_{\text{er}}(\mathbf{x}) = \Phi_{\text{er}}(\mathbf{x})\Gamma_{\text{er}}(\mathbf{x})$ $j_{\text{er}}(\mathbf{x}) = \sigma^2\Gamma_{\text{er}}(\mathbf{x})\nabla\Phi_{\text{er}}(\mathbf{x})$ (observable are time independent)

In the cylinder frame

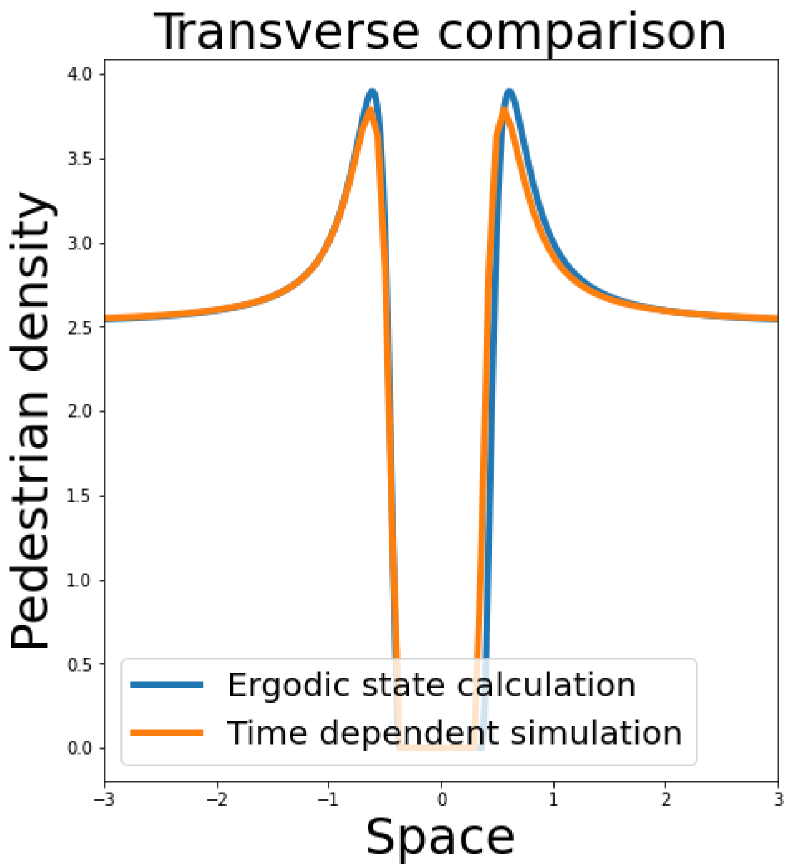
$$\begin{cases} \frac{\hbar_{\text{eff}}^2}{2m}\Delta\Phi_{\text{er}}^{\text{cf}} - \hbar_{\text{eff}}\mathbf{v}_c\cdot\vec{\nabla}\Phi_{\text{er}}^{\text{cf}} + [V_0(|\mathbf{x}|) + g\rho_{\text{er}}]\Phi_{\text{er}}^{\text{cf}} = -\lambda\Phi_{\text{er}}^{\text{cf}} \\ \frac{\hbar_{\text{eff}}^2}{2m}\Delta\Gamma_{\text{er}}^{\text{cf}} + \hbar_{\text{eff}}\mathbf{v}_c\cdot\vec{\nabla}\Gamma_{\text{er}}^{\text{cf}} + [V_0(|\mathbf{x}|) + g\rho_{\text{er}}]\Gamma_{\text{er}}^{\text{cf}} = -\lambda\Gamma_{\text{er}}^{\text{cf}} \end{cases}$$

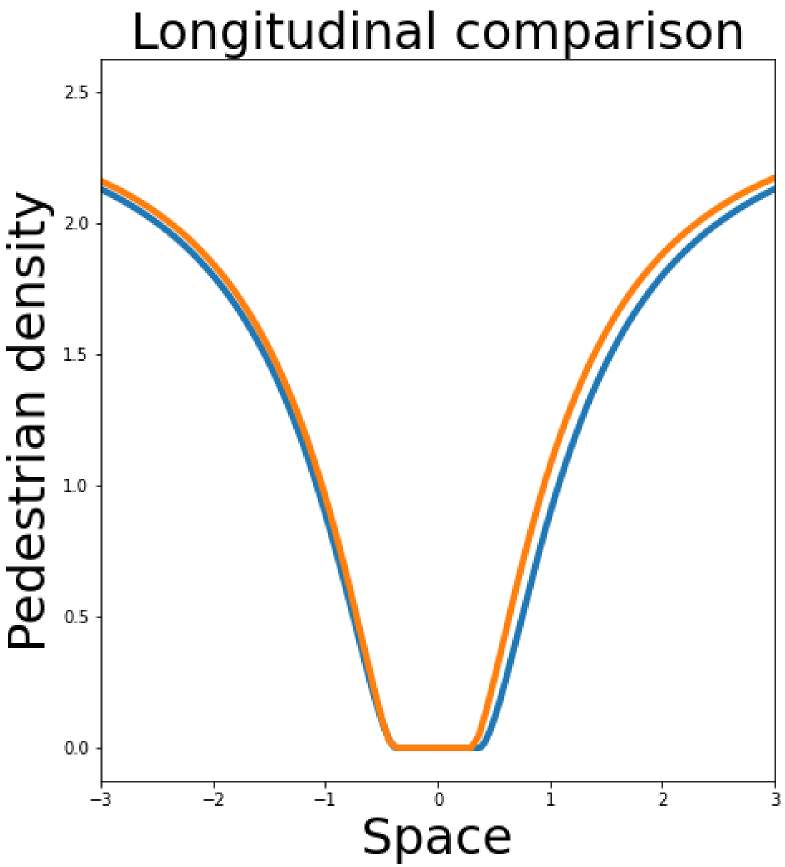
cylinder velocity

$\boxed{\lambda = -g\rho_0}$

And indeed

NB : $(\Gamma \leftrightarrow \phi)$ $[\Leftrightarrow TR]$ + $(y \leftrightarrow -y)$
= exact symmetry for ergodic case





Comparison between time dependent and ergodic simulations along the transverse (x) and parallel (y) directions [$\nu = 0.22, \xi = 0.26$]

Characteristic length and velocity scales

Cylinder :

Radius : R

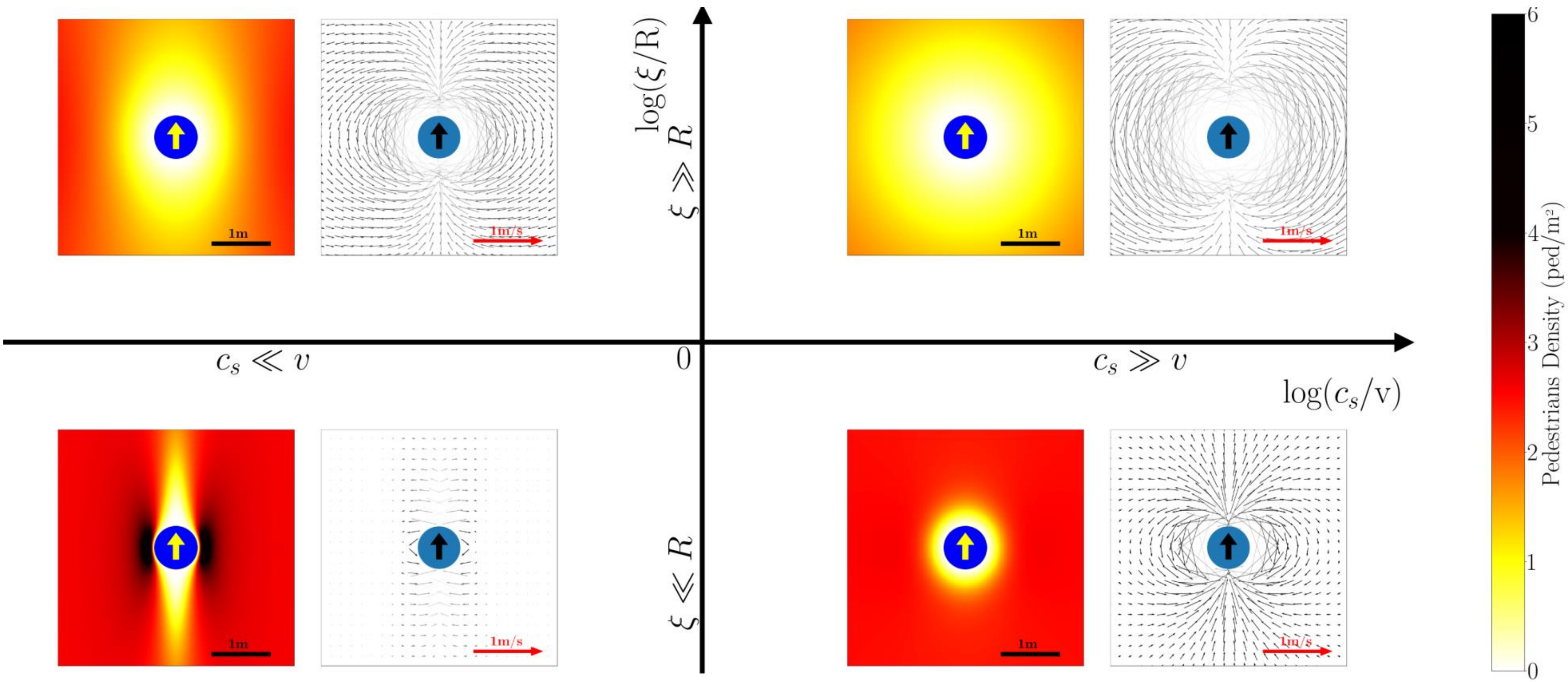
Velocity : v_c

Pedestrians :

Healing length : $\xi = \sqrt{|m\sigma^4/2g\rho_0|}$

Sound velocity : $c_s = \sqrt{|g\rho_0/2m|}$

Up to a scaling, solutions of the ergodic MFG equations depend only the ratios ξ/R and c_s/v .



Comparison between experimental and Mean Field Game simulation

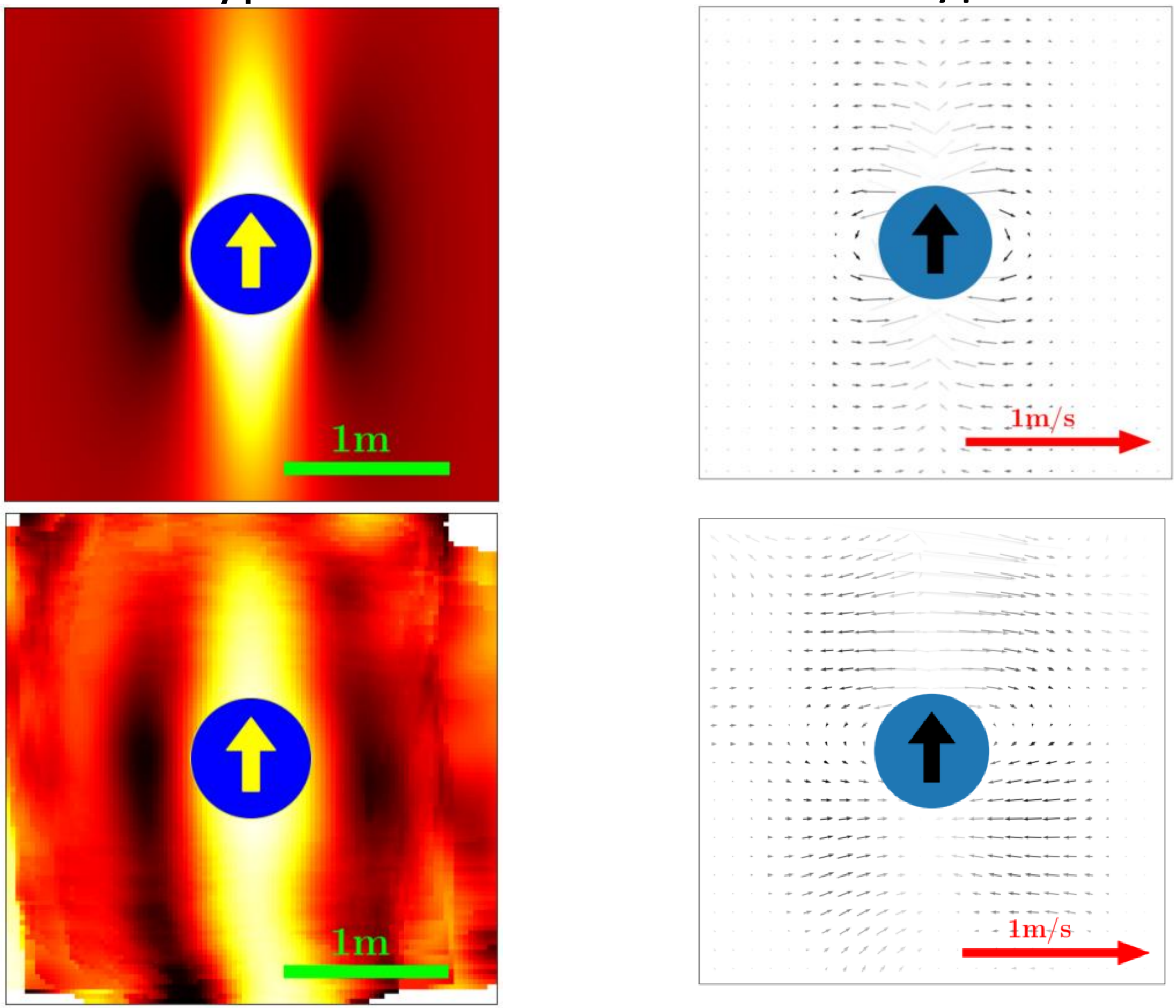
[ν=0.22, ξ=0.26, v=0.6, R=0.32m]

density plot

velocity plot

MFG

Exp



Conclusion for the cylinder crossing experiment

- The Mean Field Game approach reproduces naturally the important qualitative feature of the "crossing cylinder" experiment .

- It does significantly better than the "dynamical" approaches which are generally used to describe crowd behavior. This is especially true for the version found in most commercial software, but also for their more modern/research oriented version.

- This was obtained for the simplest of MFG model.
  Natural evolution could include "discount ratio" and "congestion effects".

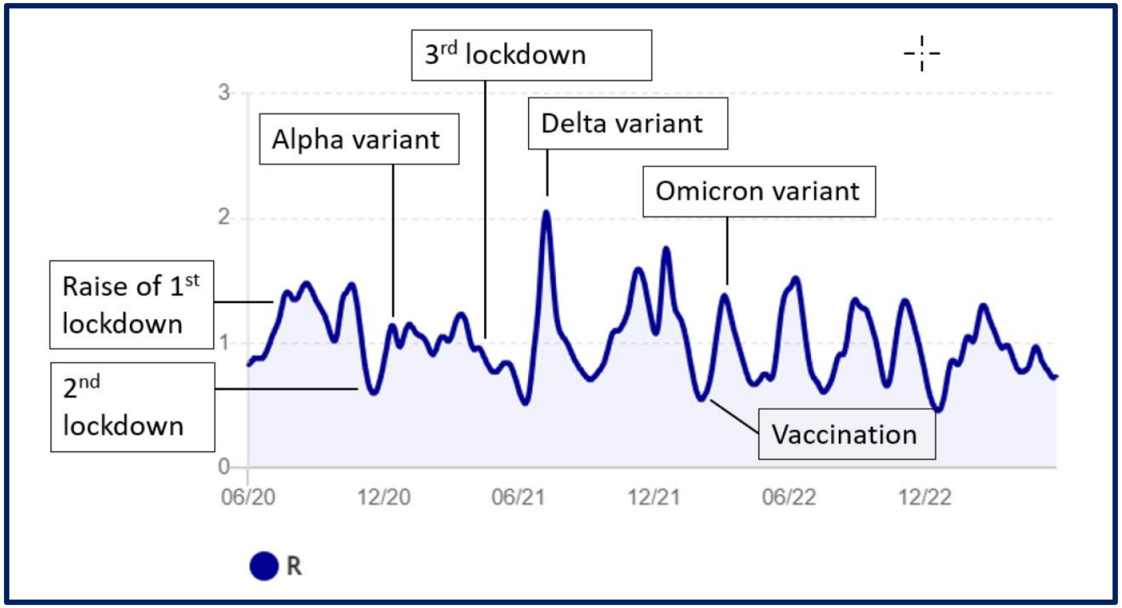
⇨

- MFG appear as a natural tool to model pedestrian crowd dynamics when anticipation is a leading driving force

V - Mean Field Game model of Epidemics

Evolution of the "effective reproduction number" during the Covid-19 pandemic between June 2020 and June 2023

R_{eff} = average number of infected persons by a sick individual.



⇛ Significant variations

- Some with easily identified causes.
- Some of theses causes are biological in nature
- Some others are due to changes in behavior

our focus

A classical model to describe epidemics, the SIR model

$\beta(t)$: transmission rate
γ : recovery rate

- $\beta(t)$: **extrinsic time dependent** functional parameter of the model.
- Hard to fit with experimental data (the dynamics of $\beta(t)$ is coupled to the one of the epidemic itself)

- We would like to **make this parameter intrinsic** (i.e. an output, rather than an input of the model)
  $\Rightarrow$ **Mean Field Game** description
- State variable = status of the agent $\in \{\text{suceptible, infected; recovered}\}$
  = discrete variable
  $\Rightarrow$ **slightly different formalism for the Mean Field Game**

[Elie et al., Mathematical Modelling of Natural Phenomena ***15*** *(2020)]*

Mean Field Game description

State variable : $x \in \{S, I, R\}$

$(S_\alpha, I_\alpha, R_\alpha)$: proportions of (Susceptible, Infected, Recover $[S_\alpha + I_\alpha + R_\alpha = 1]$

Dynamical equation for the epidemics

$$\begin{aligned} \dot{S} &= -\rho\bar{\chi}(t)S(t)I(t) \\ \dot{I} &= \rho\bar{\chi}(t)S(t)I(t) - \xi I(t) \\ \dot{R} &= \xi I(t)\,. \end{aligned}$$

Control variable

- $\beta(t) = \rho\,\chi(t)$
- $\bar{\chi}(t)dt$: pb of encounter in $[t, t+dt]$
  $\Rightarrow$ Control variable
- ρ: proba of infection / encounter

Cost function (Cost paid by individual k susceptible at time t if infected at time τ)

$$C\left(\chi_k(.), t, \tau\right) \equiv \mathfrak{r}_{\mathrm{I}}\mathbb{1}_{\tau<T} + \int_t^{\min(\tau,T)} f\left(\chi_k(s)\right)ds$$

Strategy of agent k

cost of infection

(social) cost of effort

NB : $\tau = F[\bar{\chi}(\cdot)]$ $\Rightarrow$ $\langle C\left(\chi_k(.), t, \tau\right)\rangle_\tau \equiv \mathfrak{C}\left(\chi_k(\cdot), \bar{\chi}(\cdot), t\right)$

Bellman equation

Value function

$$u_k(t) = \begin{cases} \min\limits_{\chi_k(\cdot)} \mathfrak{C}\left(\chi_k(\cdot), t\right), & k \quad \text{susceptible at } t \\ 0, & k \quad \text{infected (or recovered) at } t. \end{cases}$$

Bellman

$$u_k(t) = \min_{\chi(t)} \mathbb{E}_{x_k(t+dt)} \left[u_k(t+dt) + c(t, \chi_k(t))\right]$$

$$c(t) = \begin{cases} f(\chi_k(t))dt & \text{if susceptible at } t+dt \\ \mathfrak{r}_\mathrm{I} & \text{if infected at } t+dt \end{cases}$$

HJB

$$-\frac{du_k}{dt} = \left[\rho\chi_k^*(t)\left(\mathfrak{r}_\mathrm{I} - u_k(t)\right) + f(\chi_k(t))\right]$$

$$\chi_k^*(t) = \underset{\chi_k(t)}{\operatorname{argmin}} \left[\rho\chi_k(t)\left(\mathfrak{r}_\mathrm{I} - u_k(t)\right) + f(\chi_k(t))\right]$$

Optimization at t only

Mean Field Game equations

Dynamics
("Kolmogorov")

$$\dot{S} = -\rho\bar{\chi}(t)S(t)I(t)$$
$$\dot{I} = \rho\bar{\chi}(t)S(t)I(t) - \xi I(t)$$
$$\dot{R} = \xi I(t)\,.$$

Optimization
("HJB")

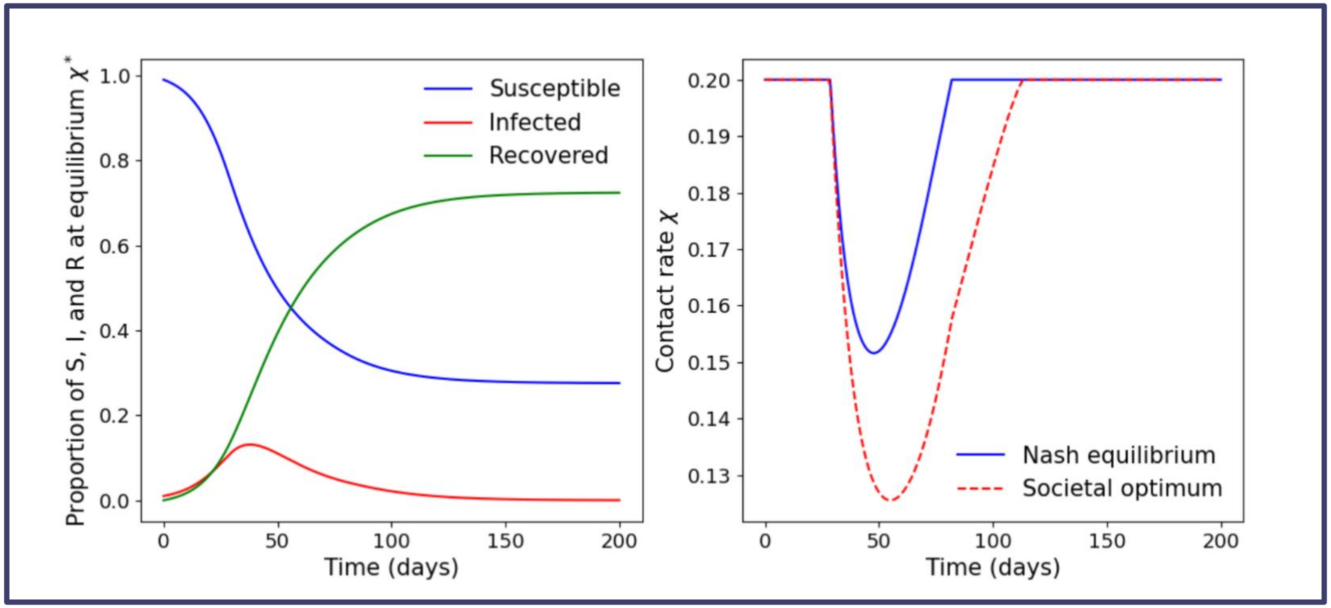
$$-\frac{du_k}{dt} = \left[\rho\chi_k^*(t)\left(\mathfrak{r}_{\mathrm{I}} - u_k(t)\right) + f(\chi_k(t))\right]$$

$$\chi_k^*(t) = \underset{\chi_k(t)}{\operatorname{argmin}}\left[\rho\chi_k(t)\left(\mathfrak{r}_{\mathrm{I}} - u_k(t)\right) + f(\chi_k(t))\right]$$

Self consistence
(Nash equilibrium)

$$\chi_k^*(t) = \bar{\chi}(t)$$

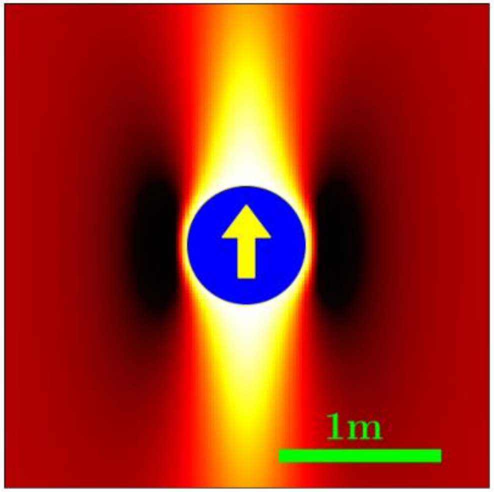
Illustration

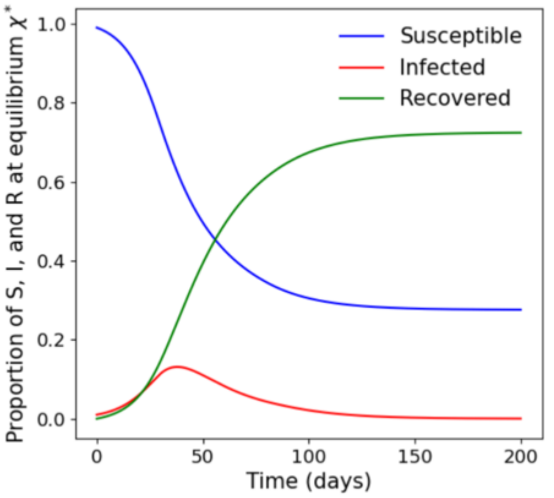


Left : epidemics dynamics of the MFG system at Nash equilibrium, Right : evolution of $\chi_N(t)$ (blue solid) compare to the evolution of $\chi_{SO}(t)$ (red dotted). Figure adapted from [Elie et al (2020)]

- Reverse anticipation
- Difference between Nash and Societal Optimum

To conclude : the threefold ways of Mean Field Games





1. “Learned configuration” : eg pedestrians
2. “Pure case” : eg smart cars / domotique
3. “Coordination through national agency”: eg epidemics