Non-Linear Schrödinger approach to Mean Field Games

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Collaboration with

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[& arXiv:1708.07730 (August 2017)]

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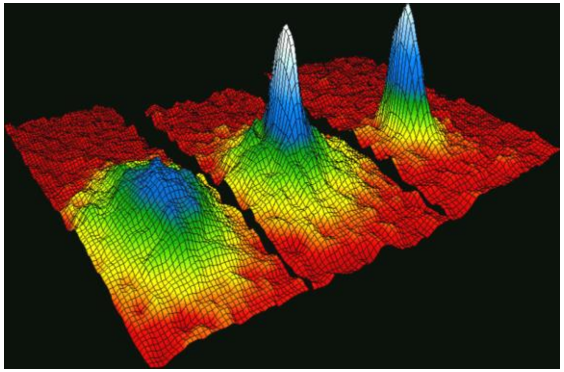
First very general question : What can physicists bring to the study of Mean Field Games ?

Seen from a physics laboratory, it seems there are two main avenues of research for MFG :

- Internal consistency of the theory, existence and uniqueness of solutions to the MFG equations, introduction of new tools making it possible to extend the theory to more complex setups
  [cf Monday to Wednesday morning sessions]
- Exact solutions, either through numerical schemes or for simple models
  [eg: yesterday morning session, Luca Nenna's talk, etc..]

Approach we (physicist) try to promote:

- develop a more "qualitative" understanding of the solutions of the MFG equations :
  - extract characteristic scales,
  - find explicit solutions in limiting regimes,
  - etc..
- Facilitated for "*quadratic*" MFG thanks to the connection with Non-linear Schrödinger equation.



Rubidium atoms (170 nK)

- Interacting bosons in the mean field approximation
- Non-linear optic
- Superconductivity
- Etc ..

Outline

A. Mapping to the Non-Linear Schrödinger equation for Quadratic mean field games *[prl]*

B. A case study: a quadratic mean field game in the strong positive coordination regime *[prl]*

C. Fine points *[arXiv:1708.07730]*

- Collapse
- Mutimodal initial densities

D. Perturbations *[arXiv:1708.07730]*

A. Quadratic mean field game & non-linear Schrödinger equation

Quadratic mean field games

control

white noise

- N agents, state $\mathbf{X}^i \in \mathbb{R}^n$ with Langevin dynamics $d\mathbf{X}^i_t = \mathbf{a}^i_t dt + \sigma d\mathbf{w}^i_t$
- cost function $\int_t^T d\tau \left[\frac{\mu}{2}(\mathbf{a}^i_\tau)^2 - V[m](\mathbf{X}^i_\tau, \tau)\right] + c_T(\mathbf{X}^i_T)$
- System of coupled pde's $[a(\mathbf{x},t) = -\nabla_{\mathbf{x}} u(\mathbf{x},t)$, $m(\mathbf{x},t) \equiv$ density of agents$]$

$$\begin{cases} \partial_t m + \nabla_{\mathbf{x}}(am) - \dfrac{\sigma^2}{2}\Delta_{\mathbf{x}} m = 0 \\ m(x, t{=}0) = m_0(x) \end{cases} \qquad \text{(Kolmogorov)} .$$

$$\begin{cases} \partial_t u - \dfrac{1}{2\mu}(\nabla_{\mathbf{x}} u)^2 + \dfrac{\sigma^2}{2}\Delta_{\mathbf{x}} u = \nabla_{\mathbf{x}} V[m](x,t) \\ u(x, t{=}T) = c_T(x) \end{cases} \qquad \text{(HJB)} .$$

Quadratic MFG represent clearly a small subclass of Mean Field Games, but, this subclass is large enough that :

- One cannot expect explicit solutions for all them
- It includes monotone systems as well as non-monotone systems
- It includes potential MFG as well as non-potential MFG

⇨

- **A priori, a non trivial problem**
- **There is a possibility to be at some level representative of a larger class of MFG**

Particular interest for long optimization time limit & relaxation to « ergodic » state

Th : [*Cardaliaguet, Lasry, Lions,Porretta (2013)*]

- No explicit time dependence: $V[m](\boldsymbol{x},\cancel{t})$
- Long time limit for the optimization : $T \rightarrow \infty$
- … + other conditions ….

$\Rightarrow$

$\exists$ an *ergodic* state $(m_e(\mathbf{x}), u_e(\mathbf{x}), \lambda)$ such that,

$$\text{for } 0 \ll t \ll T \qquad \left| \begin{array}{l} m(\mathbf{x},t) \simeq m_e(\mathbf{x}) \\ u(\mathbf{x},t) \simeq u_e(\mathbf{x}) + \lambda t \end{array} \right.$$

$$(m_e, u_e, \lambda) \text{ such that } \quad \left\{ \begin{array}{r} \lambda - \dfrac{1}{2\mu}\left(\nabla_{\mathbf{x}} u_e\right)^2 + \dfrac{\sigma^2}{2}\Delta_{\mathbf{x}} u_e = V[m_e](x) \\ \nabla_{\mathbf{x}}(\bar{m}(\nabla_{\mathbf{x}} u_e)) - \dfrac{\sigma^2}{2}\Delta_{\mathbf{x}} m_e = 0 \end{array} \right. .$$

Transformation to NLS

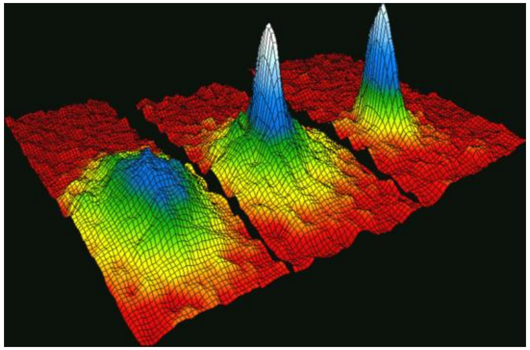
- Cole-Hopf transform: $\Phi(\mathbf{x},t) = \exp\left(-\frac{1}{\mu\sigma^2}u(\mathbf{x},t)\right)$

$$\Longrightarrow \quad \boxed{-\mu\sigma^2\partial_t\Phi = \frac{\mu\sigma^4}{2}\Delta_{\mathbf{x}}\Phi + V[\mathbf{x},m]\Phi}$$

- "Hermitization" of Kolmogorov: $\Gamma(\mathbf{x},t) \equiv m(\mathbf{x},t)\exp\left(u(\mathbf{x},t)/(\mu\sigma^2)\right)$ (i.e. $m(\mathbf{x},t) = \Gamma(\mathbf{x},t)\Phi(\mathbf{x},t)$)

$$\sigma^2\partial_t\Gamma - \frac{\sigma^4}{2}\Delta_{\mathbf{x}}\Gamma = \frac{\Gamma}{\mu}\underbrace{\left(\frac{\partial u}{\partial t} - \frac{1}{2\mu}\left(\nabla_{\mathbf{x}}u\right)^2 + \frac{\sigma^2}{2}\Delta_{\mathbf{x}}u\right)}_{V[\mathbf{x},m] \quad !!!}$$

$$\Longrightarrow \quad \boxed{\mu\sigma^2\partial_t\Gamma = \frac{\mu\sigma^4}{2}\Delta_{\mathbf{x}}\Gamma + V[\mathbf{x},m]\Gamma}$$



$$\boxed{i\hbar\partial_t\Psi = -\frac{\hbar^2}{2\mu}\Delta_{\mathbf{x}}\Psi + U_0(\mathbf{x})\Psi + g|\Psi|^2\Psi}$$

Non-Linear Schrödinger

- MFG equations, specifying to $V[m](\mathbf{x}) \equiv U_0(\mathbf{x}) + gm(\mathbf{x},t)$

$$\begin{cases} \mu\sigma^2\partial_t\Gamma = \dfrac{\mu\sigma^4}{2}\Delta_{\mathbf{x}}\Gamma + U_0(\mathbf{x})\Gamma + g\,m\Gamma \\ -\mu\sigma^2\partial_t\Phi = \dfrac{\mu\sigma^4}{2}\Delta_{\mathbf{x}}\Phi + U_0(\mathbf{x})\Phi + g\,m\Phi \end{cases}$$

$m = \Gamma\Phi$

Formal change $(\Psi, \Psi^*, \hbar) \to (\Phi, \Gamma, i\mu\sigma^2)$ maps NLS to MFG !!!

Why the excitement ?

- Man Field Games exist since 2005-2006,   the Non-Linear Schrödinger equation since at least the work of Landau and Ginzburg on superconductivity in 1950.
- NSL applies to many field of physics : superconductivity, non-linear optic, gravity waves in inviscid fluids, Bose-Einstein condensates, etc..

  → huge literature on the subject
- We feel we have a good qualitative understanding of the "physics" of NLS, together with a large variety of technical tools to study its solutions.

[NB : Change of variable giving NLS known by Guéant, (2011)]

B. case study: a quadratic mean field game in the strong positive coordination regime

To illustrate how this "transfer of knowledge" works, consider a simple (but non-trivial) quadratic mean field game:

- $d = 1$; local interaction $V[m](x) = U_0(x) + gm(x)$
- Strong positive coordination (large positive g)

(If it helps, think of it as a population dynamics model for an aquatic specie living in a river:

- $U_0(x) \equiv$ intrinsic quality of the location (e.g. for food gathering).
- g measures the protection from predator by other members of the group.
- $T =$ daylight duration, $m_0(x) =$ initial distribution in the morning, $c_T(x) =$ quality of shelter for the night.)

Tool #1 : Heisenberg representation & Ehrenfest relations

Quantum mechanics

- State of the system $\equiv$ wave function $\Psi(x,t)$
- Observables $\equiv$ operators: $\hat{O} = f(\hat{p}, \hat{x})$
- Average $\langle \hat{O} \rangle \equiv \int dx \Psi^*(x) \hat{O} \Psi(x)$
- Hamiltonian $\equiv$ $\hat{H} = \frac{\hat{p}^2}{2\mu} + V(x) = -\frac{\hbar^2}{2\mu}\Delta_x + V(x)$

$$\hat{x} \equiv x\times$$
$$\hat{p} \equiv i\hbar\partial_x$$

$$i\hbar\partial_t\Psi = \hat{H}\Psi \qquad \Rightarrow \qquad i\hbar\frac{d}{dt}\langle \hat{O} \rangle = \langle [\hat{H}, \hat{O}] \rangle$$

$$\Longrightarrow \quad \begin{cases} \frac{d}{dt}\langle \hat{x} \rangle = \frac{1}{\mu}\langle \hat{p} \rangle \\ \frac{d}{dt}\langle \hat{p} \rangle = -\langle \nabla_x V(\hat{x}) \rangle \end{cases} \qquad \text{(Ehrenfest)}$$

Quadratic Mean Field Games

- Operators: $\hat{X} \equiv x \times \qquad \hat{\Pi} \equiv \mu\sigma^2 \partial_x \qquad \hat{O} = f(\hat{\Pi}, \hat{X})$

- Average: $\langle \hat{O} \rangle (t) \equiv \int dx \Gamma(x,t) \hat{O} \Phi(x,t)$

$m = \Gamma\Phi$

$$\Rightarrow \qquad \text{if } \hat{O} = O(\cancel{\hat{\Pi}}, \hat{X}) \qquad \langle \hat{O} \rangle \equiv \int dx\, m(x) O(x)$$

$$\left(\langle \hat{1} \rangle \equiv \int dx\, m(x) = 1 \qquad \langle \hat{X} \rangle \equiv \int dx\, x m(x) \right)$$

- Hamiltonian $\equiv \qquad \hat{H} = -\left(\frac{\hat{\Pi}^2}{2\mu} + V[m](x) \right)$

$$\begin{cases} -\mu\sigma^2 \partial_t \Gamma = \hat{H} \Gamma \\ +\mu\sigma^2 \partial_t \Phi = \hat{H} \Phi \end{cases} \qquad \Rightarrow \qquad \mu\sigma^2 \frac{d}{dt} \langle \hat{O} \rangle = \langle [\hat{O}, \hat{H}] \rangle$$

Exact relations

$$\text{Force operator : } \hat{F}[m_t] \equiv -\nabla_x V[m_t](\hat{X})$$

$$\Sigma^2 \equiv \langle(\hat{X}^2\rangle - \langle\hat{X}\rangle^2) \qquad \Lambda \equiv (\langle\hat{X}\hat{\Pi} + \hat{\Pi}\hat{X}\rangle - 2\langle\hat{\Pi}\rangle\langle\hat{X}\rangle)$$

$$\begin{cases} \dfrac{d}{dt}\langle\hat{X}\rangle = \dfrac{1}{\mu}\langle\hat{\Pi}\rangle \\ \dfrac{d}{dt}\langle\hat{\Pi}\rangle = \langle F[m_t]\rangle \end{cases} \qquad \begin{cases} \dfrac{d}{dt}\Sigma^2 = \dfrac{1}{\mu}\Big(\langle\hat{X}\hat{\Pi} + \hat{\Pi}\hat{X}\rangle - 2\langle\hat{\Pi}\rangle\langle\hat{X}\rangle\Big) \\ \dfrac{d}{dt}\Lambda = -2\langle\hat{X}\hat{F}[m_t]\rangle + 2\langle\hat{\Pi}^2\rangle \end{cases}$$

<u>Local interactions</u> $\quad V[m_t](\mathbf{x}) = U_0(\mathbf{x}) + f[m_t(\mathbf{x})]$

$$\rightarrow \hat{F}[m_t] \equiv \underbrace{\hat{F}_0}_{-\nabla_x U_0} - g\,\nabla_x m_t f'(m_t))$$

$$\langle\hat{F}\rangle = \langle\hat{F}_0\rangle$$

$$\langle\hat{X}\hat{F}\rangle = \langle\hat{X}F_0\rangle - \int d\mathbf{x}\,\mathbf{x} m_t(\mathbf{x}) f'[m_t(\mathbf{x})]$$

Tool #2 : solitons

Ergodic solution

Let $\Psi_{\rm e}(x)$ the solution of the stationary NLS

$$\lambda\Psi_{\rm e} = \frac{\mu\sigma^4}{2}\Delta_x\Psi_{\rm e} + U_0(x)\Psi_{\rm e} + g\,|\Psi_{\rm e}|^2\Psi_{\rm e}$$

$$\text{Define}\quad \begin{cases} \Gamma_{\rm e}(x,t) \equiv \exp\left(+\frac{\lambda}{\mu\sigma^2}t\right)\Psi_{\rm e}(x) \\ \Phi_{\rm e}(x,t) \equiv \exp\left(-\frac{\lambda}{\mu\sigma^2}t\right)\Psi_{\rm e}(x) \end{cases}$$

$$\Rightarrow \quad \text{solution of} \quad \begin{cases} \mu\sigma^2\partial_t\Gamma = \dfrac{\mu\sigma^4}{2}\Delta_{\mathbf{x}}\Gamma + U_0(\mathbf{x})\Gamma + g\,m\Gamma \\ -\mu\sigma^2\partial_t\Phi = \dfrac{\mu\sigma^4}{2}\Delta_{\mathbf{x}}\Phi + U_0(\mathbf{x})\Phi + g\,m\Phi \end{cases}$$

with $m_{\rm e}(x) \equiv \Gamma_{\rm e}(x,t)\Phi_{\rm e}(x,t) = |\Psi_{\rm e}(x)|^2 = \text{const.}$

⇨ **Ergodic solution of the MFG problem**

Limiting case $U_0(x) \equiv 0$ (NB: $g > 0$)

In that case solution of stationary NLS known (bright soliton)

$$\Psi_{\mathrm{e}}(x) = \frac{\sqrt{\eta}}{2} \frac{1}{\cosh\left(\frac{x}{2\eta}\right)}$$

$$\eta \equiv 2\mu\sigma^4/g$$

caracteristic length scale

"Strong coordination" regime

- meaning : variations of $U_0(x)$ on the scale η are small
- ergodic state

$$m_{\mathrm{e}}(x) \simeq \frac{\eta}{4} \frac{1}{\cosh^2\left(\frac{x - x_{\max}}{2\eta}\right)}$$

$$x_{\max} = \mathrm{argmax}[U_0]$$

Tool #3 : action and variational approach

Action

$$S[\Gamma(x,t),\Phi(x,t)] \equiv \int dt\,dx\,\left[\frac{\mu\sigma^2}{2}(\partial_t\Phi\,\Gamma - \Phi\partial_t\Gamma) - \frac{\mu\sigma^4}{2}\nabla\Phi.\nabla\Gamma + U_0(x)\Phi\Gamma + \frac{g}{2}\Phi^2\Gamma^2\right]$$

$$\left[\frac{\delta S}{\delta\Gamma} = 0\right] \Leftrightarrow -\mu\sigma^2\partial_t\Phi = \frac{\mu\sigma^4}{2}\Delta_{\mathbf{x}}\Phi + V[\mathbf{x},m]\Phi$$

$$\left[\frac{\delta S}{\delta\Phi} = 0\right] \Leftrightarrow +\mu\sigma^2\partial_t\Gamma = \frac{\mu\sigma^4}{2}\Delta_{\mathbf{x}}\Gamma + V[\mathbf{x},m]\Gamma$$

- Conserved quantity: $\mathcal{E}_{\text{tot}}\cancel{(t)} \equiv \frac{1}{2\mu}\langle\hat{\Pi}^2\rangle + \langle U_0(\hat{X})\rangle + \langle\hat{H}_{\text{int}}\rangle$
- Variational anzatz $\Longrightarrow$ Ordinary Differential Equations

$$\langle\hat{H}_{\text{int}}\rangle \equiv \frac{g}{2}\int dx\, m_t(x)^2$$

Resulting Generic scenario [for strong positive coordination]

1) Herd formation: extension η, mean position $x_0 = \langle x \rangle_{m_0}$ (very short time process)

2) Propagation of the herd :

   - as a classical particle of mass μ in pot $U_0(x)$
   - initial position: $X(0) = x_0$
   - final momentum: $P(T) = -\partial_x c_T(X(T))$

3) Herd dislocation near $t = T$ (again very short process)

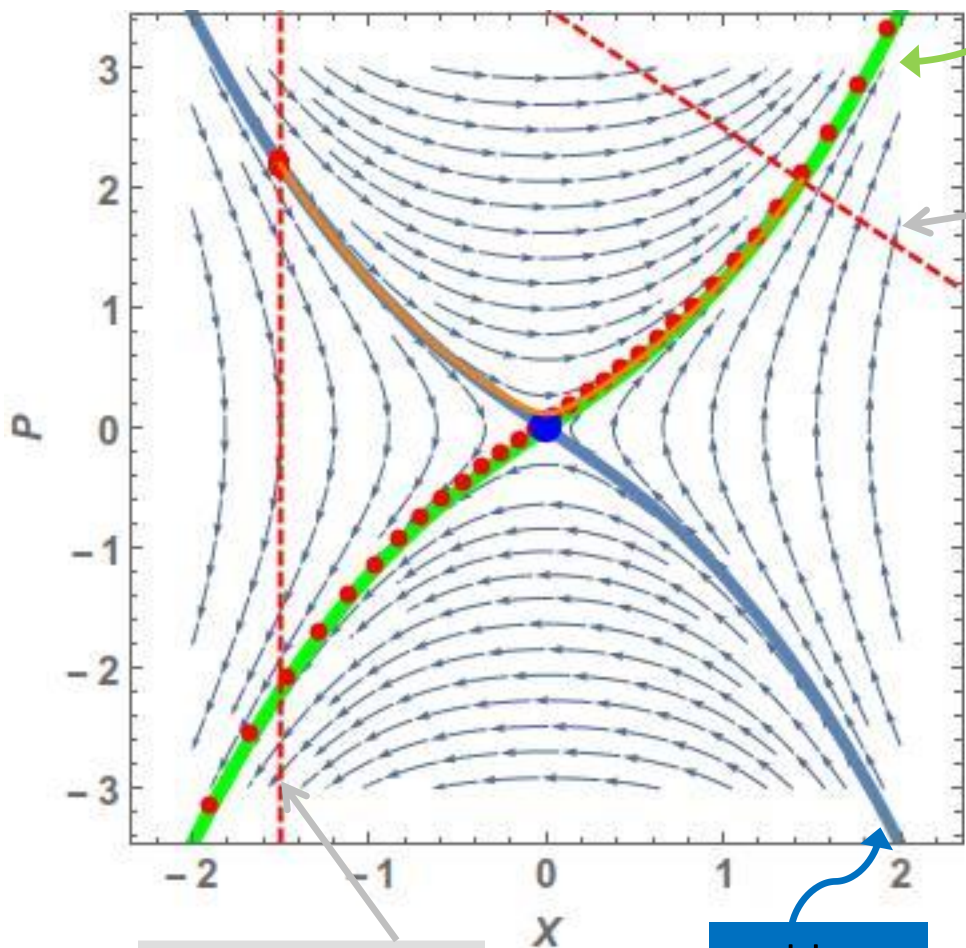
NB: Boundary pb rather than initial valuer pb

- **possibly more than one solution**
- $[T \to \infty]$ **motion governed by unstable fixed points**

Propagation phase in the long time limit : role of the unstable fix points

unstable manifold

Final condition



$$C_T(x) = -\frac{7}{2}x + \frac{x^2}{2}$$
$$\Downarrow$$
$$P_T = \frac{7}{2} - X_T$$

$$U_0(x) = -\frac{x^4}{4} - \frac{x^2}{2}$$

Initial condition

$[X_0 = -1.5]$

stable manifold

Herd formation

First stage of dynamic = herd formation.

- **It takes place on a short time scale.**
- **Can we be more precise ?**

- Assume initial distribution $m_0(x)$ "featureless", i.e. well characterized by its mean x_0 and variance Σ^2
- Neglect U_0 during the herd formation phase

$\Rightarrow$ variational Ansatz :

$$\Gamma(x,t) = e^{-\gamma(t)/\mu\sigma^2} \frac{\exp\left[-\frac{(x-x_0)^2}{4\Sigma_t^2}(1-\frac{\Lambda_t}{\mu\sigma^2})\right]}{\sqrt{2\pi\Sigma_t}}$$

$$\Phi(x,t) = e^{+\gamma(t)/\mu\sigma^2} \frac{\exp\left[-\frac{(x-x_0)^2}{4\Sigma_t^2}(1+\frac{\Lambda_t}{\mu\sigma^2})\right]}{\sqrt{2\pi\Sigma_t}}$$

Action :

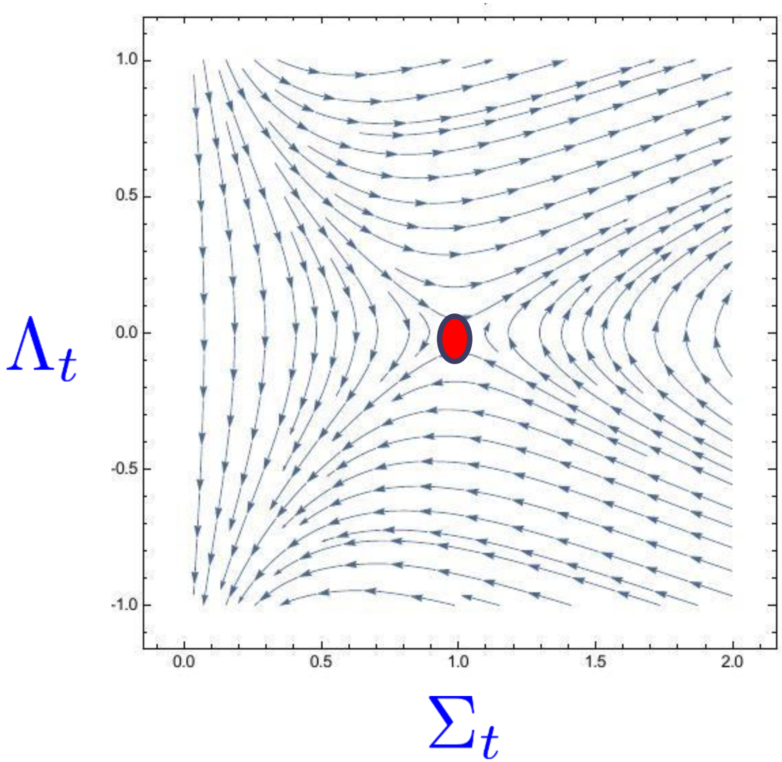
$$S[\Gamma(x,t),\Phi(x,t)] \equiv \int dt\,dx\,\Big[\frac{\mu\sigma^2}{2}(\partial_t\Phi\,\Gamma - \Phi\partial_t\Gamma) - \frac{\mu\sigma^4}{2}\nabla\Phi.\nabla\Gamma + U_0(x)\Phi\Gamma + \frac{g}{2}\Phi^2\Gamma^2\Big]$$

$$\Longrightarrow \begin{cases} \dot{\Sigma}_t^2 = \dfrac{\Lambda_t}{\mu} \\ \dot{\Lambda}_t = -\dfrac{\sigma^4}{2\mu}(1-\dfrac{\Lambda_t^2}{4})\dfrac{1}{\Sigma_t^2} + \dfrac{g}{2\sqrt{\pi}\Sigma_t} \end{cases}$$

$\Longrightarrow$ hyperbolic fixed point : $\Lambda^* = 0 \quad \Sigma^* = \sqrt{\pi}\dfrac{\mu\sigma^4}{g}$

~ soliton scale η

Flow near the fix point



Large T : need to stay on stable and unstable manifold of the fixed point.

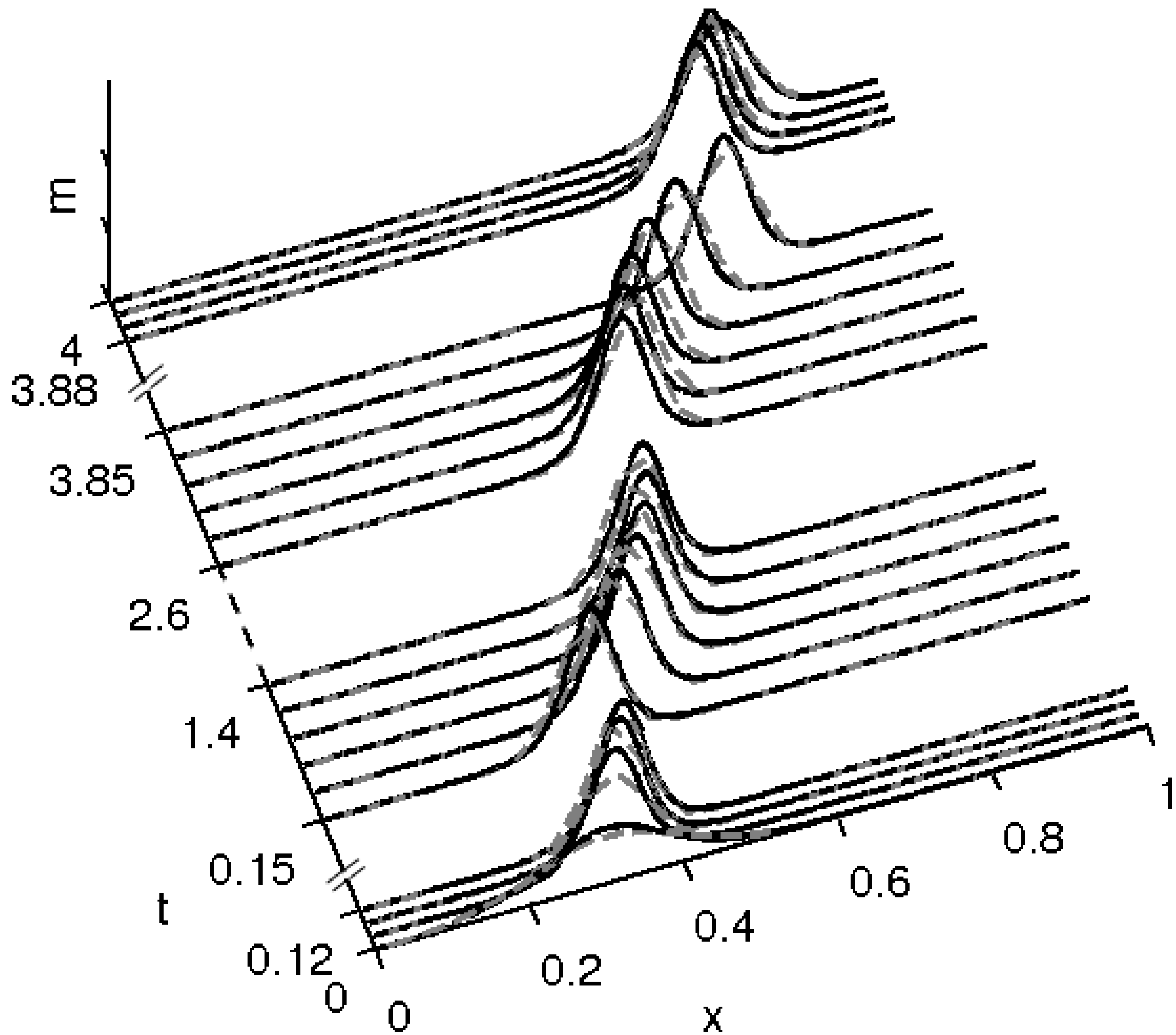
$$\frac{d^2}{dt^2}\Sigma_t^2 = \frac{g}{2\mu\sqrt{\pi}}\left(\frac{1}{\Sigma_*} - \frac{1}{\Sigma_t}\right)$$

$$\boxed{-(z_t - z_i) - \log\left(\frac{1-z_t}{1-z_i}\right) = \frac{t}{\tau^*}}$$

$$z_t \equiv \frac{\Sigma_t}{\Sigma^*} \qquad z_i \equiv \frac{\Sigma_0}{\Sigma^*}$$

$$\tau^* \sim \frac{\Sigma_*}{v_g} \qquad v_g \equiv \frac{\mu\sigma^2}{g}$$

Comparison with numerical simulation



Intermezzo : notion of “qualitative” description

- We clearly can describe in plain English what is happening to the agents (here the fishes) playing the mean field game :
  - Initial formation and final destruction of the heard (short time scale).
  - Beyond this motion as a classical particle in potential $U_0(x)$.
  - Role of the unstable fixed points and of the associated stable and unstable manifold.
- Understanding associated with *accurate* approximation scheme

<u>How much did we actually learn ?</u>

Maybe one could have “guessed” that once the heard will be formed, interaction would become irrelevant, and then

- optimization → classical evolution
- Time scale could be get from dimensional analysis

C. Fine Points [things harder to guess]

1) Collapse

Generalization of the model :

- Higher dimensionality $(d \geq 1)$
- Non linear interaction : $\tilde{V}[m](\mathbf{x}) = U_0(\mathbf{x}) + g\,[m(x)]^{\alpha}$,

Generalized variational ansatz :

$$\Phi(\mathbf{x},t) = \exp\left\{\frac{-\gamma_t + \mathbf{P}_t \cdot \mathbf{x}}{\mu\sigma^2}\right\} \prod_{\nu=1}^{d} \left[\frac{1}{(2\pi(\Sigma_t^{\nu})^2)^{1/4}} \exp\left\{-\frac{(x^{\nu} - X_t^{\nu})^2}{(2\Sigma_t^{\nu})^2}(1 - \frac{\Lambda_t^{\nu}}{\mu\sigma^2})\right\}\right]$$

$$\Gamma(\mathbf{x},t) = \exp\left\{\frac{+\gamma_t - \mathbf{P}_t \cdot \mathbf{x}}{\mu\sigma^2}\right\} \prod_{\nu=1}^{d} \left[\frac{1}{(2\pi(\Sigma_t^{\nu})^2)^{1/4}} \exp\left\{-\frac{(x^{\nu} - X_t^{\nu})^2}{(2\Sigma_t^{\nu})^2}(1 + \frac{\Lambda_t^{\nu}}{\mu\sigma^2})\right\}\right]$$

Center of mass coordinates:

$$\left.\begin{aligned}\dot{X}_t^{\nu} &= \frac{P_t^{\nu}}{\mu} \\ \dot{P}_t^{\nu} &= -\langle \partial^{\nu} U_0(\mathbf{x}) \rangle_t \simeq -\partial^{\nu} U_0(\mathbf{X}_t)\end{aligned}\right\} \Longrightarrow \text{classical motion}$$

Variances and position-momentum correlators

$$\dot{\Sigma}^{\nu} = \frac{\Lambda^{\nu}}{2\mu\Sigma^{\nu}} \,,$$

$$\dot{\Lambda}_t^{\nu} = \frac{(\Lambda_t^{\nu})^2 - \mu^2\sigma^4}{2\mu(\Sigma_t^{\nu})^2} + \frac{2g\alpha}{\alpha+1} \prod_{\nu'=1}^{d} \left[\frac{1}{\sqrt{\alpha+1}(2\pi)^{\alpha/2}} \left(\frac{1}{\Sigma_t^{\nu'}} \right)^{\alpha} \right] .$$

$$\text{fixed point} \quad \begin{cases} \Lambda_*^{\nu} = 0 \\ \Sigma_*^{\nu} = \left[\frac{4\alpha}{\alpha+1} \left(\frac{1}{(\alpha+1)(2\pi)^{\alpha}} \right)^{d/2} \frac{g}{\mu\sigma^4} \right]^{-1/(2-\alpha d)} \end{cases} .$$

Back to $d = 1$ $(\longrightarrow$ critical $\alpha = 2)$

canonical coordinates : $$\begin{cases} \mathsf{q}_t = \dfrac{\Sigma_t}{\Sigma_*}\,, \\ \mathsf{p}_t = -\dfrac{\Sigma_*}{2}\dfrac{\Lambda_t}{\Sigma_t} \end{cases}$$

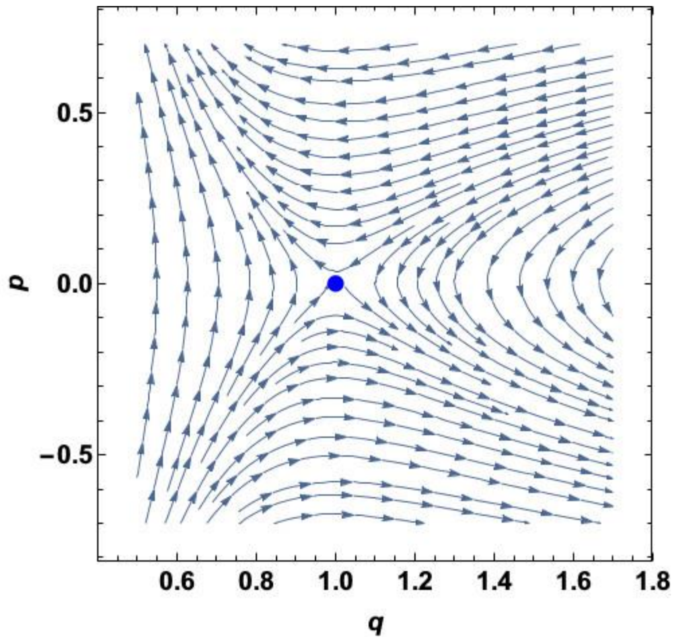
Hamitonian : $$h(\mathsf{p},\mathsf{q}) = -\frac{\mathsf{p}^2}{2\mu\Sigma_*^2} + \frac{\mu\sigma^4}{4\Sigma_*^2}\left[\frac{1}{2\mathsf{q}^2} - \frac{1}{\alpha\mathsf{q}^\alpha}\right]$$

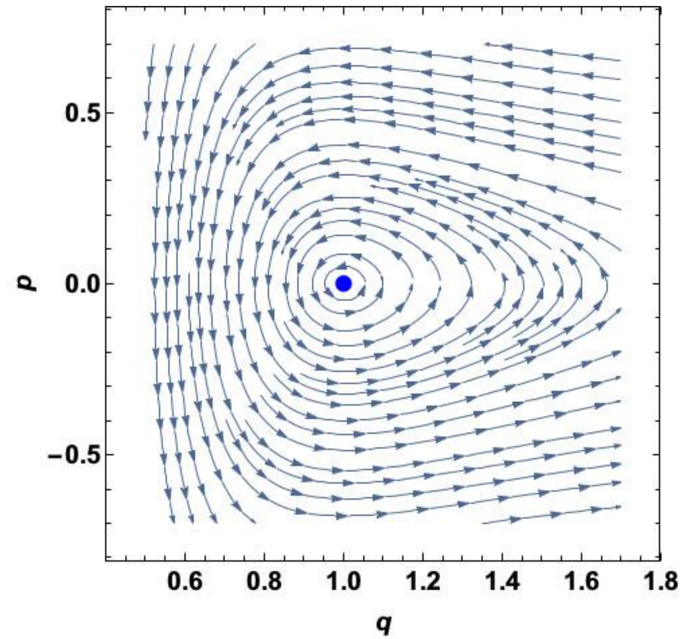
Equation of motion : $$\begin{cases} \dot{\mathsf{q}} = +\dfrac{\partial h(p,q)}{\partial \mathsf{p}} = -\dfrac{p}{\mu\Sigma_*^2} \\ \dot{\mathsf{p}} = -\dfrac{\partial h(p,q)}{\partial \mathsf{q}} = \dfrac{\mu\sigma^4}{4\Sigma_*^2}\left[\dfrac{1}{q^3} - \dfrac{1}{q^{(\alpha+1)}}\right] \end{cases}$$

Fixed point : $(\mathsf{q}_* = 1, \mathsf{p}_* = 0)$

$$\left.\frac{\partial^2 h}{\partial^2 \mathsf{p}}\right|_{\binom{\mathsf{q}_*}{\mathsf{p}_*}} = \frac{-1}{\mu\Sigma_*^2}, \quad \left.\frac{\partial^2 h}{\partial \mathsf{p}\partial \mathsf{q}}\right|_{\binom{\mathsf{q}_*}{\mathsf{p}_*}} = 0, \quad \left.\frac{\partial^2 h}{\partial^2 \mathsf{q}}\right|_{\binom{\mathsf{q}_*}{\mathsf{p}_*}} = \frac{\mu\sigma^4}{4\Sigma_*^2}(2-\alpha)$$

Stability of the fixed point





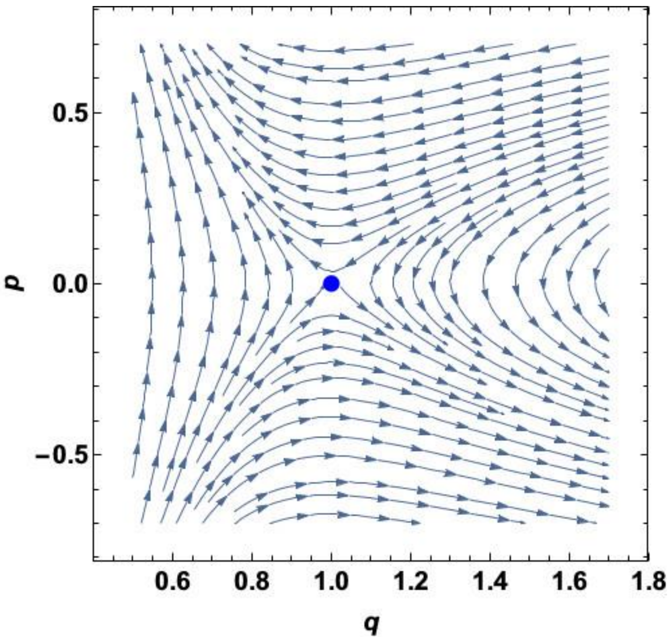
$\alpha < 2$: fixed point = saddle
$\Longrightarrow$ hyperbolic f.p.

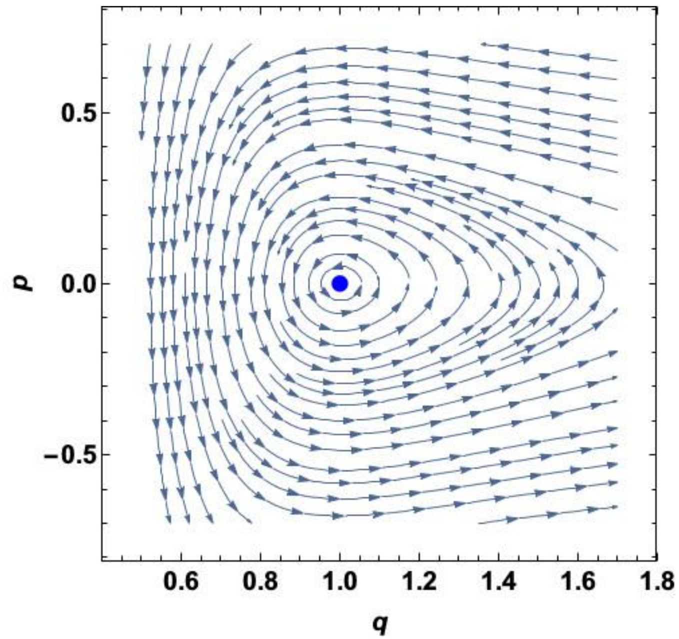
$\alpha > 2$: fixed point = maxima
$\Longrightarrow$ elliptic f.p.

When $\alpha d < 2$

$$\left.\begin{array}{l} \bullet \text{ Interactions dominate at large distances} \\ \bullet \text{ Diffusion dominates at small distances} \end{array}\right\} \Longrightarrow \quad \text{stability}$$

Stability of the fixed point





$\alpha < 2$: fixed point = saddle
$\Longrightarrow$ hyperbolic f.p.

$\alpha > 2$: fixed point = maxima
$\Longrightarrow$ elliptic f.p.

When $\alpha d > 2$

$$\left.\begin{array}{l}\bullet \text{ Interactions dominate at small distances} \\ \bullet \text{ Diffusion dominates at large distances}\end{array}\right\} \Longrightarrow \quad \text{collapse/spreading}$$

Stabilisation of the collapse by a finite range interaction

$(d = 1, \alpha = 3 > 2)$

$V[m](x) = gm(x)^3$ can be seen as the $\xi \to 0$ limit of

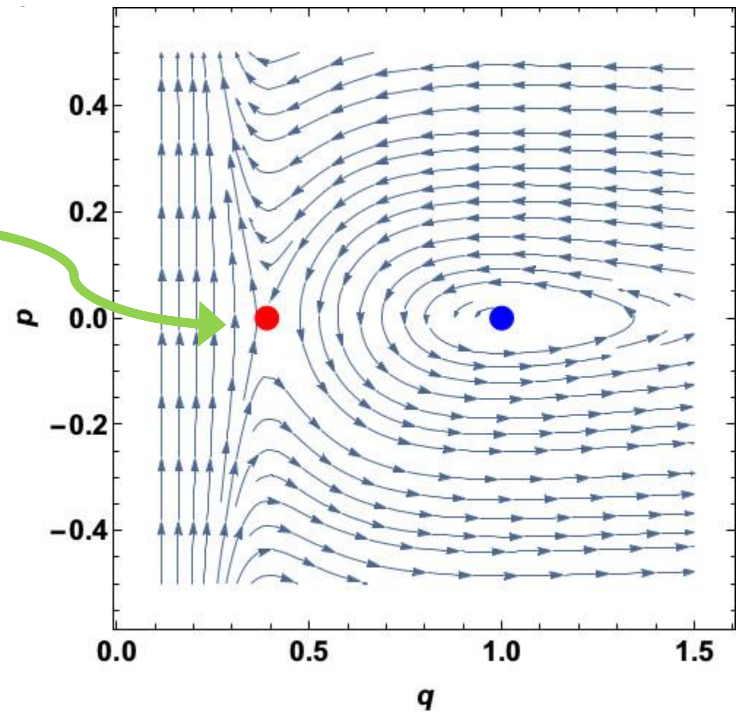
$$V[m](x) = g \int dy_2 dy_3 dy_4 K(x, y_2, y_3, y_4) m(y_2) m(y_3) m(y_4)$$

with $$K(y_1, y_2, y_3, y_4) \equiv \frac{1}{2(\sqrt{2\pi}\xi)^3} \exp\left[-\frac{1}{16\xi^2} \sum_{i \neq j} (y_i - y_j)^2\right]$$

$$\xrightarrow[\xi \to 0]{} \delta(y_1 - y_2)\delta(y_2 - y_3)\delta(y_3 - y_4)$$

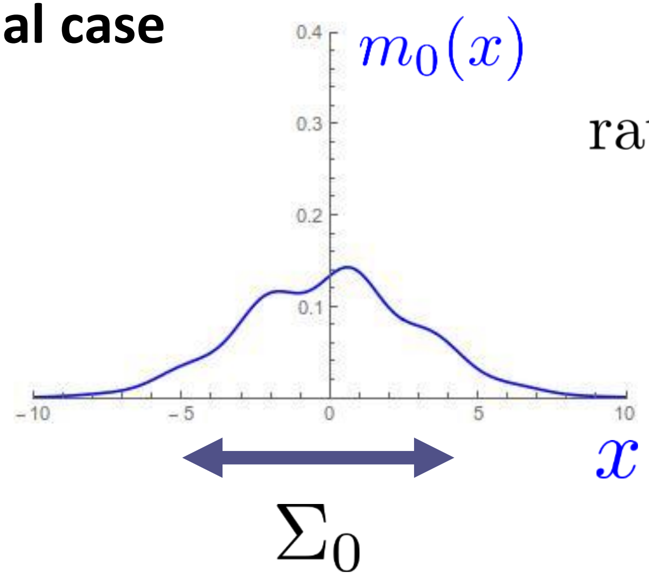
New hyperbolic fixed point

(here $\xi = .3\Sigma_*$)

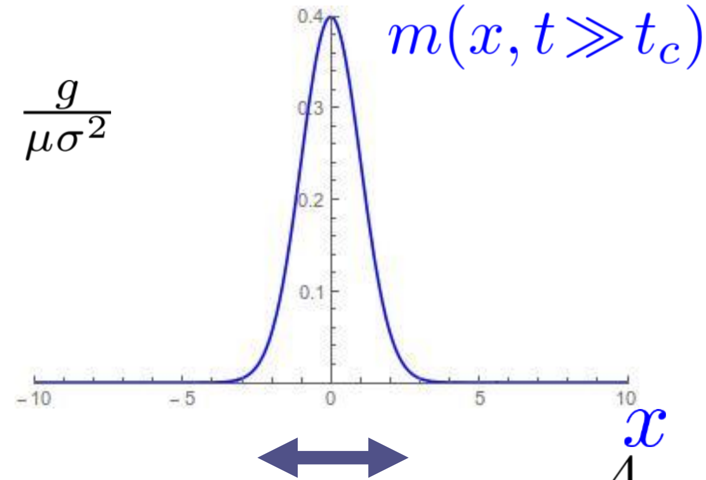


2) Multi-modal initial conditions

Mono-modal case

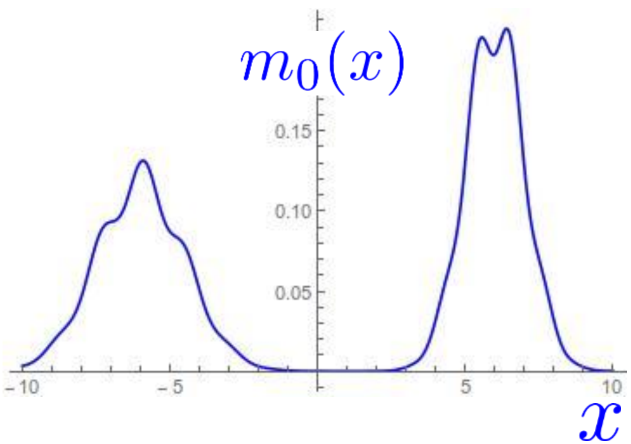


rate $v_g \sim \frac{g}{\mu\sigma^2}$



$$\Sigma_* \sim \frac{\mu\sigma^4}{g}$$

? What if :



Bi-modal case:

$$m_0(x) = \underbrace{m_0^a(x)}_{\rho^a} + \underbrace{m_0^b(x)}_{\rho^b}$$

$$(\rho^a + \rho^b = 1)$$

Variational ansatz:

$$\begin{cases} \Phi(x,t) = \Phi^a(x,t) + \Phi^b(x,t) \ , \\ \Gamma(x,t) = \Gamma^a(x,t) + \Gamma^b(x,t) \ , \end{cases}$$

$$\begin{cases} \Phi^k(x,t) = \sqrt{\rho^k} \exp\left[\frac{-\gamma_t + P_t^k \cdot x}{\mu\sigma^2}\right] \frac{1}{\left(2\pi(\Sigma_t^k)^2\right)^{1/4}} \exp\left[-\frac{(x - X_t^k)^2}{(2\Sigma_t^k)^2}(1 - \frac{\Lambda_t^k}{\mu\sigma^2})\right] \\ \Gamma^k(x,t) = \sqrt{\rho^k} \exp\left[\frac{+\gamma_t - P_t^k \cdot x}{\mu\sigma^2}\right] \frac{1}{\left(2\pi(\Sigma_t^k)^2\right)^{1/4}} \exp\left[-\frac{(x - X_t^k)^2}{(2\Sigma_t^k)^2}(1 + \frac{\Lambda_t^k}{\mu\sigma^2})\right] \end{cases}$$

Dynamics [before the two subgroups merge]

- Variances and position-momentum correlators

  Same dynamics as for the mono-modal case, except for

$$g \to g^{(a,b)} = \rho^{(a,b)} g$$

  (lighter groups contract more slowly and remain more extended)

- Center of mass motion

  – obtained through conservation of energy and momentum

$$P_{\text{tot}} = \rho^a P^a + \rho^b P^b = \mu(\rho^a v^a + \rho^b v^b) = 0$$

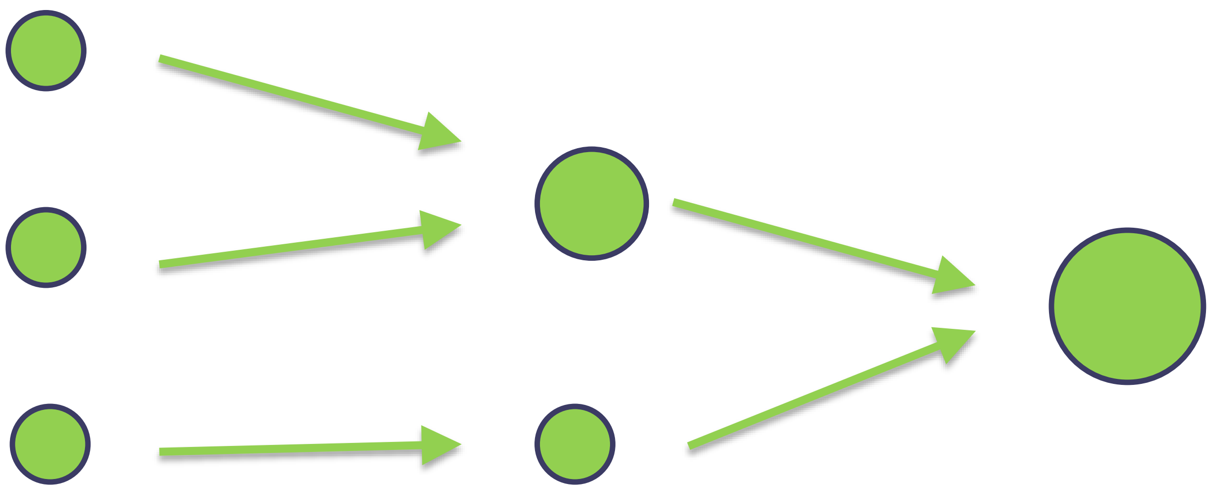
$$\sum_{k=a,b} \rho^k \left\{ \frac{1}{2}\mu [v^k]^2 + [\rho^k]^2 \tilde{E}^*_{\text{tot}} \right\} = \tilde{E}^*_{\text{tot}} = \frac{1}{8\pi} \frac{g^2}{\mu\sigma^4}$$

$$\Longrightarrow \qquad |v^{a,b}| = \sqrt{\frac{3}{4\pi}\rho^{b,a}} v_g$$

  (lighter groups move more quickly)

Muti-modal case:

- Can be obtained from a generalization of the two-modal case as long as the groups are well separated enough
- Until the last merging, total momentum of each subgroup is nonzero (even if $U_0$is neglected).
- The order in which the mergings occur is non-trivial



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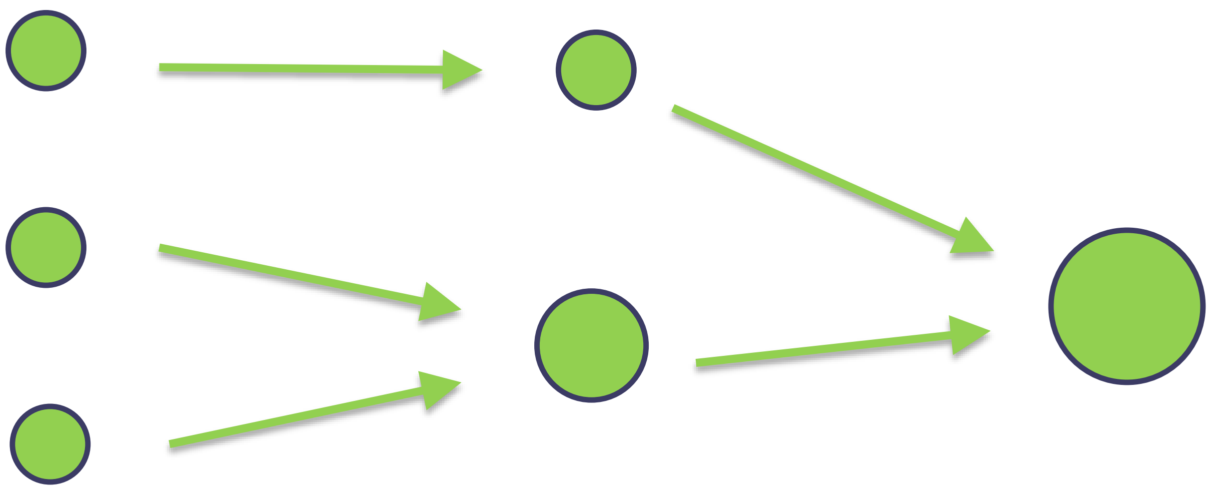


D. First order perturbations theory

Oposit regime of weak interaction between the agents
- Main forces : external potential $U_0(x)$ and noise
- Interaction term $gm(x)$ weak, treated perturbatively

1) Non-interacting limit

$$\boxed{\hat{H}_0 = -\frac{\mu\sigma^4}{2}\Delta_x - U_0(\mathbf{x})}$$

$$\begin{cases} \mu\sigma^2\partial_t\Phi = +\hat{H}_0\Phi, \\ \mu\sigma^2\partial_t\Gamma = -\hat{H}_0\Gamma, \end{cases}$$

- eigenvectors $\psi_0(\mathbf{x}), \psi_1(\mathbf{x}), \ldots$
- eigenvalues $\lambda_0 \leq \lambda_1 \leq \ldots$
- $G_o(x,x',t) \equiv \sum_{n\geq 0} e^{-\lambda_n\, t/\mu\sigma^2}\psi_n(x)\psi_n(x')$

$$\begin{cases} \Phi(x,t) = \int dx\, \Phi(x',t')\, G_o(x',x,t'-t) \quad t \leq t' & (\Phi(x,T) = \Phi_T(x)) \\ \Gamma(x,t) = \int dx'\, G_o(x,x',t-t')\, \Gamma(x,t') \quad t \geq t' & (\Gamma(\mathbf{x},0) = \dfrac{m_0(\mathbf{x})}{\Phi(\mathbf{x},0)}) \end{cases}$$

2) Long optimization time

$$\left(T \gg t_{\text{erg}} \equiv \frac{\mu\sigma^2}{\lambda_0 - \lambda_1}\right)$$

- Still non-interacting
- Focus on ($t \;\leq$ "T/2")

Ergodic state

$$\begin{cases} \Phi_e(x,t) \equiv \; C \; e^{+\lambda_0 t/\mu\sigma^2} \psi_0(x) \\ \Gamma_e(x,t) \equiv C^{-1} e^{-\lambda_0 t/\mu\sigma^2} \psi_0(x) \end{cases} \implies \boxed{m_e(x,t) \equiv \psi_0^2(x)}$$

$$[C \equiv e^{-\lambda_0 T/\mu\sigma^2} \langle \psi_0 | \Phi_T \rangle]$$

Density propagator

$$m(x,t) = \int dx' F_0(x,x',t) m_0(x')$$

$$\boxed{F_0(x,x',t) \equiv \psi_0(x) G_0(x,x',t) \frac{e^{+\lambda_0 t/\mu\sigma^2}}{\psi_0(x')}}$$

2) Weak interactions

unperturbed density

- perturbation : $\delta U(t) \equiv g m^{(0)}(x,t)$

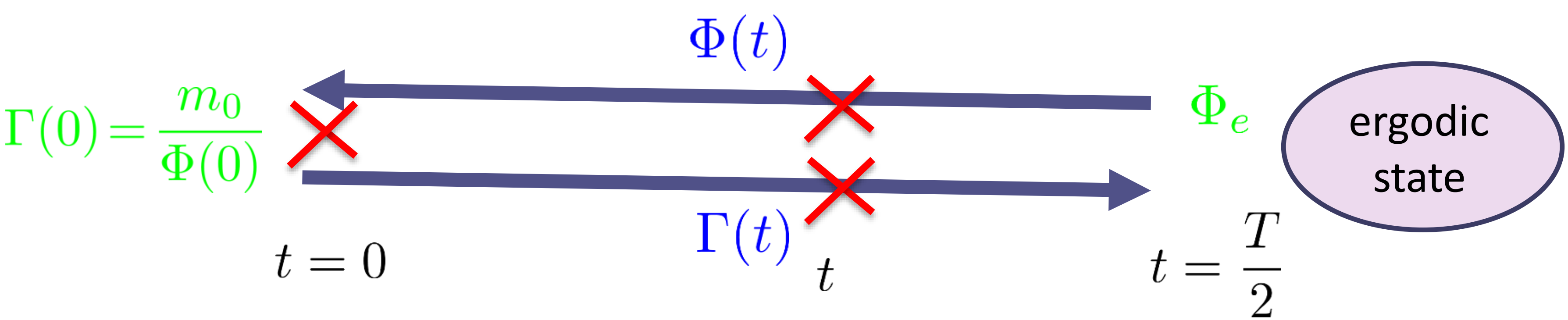
- basic tool q.m. time dependent perturbation theory

$$\hat{G} = \hat{G}^0 + \hat{G}^0 \delta U \hat{G}^0 + \ldots\ldots$$

- two catches :

  - Need to perform time dependent perturbation theory around the static perturbed potential $U_0(x) + g m_e^{(0)(x)}$

  - Perturbation acts in three places :



<u>first order solution to the Mean Field Game equations</u>

$$\hat{H}_e = -\frac{1}{2\mu}\Pi_x^2 - U_0(\mathbf{x}) - g m_e(\mathbf{x}) \qquad \hat{H}_0 = -\frac{1}{2\mu}\Pi_x^2 - U_0(\mathbf{x})$$

$\Gamma(t)$

$$\begin{aligned} m(x,t) = {} & m^{H_0}(x) + \int dx' (F_e(x,x',t) - F_{H_0}(x,x',t))\, m_0(x') \\ & + \frac{g}{\mu\sigma^2}\int_0^t ds \int dy\, dx' \left[F_{H_0}(x,y,t-s) - F_{H_0}(x,x',t)\right] \\ & \qquad \times \left[m^{H_0}(y,s) - m_e^{H_0}(y)\right] F_{H_0}(y,x',s) m_0(x') \\ & + \frac{g}{\mu\sigma^2}\int_t^{T_e} ds \int dy\, dx' \left[m^{H_0}(y,s) - m_e^{H_0}(y)\right] F_{H_0}(x,x',t) \\ & \qquad \times \left[F_{H_0}(y,x,s-t) - F_{H_0}(y,x',\, s\,)\right] m_0(x') \end{aligned}$$

$\Gamma(0)$ $\Phi(t)$

Conclusion

- Formal, but deep, relation between a class of mean field games and the Non-Linear Schrödinger equation dear to the heart of physicists
- Classical tools developed in that context (Ehrenfest relations, solitons, variational methods, etc ..) can be used to analyze the solutions of the mean field games equations
- Here: application to a simple population dynamics model

  → rather thorough understanding of this model (including more structured initial conditions, collapse of the soliton,..)

- It seems rather clear that the connection with NLS will eventually provide a good level of understanding for a large class of quadratic mean field games :
  - repulsive interaction (Thomas-Fermi approximation, etc…)
  - two-populations [Schelling-like] models (domain formation, tunneling effect, etc ..)

Question : how much is this useful ?

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If you are physicist …

- Mildly useful but a lot of fun

Question : how much is this useful ?

If you are an economist / sociologist / etc …

- Understanding the qualitative properties of the solution of MFG equations is presumably more important than quantitative accuracy (no point in being more precise than the model itself).
- There may be some quadratic MFG actually relevant to a practical problem (but not necessarily to the one you are interested in).
- However many of the approximation scheme (variational approximations, Ehrenfest relations) etc .. do not necessarily rely in a fundamental way on the transformation to NLS

Question : how much is this useful ?

If you are a mathematician …

- Understanding deeply a class of MFG could help gaining intuition for the more general setting (cf Ising model).
- Eg : non-monotone systems :
  - Non-uniqueness of the solution appear more as a feature than as a bug. Quadratic MFG may represent a good setup to think about this.
  - Same thing for the existence of the ergodic state [eg : relation between the local point of view that emerge from the variational approximation and the more general constraints of monotonicity]
- Eg : monotone systems : for large interactions, noise may become largely irrelevant for most of the dynamics (Thomas Fermi approximation) → may justify simplified description.
- Models with two different kind of small players (eg: Schelling).
- Etc …