Mean Field Games in the weak noise limit : a semiclassical description

Denis Ullmo
(LPTMS, Orsay & Institut Pascal)

Collaboration with
Thierry Gobron (LPTM-Cergy)   Thibault Bonnemain (LPTM(S))

[Physica A. ***532****, 310 (2019) & "in preparation"]*

Mean Field Games in the weak noise limit : a semiclassical description

Denis Ullmo
(LPTMS, Orsay & Institut Pascal)

Collaboration with
Thierry Gobron (LPTM-Cergy) Thibault Bonnemain (LPTM(S))

*[Physica A. **532**, 310 (2019) & "in preparation"]*

Mean Field Games in the weak noise limit : a semiclassical description

Denis Ullmo
(LPTMS, Orsay & Institut Pascal)

Collaboration with
Thierry Gobron (LPTM-Cergy)   Thibault Bonnemain (LPTM(S))

*[Physica A. **532**, 310 (2019) & "in preparation"]*

Mean Field Games

A mean field game paradigm : model of population dynamics

[Guéant, Lasry, Lions (2011)]

- N agents $i = 1, 2, \cdots, N \quad (N \gg 1)$

- state of agent $i \longrightarrow$ real vector $\mathbf{X}^i$ (here just physical space)

$$m(\mathbf{x}, t) \equiv \frac{1}{N} \sum_{1}^{N} \delta(\mathbf{x} - \mathbf{X}_t^i) \qquad \text{density of agents}$$

- agent's dynamic

$$d\mathbf{X}_t^i = \mathbf{a}_t^i dt + \sigma d\mathbf{w}_t^i$$

$d\mathbf{w}_t^i \equiv$ white noise
drift $\mathbf{a}_t^i \equiv$ control parameter

- agent tries to optimize (by the proper choice of $\mathbf{a}_t^i$) the cost function

$$\int_t^T d\tau \left[\frac{\mu}{2} (\mathbf{a}_\tau^i)^2 - V[m](\mathbf{X}_\tau^i) \right] \; + \; c_T(\mathbf{X}_T^i)$$

Mean Field Games

A mean field game paradigm : model of population dynamics

[Guéant, Lasry, Lions (2011)]

- N agents $i = 1, 2, \cdots, N \qquad (N \gg 1)$
- state of agent $i \longrightarrow$ real vector $\mathbf{X}^i$ (here just physical space)

$$m(\mathbf{x}, t) \equiv \frac{1}{N} \sum_{1}^{N} \delta(\mathbf{x} - \mathbf{X}_t^i) \qquad \text{density of agents}$$

- agent's dynamic

$$d\mathbf{X}_t^i = \mathbf{a}_t^i dt + \sigma d\mathbf{w}_t^i$$

$d\mathbf{w}_t^i \equiv$ white noise
drift $\mathbf{a}_t^i \equiv$ control parameter

- agent tries to optimize (by the proper choice of $\mathbf{a}_t^i$) the cost function

$$\int_t^T d\tau \left[\frac{\mu}{2} (\mathbf{a}_\tau^i)^2 - V[m](\mathbf{X}_\tau^i) \right] \; + \; c_T(\mathbf{X}_T^i)$$

$$U_0(x) + gm(x)$$

Mean Field Game = coupling between a (collective) stochastic motion and an (individual) optimization problem through a mean field $V[m](\mathbf{x}, t)$

- Langevin dynamic $d\mathbf{X}_t^i = \mathbf{a}_t^i dt + \sigma d\mathbf{w}_t^i$ leads to a *(forward)* diffusion equation for the density $m(x, t)$

$$\begin{cases} \partial_t m + \nabla_{\mathbf{x}}(am) - \dfrac{\sigma^2}{2}\Delta_{\mathbf{x}} m = 0 \\ m(x, t{=}0) = m_0(x) \end{cases} \qquad \text{(Kolmogorov)}\ .$$

- Optimization problem, through linear programming, leads to a *(backward)* Hamilton-Jacobi-Bellman equation for the value function $u(\mathbf{x}, t)$

$$\begin{cases} \partial_t u + \dfrac{1}{2\mu}\left(\nabla_{\mathbf{x}} u\right)^2 + \dfrac{\sigma^2}{2}\Delta_{\mathbf{x}} u = V[m](x, t) \\ u(x, t{=}T) = c_T(x) \end{cases} \qquad \text{(HJB)}\ .$$

- Kolmogorov coupled to HJB through the drift $a(x, t) = -\partial_x u(x, t)$

- HJB coupled to Kolmogorov through the mean field $V[m](x, t)$

Mean Field Game = coupling between a (collective) stochastic motion and an (individual) optimization problem through a mean field $V[m](\mathbf{x}, t)$

- Langevin dynamic $d\mathbf{X}_t^i = \mathbf{a}_t^i dt + \sigma d\mathbf{w}_t^i$ leads to a *(forward)* diffusion equation for the density $m(x,t)$

$$\begin{cases} \partial_t m + \nabla_{\mathbf{x}}(am) - \frac{\sigma^2}{2}\Delta_{\mathbf{x}} m = 0 \\ m(x, t=0) = m_0(x) \end{cases} \qquad \text{(Kolmogorov)}\ .$$

- Optimization problem, through linear programming, leads to a *(backward)* Hamilton-Jacobi-Bellman equation for the value function $u(\mathbf{x}, t)$

$$\begin{cases} \partial_t u + \frac{1}{2\mu}(\nabla_{\mathbf{x}} u)^2 + \frac{\sigma^2}{2}\Delta_{\mathbf{x}} u = V[m](x,t) \\ u(x, t=T) = c_T(x) \end{cases} \qquad \text{(HJB)}\ .$$

- Kolmogorov coupled to HJB through the drift $a(x,t) = -\partial_x u(x,t)$

- HJB coupled to Kolmogorov through the mean field $V[m](x,t)$

Recent, applications oriented, mean field game models

- Models for vaccination policies [Laetitia Laguzet, Ph.D. thesis, 2015]
- Price formation process in the presence of high frequency participant [Lachapelle, Lasry, Lehalle, Lions (2015)]
- Load shaping via grid wide coordination of heating-cooling electric loads [Kizilkale and Malhamé, (2015))

The weak noise limit

Mean Field Games equation is the $\sigma^2 \to 0$ limit

$$\begin{cases} \partial_t m + \nabla_{\mathbf{x}}(a m) - \dfrac{\sigma^2}{2}\Delta_{\mathbf{x}} m = 0 & \text{(FP)} \\ \partial_t u - \dfrac{1}{2\mu}\left(\nabla_{\mathbf{x}} u\right)^2 + \dfrac{\sigma^2}{2}\Delta_{\mathbf{x}} u = V[m](x,t) & \text{(HJB)} \end{cases}$$

The weak noise limit

Mean Field Games equation is the $\sigma^2 \to 0$ limit

$$\begin{cases} \partial_t m + \nabla_{\mathbf{x}}(am) - \cancel{\frac{\sigma^2}{2}\Delta_{\mathbf{x}}} m = 0 & \text{(FP)} \\ \partial_t u - \frac{1}{2\mu}(\nabla_{\mathbf{x}} u)^2 + \cancel{\frac{\sigma^2}{2}\Delta_{\mathbf{x}}} u = V[m](x,t) & \text{(HJB)} \end{cases} .$$

$$\Rightarrow \quad \begin{cases} \partial_t m + \nabla_{\mathbf{x}}(am) = 0 & \text{(Transport)} \\ \partial_t u - \frac{1}{2\mu}(\nabla_{\mathbf{x}} u)^2 = V[m](x,t) & \text{(Hamilton Jacobi)} \end{cases}$$

- The $\sigma^2 = 0$ limit of MFG equations leads to a system of coupled **classical** equations, which are more “intuitive”.
- However this limit is often singular, and a small noise is mandatory to **regularize** the theory

- ○ Genuine interest in the small σ^2 limit of the theory
- ○ $\sigma^2 \to 0$ limit formally similar to the quantum $\hbar \to 0$ limit

Outline

A. WKB approximation for the Fokker Planck equation *[Physica A. **532**, 310 (2019)"]*
   1. Maslov approach for the Fokker Planck equation
   2. Application to the seminar problem

B. Thomas Fermi approximation and integrability
   1. Scaling solutions in the long optimization time limit
   2. Schrödinger representation and hydrodynamic formalism
   3. Hodograph transform and integrability

A. The time dependent WKB approximation for the Fokker-Planck equation

Fokker-Planck equation :

$$\partial_t m(\mathbf{x},t) + \nabla(\mathbf{a}(\mathbf{x},t)m(\mathbf{x},t)) - \frac{\sigma^2}{2}\Delta m(\mathbf{x},t) = 0$$

$$(\mathrm{FP})\ \times\sigma^2 \quad \Rightarrow \quad \hat{L}m = 0$$

$$\text{with} \quad \hat{L} \equiv [\lambda^{-1}\partial_t \cdot + \lambda^{-1}\partial_x(a\cdot) - \tfrac{1}{2}(\lambda^{-1}\partial_x)^2\cdot]$$

$$\equiv \lambda\text{-pseudo differential operator}$$

$$(\lambda \equiv \sigma^{-2} \text{ assumed large})$$

Classical symbol : $L(x,t;p,E) = E + pa(x,t) - p^2/2$

$$\boxed{\begin{cases} \dot{t} = \partial_E L = 1 & \dot{E} = -\partial_t L = -p\partial_t a \\ \dot{x} = \partial_p L = a(t,x) - p & \dot{p} = -\partial_x L = -p\partial_x a \end{cases}}$$

A. The time dependent WKB approximation for the Fokker-Planck equation

Fokker-Planck equation :

$$\sigma^2\,\partial_t m(\mathbf{x},t) + \sigma^2\,\nabla(\mathbf{a}(\mathbf{x},t)m(\mathbf{x},t)) - \frac{\sigma^4}{2}\Delta m(\mathbf{x},t) = 0$$

$$\text{(FP)}\ \times\sigma^2 \qquad \Rightarrow \qquad \hat{L}m \;=\; 0$$

$$\text{with}\quad \hat{L} \equiv [\lambda^{-1}\partial_t\,\cdot + \lambda^{-1}\partial_x(a\cdot) - \tfrac{1}{2}(\lambda^{-1}\partial_x)^2\cdot]$$

$$\equiv \lambda\text{-pseudo differential operator}$$

$$(\lambda \equiv \sigma^{-2}\ \text{assumed large})$$

Classical symbol : $L(x,t;p,E) = E + pa(x,t) - p^2/2$

$$\boxed{\begin{cases} \dot{t} = \partial_E L = 1 & \dot{E} = -\partial_t L = -p\partial_t a \\ \dot{x} = \partial_p L = a(t,x) - p & \dot{p} = -\partial_x L = -p\partial_x a \end{cases}}$$

A. The time dependent WKB approximation for the Fokker-Planck equation

Fokker-Planck equation :

$$\lambda^{-1}\partial_t m(\mathbf{x},t) + \lambda^{-1}\nabla(\mathbf{a}(\mathbf{x},t)m(\mathbf{x},t)) - \frac{\lambda^{-2}}{2}\Delta m(\mathbf{x},t) = 0$$

$$\text{(FP)} \times\sigma^2 \qquad \Rightarrow \qquad \hat{L}m = 0$$

$$\text{with} \quad \hat{L} \equiv [\lambda^{-1}\partial_t \cdot + \lambda^{-1}\partial_x(a\cdot) - \tfrac{1}{2}(\lambda^{-1}\partial_x)^2\cdot]$$

$$\equiv \lambda\text{-pseudo differential operator}$$

$$(\lambda \equiv \sigma^{-2} \text{ assumed large})$$

Classical symbol : $L(x,t;p,E) = E + pa(x,t) - p^2/2$

$$\begin{cases} \dot{t} = \partial_E L = 1 & \dot{E} = -\partial_t L = -p\partial_t a \\ \dot{x} = \partial_p L = a(t,x) - p & \dot{p} = -\partial_x L = -p\partial_x a \end{cases}$$

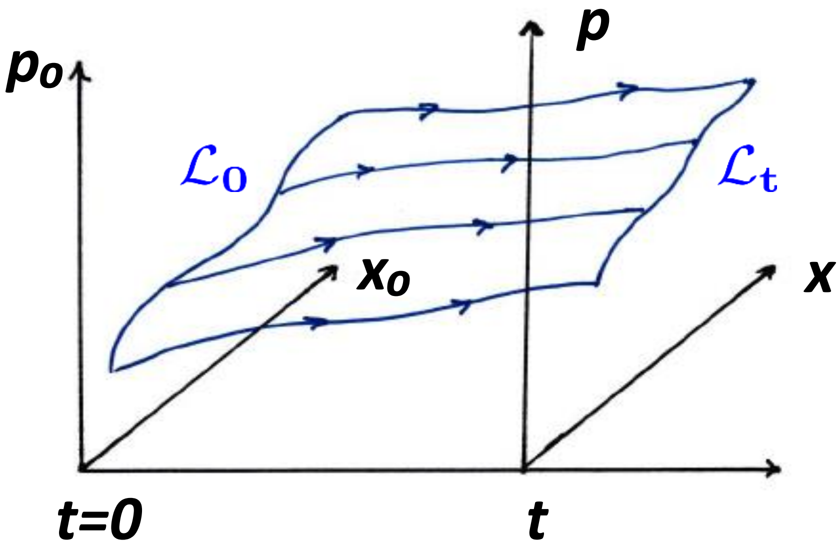
“Semiclassical” initial state : $\mathcal{N}(x_0)\exp\left[\lambda S_0(x_0)\right]$

e.g. Gaussian : $\quad m_0(x_0) = \mathcal{N}\exp\left[-\frac{\mu}{2\sigma^2}(x_0-\bar{x}_0)^2\right]$

$$S_0(x_0) \equiv -\mu\frac{(x_0-\bar{x}_0)^2}{2} \qquad \mathcal{N}\sqrt{\frac{\mu}{2\pi\sigma^2}}$$

$\rightarrow$ associate a Lagrangian manifold (just a curve in 1d)

$$\mathcal{L}_0 \equiv \{(x_0, p_0(x_0) = \partial_x S_0)\} \quad \xrightarrow{\text{Gaussian}} \quad p_0(x_0) = -\lambda(x_0-\bar{x}_0)$$



The WKB evolution is based on the classical evolution of the Lagrangian manifold

[Maslov & Fedoriuk (1981)]

Semiclassical action

$$S(t,x) \equiv \int_{[\mathcal{L}:(0,\bar{x}_0)\to(t,x)]\subset\mathcal{M}} p dx + E dt$$

Semiclassical evolution

$$m_{\mathrm{s.c.}}(t,x) = \frac{\mathcal{N}}{\sqrt{\partial_{x_0} x(t,x_0)}} \exp\left[\lambda S(t,x) - \frac{1}{2}\int_{t_0}^{t}(\partial_x a)d\tau\right] \;,$$

<u>Difference with usual WKB :</u>

- $\lambda = \frac{1}{\hbar} \to \lambda = \frac{1}{\sigma^2}$
- $i\lambda \to \lambda$
- $\hat{L}$ non-hermitic $\Rightarrow$ term $\exp\left[-\frac{1}{2}\int_{t_0}^{t}(\partial_x a)d\tau\right]$

Semiclassical action

$$S(t,x) \equiv \int_{[\mathcal{L}:(0,\bar{x}_0)\to(t,x)]\subset\mathcal{M}} p\,dx + E\,dt$$

Semiclassical evolution

$$m_{\mathrm{s.c.}}(t,x) = \frac{\mathcal{N}}{\sqrt{\partial_{x_0} x(t,x_0)}} \exp\left[\lambda S(t,x) - \frac{1}{2}\int_{t_0}^{t} (\partial_x a)\,d\tau\right] \;,$$

<u>Difference with usual WKB :</u>

- $\lambda = \frac{1}{\hbar} \to \lambda = \frac{1}{\sigma^2}$
- $i\lambda \to \lambda$
- $\hat{L}$ non-hermitic $\Rightarrow$ term $\exp\left[-\frac{1}{2}\int_{t_0}^{t}(\partial_x a)\,d\tau\right]$

Illustration with the drift field of the "seminar problem"

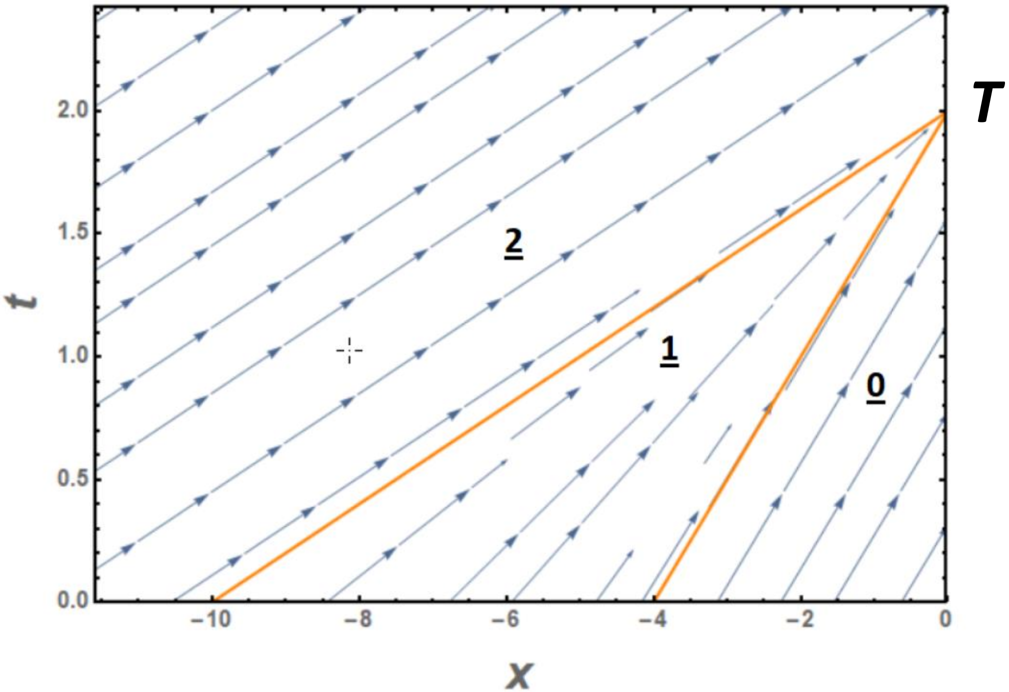


Figure 1 Regions of the (t, x) space, where $T = 2$ is the time when the seminar effectively begins, and their associated optimal drift $a(t, x)$. In regions (0) and (2)the drift stays constant and is denoted respectively $a^{(0)}$ and $a^{(2)}$ (here 5 and 2). In region (1),the drift is linear in x.

$$a(t,x) = \begin{cases} a^{(0)} & \text{for} \qquad x \le -a^{(0)}(T-t) \\ \dfrac{-x}{(T-t)} & \text{for} \qquad -a^{(0)}(T-t) \le x \le -a^{(2)}(T-t) \\ a^{(2)} & \text{for} \qquad -a^{(2)}(T-t) \le x \le 0 \end{cases}$$

Constant drift : *a = const.*

$$S(t,x) = -\left(\frac{\mu t}{1+\mu t}\right)\left(\frac{(x-\bar{x}_0-at)^2}{2t}\right)$$

$$\partial_{x_0}\boldsymbol{x}(t,x_0) = 1 + t\mu \qquad\qquad \partial_x a \equiv 0$$

$$\boxed{m(t,x) = \sqrt{\frac{\mu}{2\pi\sigma^2}}\frac{1}{\sqrt{1+t\mu}}\exp\left[-\left(\frac{\mu t}{1+\mu t}\right)\left(\frac{(x-\bar{x}_0-at)^2}{2t\sigma^2}\right)\right]}$$

$$\xrightarrow{\mu\to\infty} \quad G(t,x,\bar{x}_0) = \sqrt{\frac{1}{2\pi t\sigma^2}}\exp\left[-\frac{(x-\bar{x}_0-at)^2}{2t\sigma^2}\right]$$

- Exact result, even for finite μ
- Neumann or Dirichlet boundary conditions can be implemented

Linear drift : ***a=-x/(T-t)***

$$S(t,x) = \frac{\mu(\bar{x}_0(t-T) - Tx)^2}{2(T-t)(T-t+\mu Tt)} \,.$$

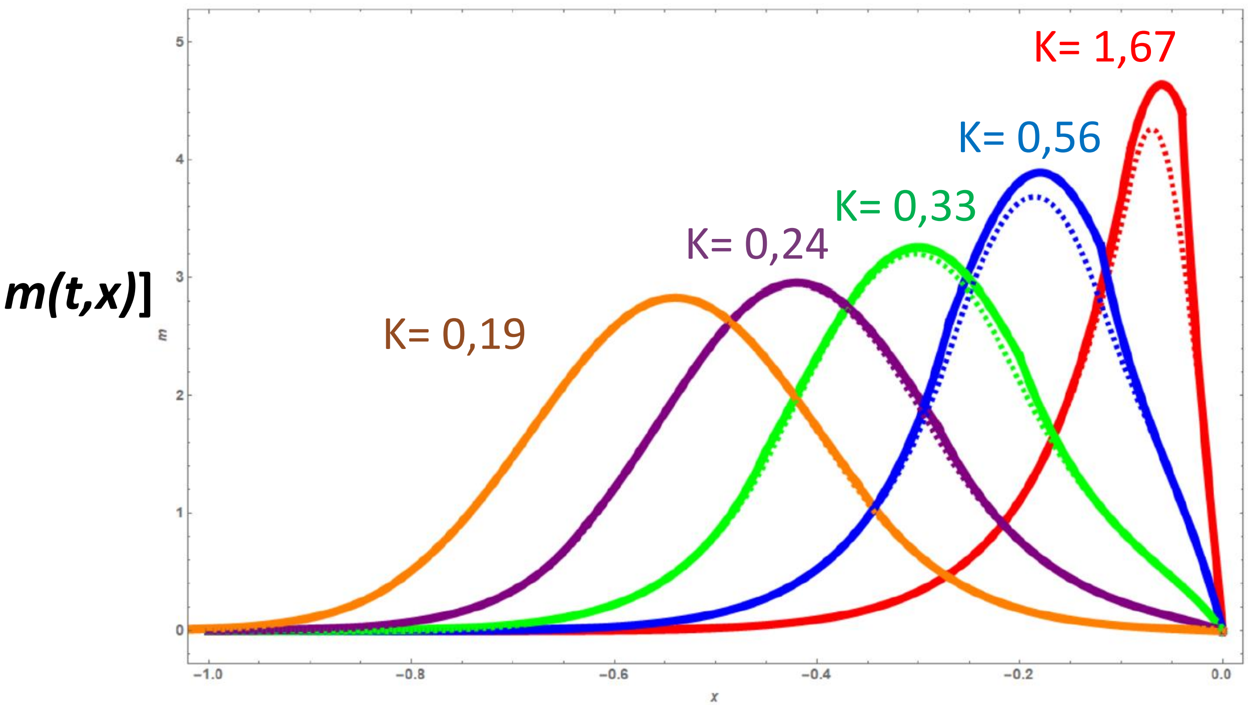
$$\partial x / \partial x_0 = (T - t + \mu t T)/T, \qquad \int_0^t (\partial_x a) dt = \log[(T-t)/T]$$

$$\boxed{m(t,x) = \sqrt{\frac{\mu}{2\pi\sigma^2}} \sqrt{\frac{T^2}{(\mu t T - T - t)(T-t)}} \exp\left[\frac{\mu(xT - (T-t)\bar{x}_0)^2}{2\sigma^2(t-T)(T-t+\mu Tt)}\right]}$$

$$\stackrel{\mu\to\infty}{\longrightarrow} G(t,x,\bar{x}_0) = \sqrt{\frac{T}{2\pi\sigma^2 t(T-t)}} \exp\left(-\frac{T(x - \frac{T-t}{T}\bar{x}_0)^2}{2\sigma^2 t(T-t)}\right)$$

- Exact result again, even for finite μ
- Neumann or Dirichlet boundary conditions can be implemented

Coupling the two solutions [with Dirichlet *m(t,0) = 0* at origin]



$$K = \frac{\tau_{\text{drift}}}{\tau_{\text{diffusion}}} = \left| \frac{\sigma^2}{a x(t, \bar{x}_0)} \right| \propto \sigma^2$$

Figure 4 Spatial distribution of the agents at fixed time, dashed lines show the numerical solution while solid lines show the approximation. From left to right, $K = 0.19$ and $t = 1.1$, $K = 0.24$ and $t = 1.3$, $K = 0.33$ and $t = 1.5$, $K = 0.56$ and $t = 1.7$, $K = 1.67$ and $t = 1.9$. In this case $T = 2$, $a^{(0)} = 0.4$, $a^{(2)} = 0.9$, $\sigma = 0.2$, $\bar{x}_0 = 1.2$ and $= 106$.

B. Thomas Fermi approximation and integrability

- Mean Field Games equations [$a = -\nabla_{\mathbf{x}} u$, $m \equiv$ agent density]

$$\begin{cases} \partial_t m + \nabla_{\mathbf{x}}(am) - \dfrac{\sigma^2}{2}\Delta_{\mathbf{x}} m = 0 \\ m(x,t=0) = m_0(x) \end{cases} \qquad \text{(Fokker-Planck)} \; .$$

$$\begin{cases} \partial_t u - \dfrac{1}{2\mu}\left(\nabla_{\mathbf{x}} u\right)^2 + \dfrac{\sigma^2}{2}\Delta_{\mathbf{x}} u = V[m](x,t) \\ u(x,t=T) = c_T(x) \end{cases} \qquad \text{(HJB)} \; .$$

- Repulsive interaction between the agents

$$V[m](x) = U_0(x) + g m(x) \; ; \quad g < 0$$

B. Thomas Fermi approximation and integrability

- Mean Field Games equations $[a = -\nabla_{\mathbf{x}} u,\ m \equiv \text{agent density}]$

$$\begin{cases} \partial_t m + \nabla_{\mathbf{x}}(am) - \dfrac{\sigma^2}{2}\Delta_{\mathbf{x}} m = 0 \\ m(x, t{=}0) = m_0(x) \end{cases} \qquad \text{(Fokker-Planck)}\ .$$

$$\begin{cases} \partial_t u - \dfrac{1}{2\mu}\left(\nabla_{\mathbf{x}} u\right)^2 + \dfrac{\sigma^2}{2}\Delta_{\mathbf{x}} u = V[m](x,t) \\ u(x, t{=}T) = c_T(x) \end{cases} \qquad \text{(HJB)}\ .$$

- Repulsive interaction between the agents

$$V[m](x) = \cancel{U_0}(x) + gm(x)\ ; \quad g < 0$$

- Strong interaction regime

$$|g\nabla_x m| \gg |\nabla U_0|\ ; \qquad \eta \equiv \frac{\mu\sigma^2}{|g|} \ll L$$

The « ergodic » state

Th : [*Cardaliaguet, Lasry, Lions,Porretta (2013)*]

- No explicit time dependence: $V[m](\boldsymbol{x},\cancel{t})$
- Long time limit for the optimization : $T \to \infty$
- … + other conditions ….

$\exists$ an *ergodic* state $(m_e(\mathbf{x}), u_e(\mathbf{x}), \lambda)$ such that,

$$\text{for } 0 \ll t \ll T \qquad \left| \begin{aligned} m(\mathbf{x},t) &\simeq m_e(\mathbf{x}) \\ u(\mathbf{x},t) &\simeq u_e(\mathbf{x}) + \lambda t \end{aligned} \right.$$

$$(m_e, u_e, \lambda) \text{ such that } \quad \left\{ \begin{aligned} \lambda - \frac{1}{2\mu}\left(\nabla_{\mathbf{x}} u_e\right)^2 + \frac{\sigma^2}{2}\Delta_{\mathbf{x}} u_e &= V[m_e](x) \\ \nabla_{\mathbf{x}}(\bar{m}(\nabla_{\mathbf{x}} u_e)) - \frac{\sigma^2}{2}\Delta_{\mathbf{x}} m_e &= 0 \end{aligned} \right. .$$

E.g. finite box (*d=1*), $U_o(x) \equiv 0$

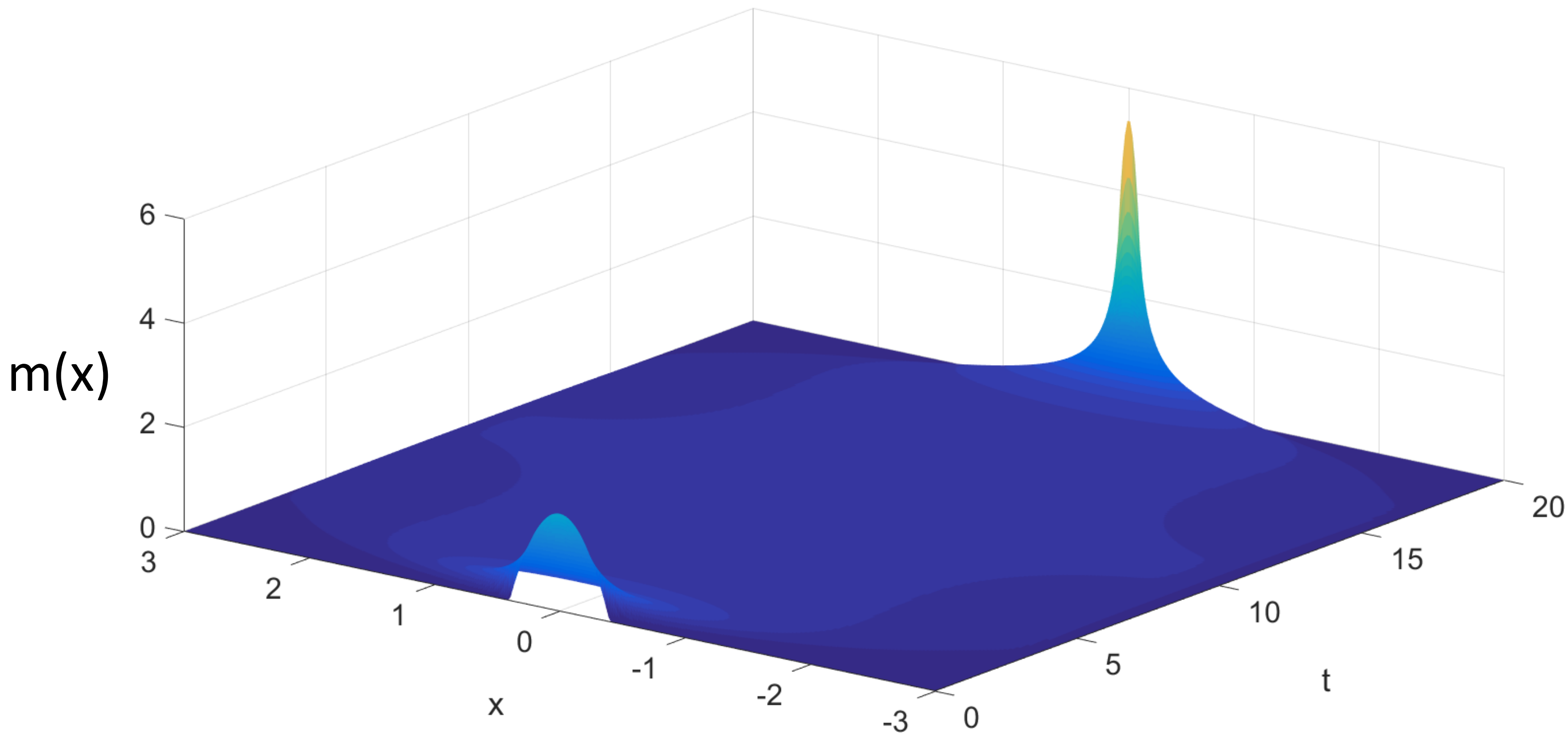


Figure 3 Distribution of the agents as a function of time and poistion, in the long optimization time limit $T \gg 1$, for negative coordination ($g < 0$) and in the abscence of "one-body potential" ($U_0(x) = 0$).

E.g. finite box (*d=1*), $U_o(x) \equiv 0$

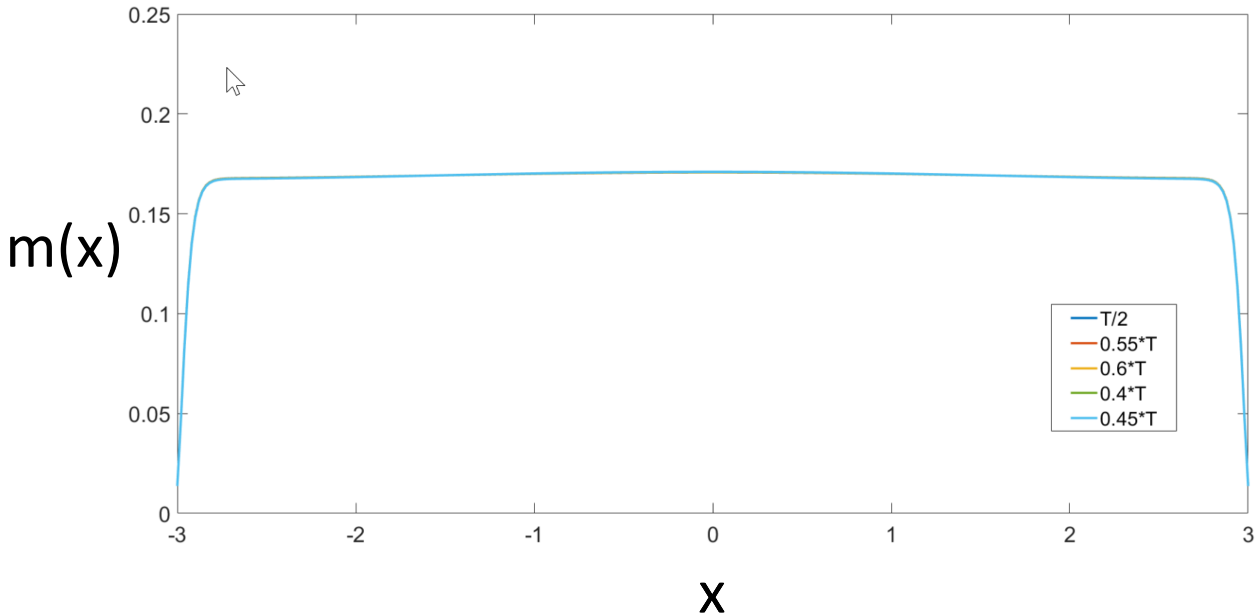
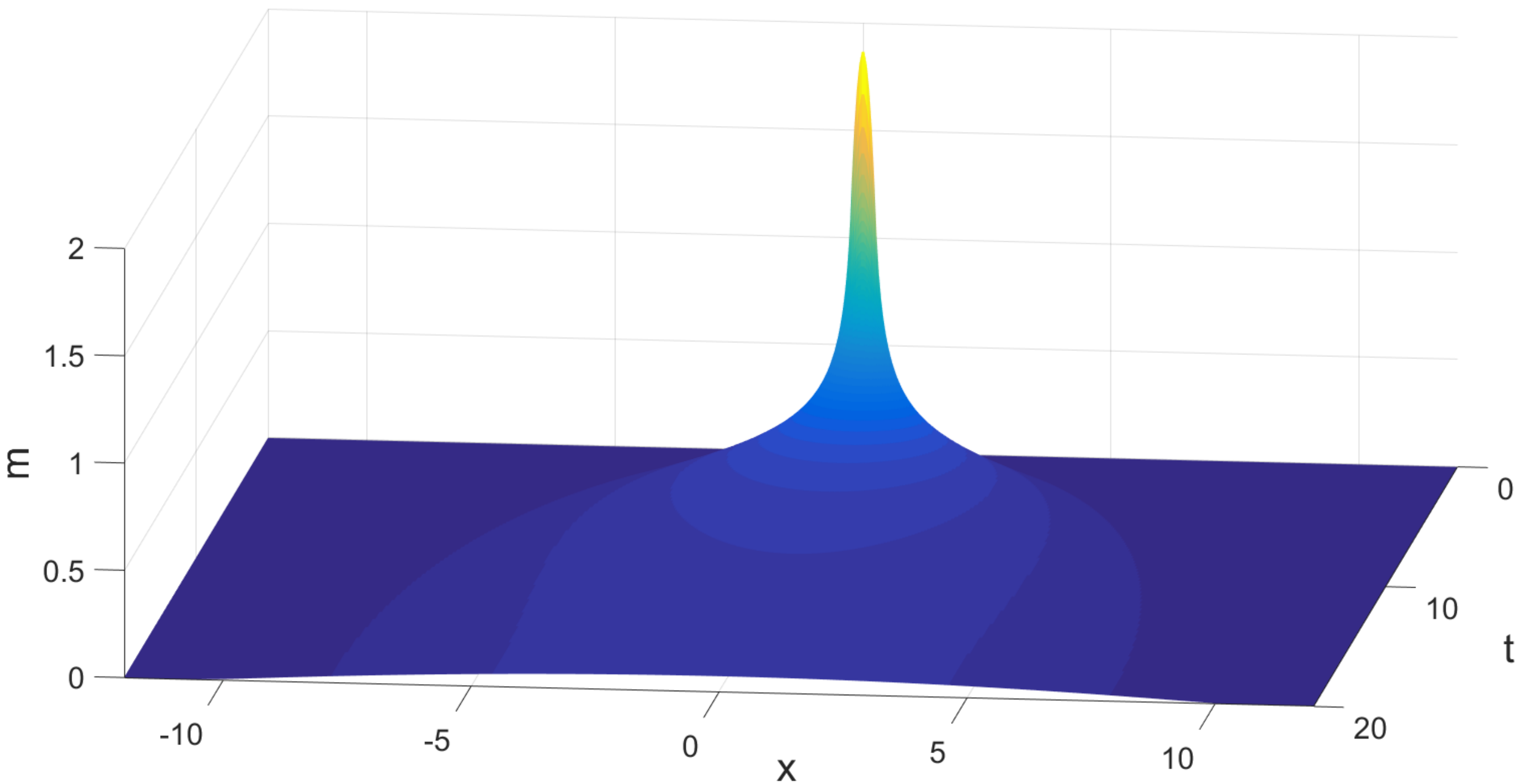


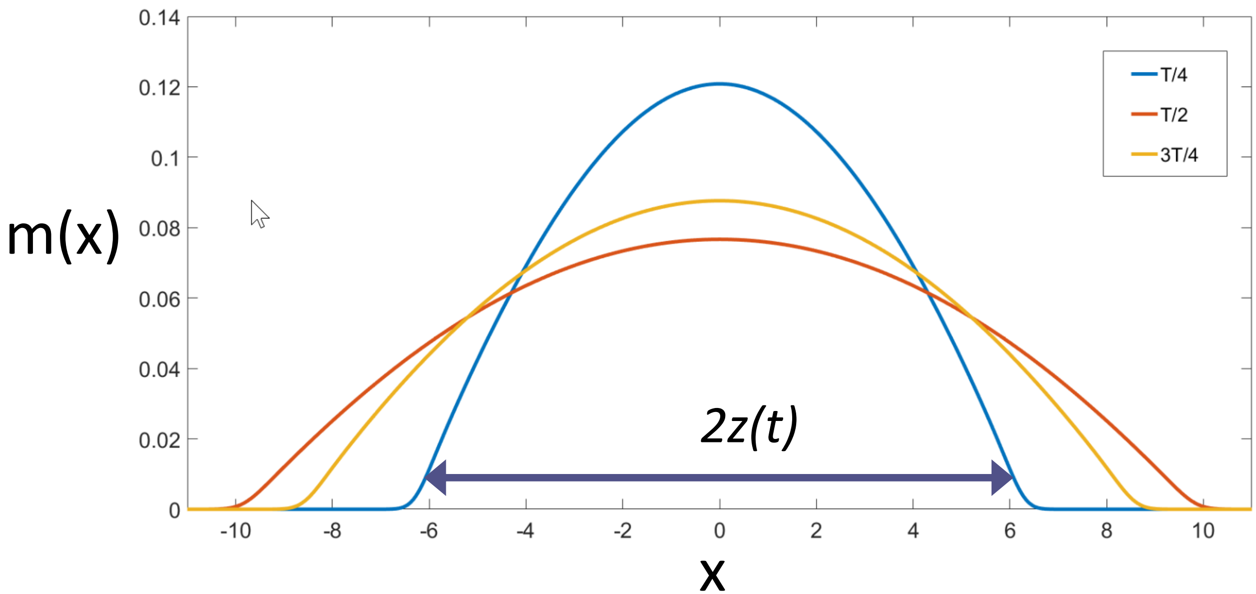
Figure 4 Spatial distribution of the agents at fixed time, in the long optimization time limit $T \gg 1$ ($U_0(x) = 0$). The various curves correspond to $t/T = 0.4, 0.45, 0.5, 0.55, 0.6$

Question : what about infinite boxes ?

"infinite" box (*d=1*), $U_o(x) \equiv 0$



“infinite” box (*d=1*), $U_o(x) \equiv 0$



$$\eta \equiv \frac{\mu\sigma^2}{|g|} \ll z$$

Numercial facts (for typical boundary conditions) :

- $m(x,t)$ looks very much like an inverted parabola

$$m(x,t) \simeq \frac{3(z(t)^2 - x^2)}{4z(t)^3}$$

- its width scale as

$$z(t) \sim t^{2/3}$$

How do we understand this ?

1rst stage : transformation into Non Linear Schrödinger

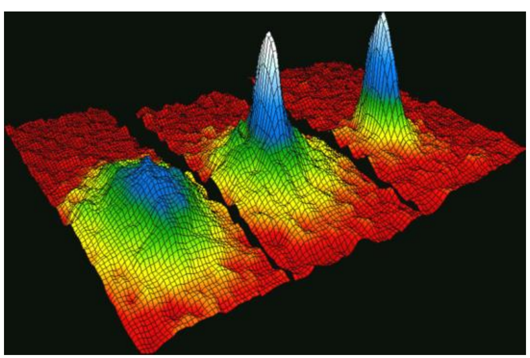
[Swiecicki, Gobron, Ullmo prl (2016), Phys Rep. (2019)]

- Introduce two new variables $\Phi(x,t)$, $\Gamma(x,t)$ defined by :

$$u(x,t) = -\mu\sigma^2 \log\left(\Phi(x,t)\right) \ , \quad m(x,t) = \Gamma(x,t)\Phi(x,t)$$

$$\Rightarrow \begin{cases} \mu\sigma^2 \partial_t \Gamma = \dfrac{\mu\sigma^4}{2}\Delta_{\mathbf{x}}\Gamma + U_0(\mathbf{x})\Gamma + g\, m\Gamma \\ -\mu\sigma^2 \partial_t \Phi = \dfrac{\mu\sigma^4}{2}\Delta_{\mathbf{x}}\Phi + U_0(\mathbf{x})\Phi + g\, m\Phi \end{cases}$$

$$m = \Gamma\Phi$$



$$\boxed{i\hbar\partial_t\Psi = -\frac{\hbar^2}{2\mu}\Delta_{\mathbf{x}}\Psi + U_0(\mathbf{x})\Psi + g|\Psi|^2\Psi}$$

Non-Linear Schrödinger

$$(\Psi, \Psi^*, \hbar) \to (\Phi, \Gamma, i\mu\sigma^2)$$

2cd stage : hydrodynamic variables

- Go back from $(\Phi(x,t), \Gamma(x,t)) \longrightarrow (m(x,t), v(x,t))$:

$$v(x,t) = \frac{\sigma^2}{2}\left(\frac{\nabla\Phi(x,t)}{\Phi(x,t)} - \frac{\nabla\Gamma(x,t)}{\Gamma(x,t)}\right)$$

$$\Longrightarrow \quad \begin{cases} \partial_t m + \nabla(mv) = 0 \\ \partial_t v + \nabla\left[\frac{\sigma^4}{2\sqrt{m}}\Delta\sqrt{m} + \frac{v^2}{2} + \frac{g}{\mu}m + U_0(x)\right] = 0 \end{cases}$$

- Riemann invariants nomenclature

$$\xi(x,t) = v(x,t)\ , \qquad \eta(x,t) = \sqrt{\frac{|g|m(x,t)}{\mu}}$$

($\lambda_\pm = \xi \pm i\eta$ are the Riemann invariants)

2cd stage : hydrodynamic variables

- Go back from $(\Phi(x,t), \Gamma(x,t)) \longrightarrow (m(x,t), v(x,t))$:

$$v(x,t) = \frac{\sigma^2}{2}\left(\frac{\nabla\Phi(x,t)}{\Phi(x,t)} - \frac{\nabla\Gamma(x,t)}{\Gamma(x,t)}\right)$$

$$\Longrightarrow \quad \begin{cases} \partial_t m + \nabla(mv) = 0 \\ \partial_t v + \nabla\left[\cancel{\dfrac{\sigma^4}{2\sqrt{m}}}\Delta\sqrt{m} + \dfrac{v^2}{2} + \dfrac{g}{\mu}m + \cancel{U_0(x)}\right] = 0 \end{cases}$$

- Riemann invariants nomenclature

$$\xi(x,t) = v(x,t)\ , \qquad \eta(x,t) = \sqrt{\frac{|g|m(x,t)}{\mu}}$$

($\lambda_\pm = \xi \pm i\eta$ are the Riemann invariants)

2cd stage : hydrodynamic variables

- Go back from $(\Phi(x,t), \Gamma(x,t)) \longrightarrow (m(x,t), v(x,t))$:

$$v(x,t) = \frac{\sigma^2}{2}\left(\frac{\nabla\Phi(x,t)}{\Phi(x,t)} - \frac{\nabla\Gamma(x,t)}{\Gamma(x,t)}\right)$$

$$\Longrightarrow \begin{cases} \partial_t m + \nabla(mv) = 0 \\ \partial_t v + \nabla\left[\cancel{\frac{\sigma^4}{2\sqrt{m}}}\Delta\sqrt{m} + \frac{v^2}{2} + \frac{g}{\mu}m + \cancel{U_0(x)}\right] = 0 \end{cases}$$

- Riemann invariants nomenclature

$$\xi(x,t) = v(x,t)\ , \qquad \eta(x,t) = \sqrt{\frac{|g|m(x,t)}{\mu}}$$

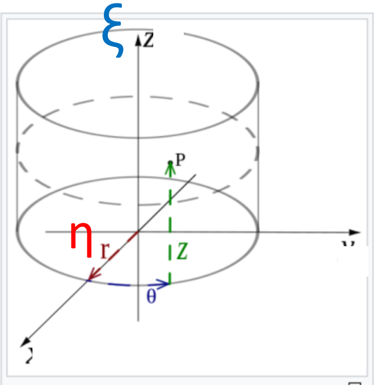
($\lambda_\pm = \xi \pm i\eta$ are the Riemann invariants)

3rd stage : hodograph transform and potential representation

- The equations governing (m, v), (i.e. (η, ξ)) are non linear, but they are first order (ie in some sens linear) in (∂_x, ∂_t)

$$\Rightarrow \text{invert :} \qquad (\eta(x,t), \xi(x,t)) \to (x(\eta,\xi), t(\eta,\xi))$$

- Potential representation : $\chi(\eta, \xi)$



$$\partial_{\xi,\xi}\chi + \partial_{\eta,\eta}\chi + \frac{1}{\eta}\partial_\eta\chi = 0$$

(Laplace equation in cylindrical coordinates)

$$\Rightarrow \qquad \vec{E} = -\nabla V \, , \qquad \begin{cases} E_\eta = -\partial_\eta \chi = -\eta t \\ E_\xi = -\partial_\xi \chi = -2(x - \xi t) \end{cases}$$

- Implementing boundary conditions is however not so straighforward : e.g. at $t = 0$

$$m(x, t\!=\!0) = m_0(x) \quad \Leftrightarrow \quad m(x(\eta,\xi) \text{ s.t. } t(\eta,\xi)\!=\!0) = m_0(x(\eta,\xi))$$

Boundary conditions (eg at *t=0*)

To meet the intial boundary condition, one needs to

- Pick a solution of

$$\partial_{\xi,\xi}\chi + \partial_{\eta,\eta}\chi + \frac{1}{\eta}\partial_\eta \chi = 0$$

- Derive $x(\xi,\eta)$ and $t(\xi,\eta)$ from

$$\begin{cases} E_\eta = -\eta t(\xi,\eta) \\ E_\xi = -2(x(\xi,\eta) - \xi t(\xi,\eta)) \end{cases} \Rightarrow \quad \begin{cases} t = -E_\eta(\xi,\eta)/\eta \\ x = -\frac{1}{2}E_\xi(\xi,\eta) + \frac{\xi}{\eta}E_\eta(\xi,\eta) \end{cases}$$

- Locate the surface $t(\xi,\eta) = 0$
- Check that on that surface that

$$x(\xi,\eta) = (m_0)^{-1}\left[\left(\frac{\mu\eta}{g}\right)^2\right] \qquad \text{since } \eta = \sqrt{\frac{|g|m}{\mu}}$$

Multipolar expansion

- View χ are originating from a distribution of charges $\rho(\xi,\eta)$ located either near the orgin $((t > T))$, or at infinity $((t < 0))$.

- Mulitpolar expansion $(r = \sqrt{\xi^2 + \eta^2})$

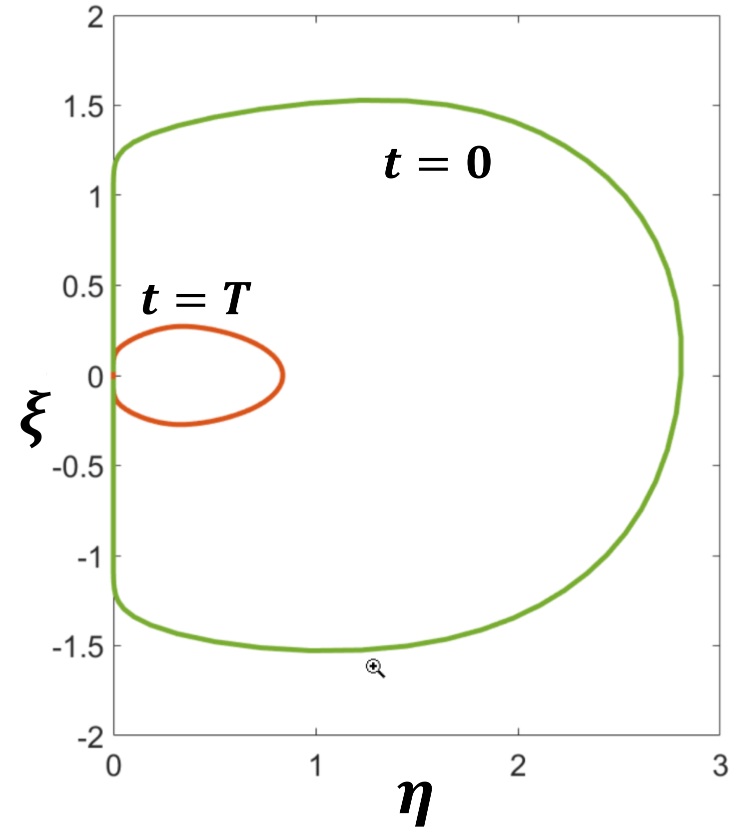
$$\chi(\xi,\eta) = \sum_{l=-\infty}^{+\infty} \frac{Q_l}{r^{l+1}} P_l\left(\frac{\xi}{r}\right)$$

$(l \geq 0 \rightarrow$ charge near origin;

$l < 0 \rightarrow$ charge at infinity$)$

- For $0 \ll t \ll T$: keep only the monopole

$$\chi(\xi,\eta) = \frac{Q_0}{r}$$



Multipolar expansion

- View χ are originating from a distribution of charges $\rho(\xi,\eta)$ located either near the orgin $((t > T))$, or at infinity $((t < 0))$.

- Mulitpolar expansion $(r = \sqrt{\xi^2 + \eta^2})$

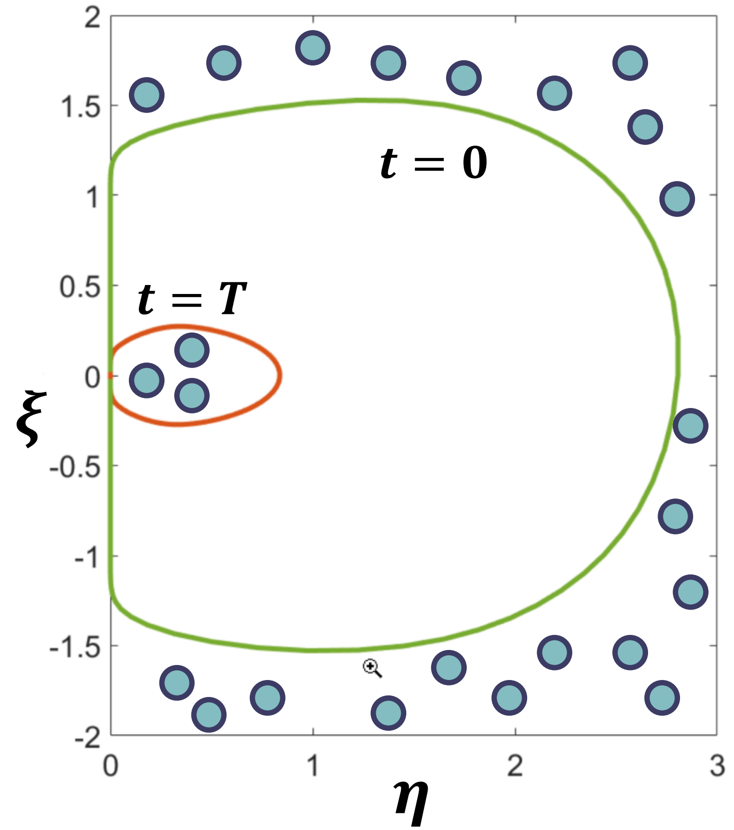
$$\chi(\xi,\eta) = \sum_{l=-\infty}^{+\infty} \frac{Q_l}{r^{l+1}} P_l\left(\frac{\xi}{r}\right)$$

$(l \geq 0 \rightarrow$ charge near origin;

$l < 0 \rightarrow$ charge at infinity$)$

- For $0 \ll t \ll T$: keep only the monopole

$$\chi(\xi,\eta) = \frac{Q_0}{r}$$



Long optimization time limit *T* → ∞

- Monopole potential

$$\chi(\xi,\eta) = \frac{Q_0}{\sqrt{\eta^2+\xi^2}}$$

- Electric field

$$\begin{cases} E_\eta = -\eta \dfrac{Q_0}{(\eta^2+\xi^2)^{3/2}} = -\eta t \\ E_\xi = -\xi \dfrac{Q_0}{(\eta^2+\xi^2)^{3/2}} = -2(x-\xi t) \end{cases} \Rightarrow \begin{cases} t = \dfrac{Q_0}{(\eta^2+\xi^2)^{3/2}} \\ \xi t = 2(x-\xi t) \Rightarrow \xi = \dfrac{2x}{3t} \end{cases}$$

- And thus (remember that $\eta^2 = |g|m/\mu$)

$$\eta^2 = \left(\frac{Q_0}{t}\right)^{2/3} - \left(\frac{2x}{3t}\right)^2 = \frac{3}{2}\frac{\left(Q_0^{2/3}\tilde{z}(t)^2 - x^2\right)}{\tilde{z}(t)^3} \qquad \tilde{z}(t) = \frac{3}{2}t^{2/3}$$

Long optimization time limit *T* → ∞

- Monopole potential

$$\chi(\xi,\eta) = \frac{Q_0}{\sqrt{\eta^2+\xi^2}}$$

- Electric field

$$\begin{cases} E_\eta = -\eta \dfrac{Q_0}{(\eta^2+\xi^2)^{3/2}} = -\eta t \\ E_\xi = -\xi \dfrac{Q_0}{(\eta^2+\xi^2)^{3/2}} = -2(x-\xi t) \end{cases} \Rightarrow \begin{cases} t = \dfrac{Q_0}{(\eta^2+\xi^2)^{3/2}} \\ \xi t = 2(x-\xi t) \Rightarrow \xi = \dfrac{2x}{3t} \end{cases}$$

- And thus (remember that $\eta^2 = |g|m/\mu$)

$$\eta^2 = \left(\frac{Q_0}{t}\right)^{2/3} - \left(\frac{2x}{3t}\right)^2 = \frac{3}{2}\frac{\left(Q_0^{2/3}\tilde{z}(t)^2 - x^2\right)}{\tilde{z}(t)^3} \qquad \tilde{z}(t) = \frac{3}{2}t^{2/3}$$

What about the charge Q_0?

Gauss $$Q_0 = \frac{1}{4\pi}\int_{S_{\tilde{t}}} (\vec{E}\cdot\vec{n})dS$$

$$\begin{cases} E_\eta = -\eta t \\ E_\xi = -2(x-\xi t) \end{cases}$$

$$dS = \eta\sqrt{(\partial_x\xi)^2 + (\partial_x\eta)^2}\,d\theta dx \qquad \hat{n} = \frac{1}{\sqrt{(\partial_x\xi)^2 + (\partial_x\eta)^2}}\Big((\partial_x\xi)\hat{\xi} - (\partial_x\eta)\hat{\eta}\Big)$$

$$Q_0 = \frac{1}{2}\int_{-\infty}^{+\infty}\Bigg[-\tilde{t}\,\underbrace{\partial_x(\eta^2\xi)}_{\to 0} + 2\,\underbrace{x\eta\partial_x\eta}_{\to x\partial_x(\eta^2)}\Bigg]dx = \frac{2g}{\mu}\int_{-\infty}^{+\infty} m(x)dx$$

$$\Rightarrow \quad \boxed{m(x,t) = \frac{3}{4}\frac{\left(z(t)^2 - x^2\right)}{z(t)^3} \qquad z(t) = \frac{3}{2}\left(\frac{2|g|}{\mu}\right)^{1/3} t^{2/3}}$$

$$\eta^2 = \frac{|g|m}{\mu}$$

Multipolar coefficients Q_n

- The Mean Field Game equations with $U_0(x) \equiv 0$ are integrable with the "charges" Q_n as constant of motions
- Q_n $n \geq 0$ ➔ "usual" integral of motion in the language of Non-Linear Schrödinger Equation (eg Q_1 → momentum, Q_2 → energy)
- Q_n $n < 0$ ➔ "new" integral of motion (eg : Q_{-2} → initial time)

Conclusion

- Limit $\sigma^2 \to 0$ of Mean Field Games equations share a lot of formal similarity with the quantum $\hbar \to 0$ limit
- For the Fokker Planck equation, a WKB approach "à la Maslov" can be constructed with minor modifications, an work beautifully well.
- In the "large repulsive interaction" limit where $U_0(x)$ can be neglected, the system of coupled Mean Field Games equations become integrable.
- The (semiclassical) Thomas Fermi regime can then be addressed using a potential representation in which the "charges" of the multipolar expansion are the integral of motion.
- In the large optimization time T limit, and for $0 \ll t \ll T$ regime, keeping only the monopolar term leads to the scaling behavior observed numerically.