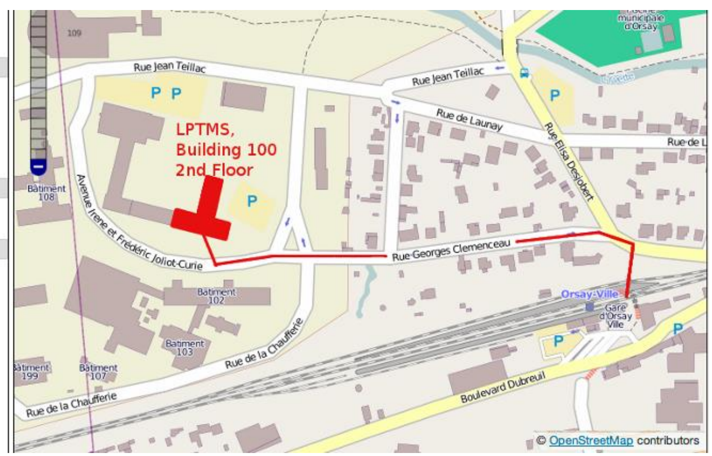
Mean Field Games:
An [imaginary time] Schrödinger approach

Denis Ullmo (LPTMS-Orsay)

Collaboration with

Thierry Gobron (LPTM-Cergy)   Igor Swiecicki (LPTM(S))



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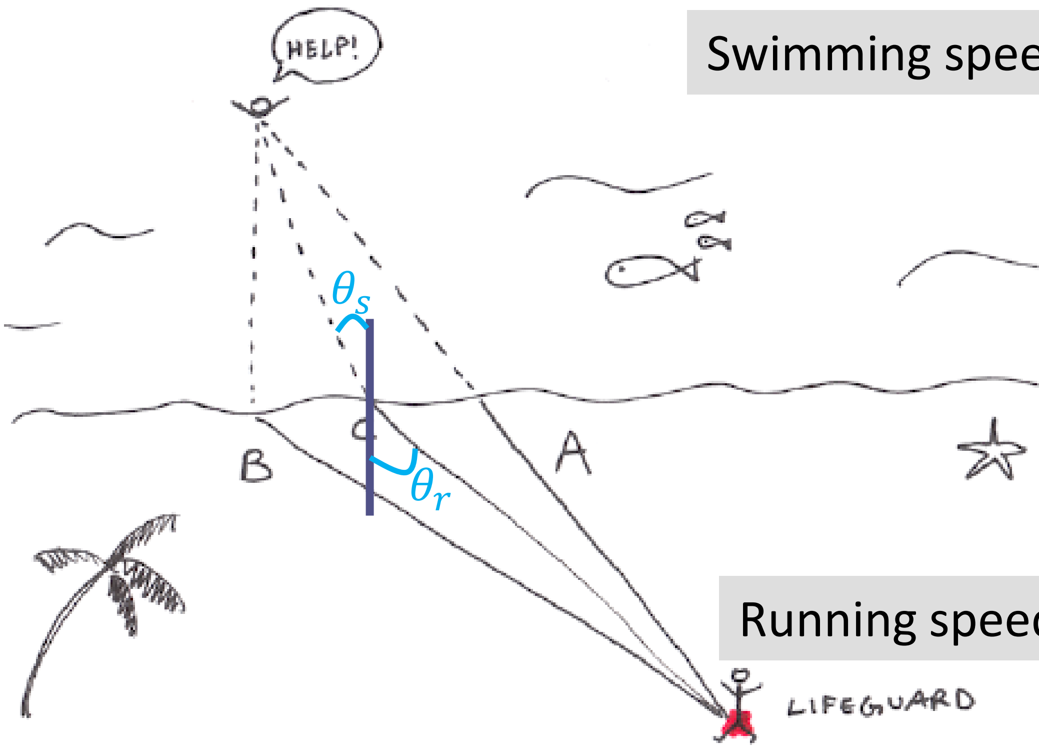
Outline

1. Introduction to mean field games
    - Optimization problems
    - Game theory
    - Mean field games
1. Quadratic mean field games and the Non-Linear Schrödinger equation
    - Mapping to NLS
    - A case study: a quadratic mean field game in the strong positive coordination regime

Part I
Introduction to mean field games

Optimization problems

The lifeguard problem



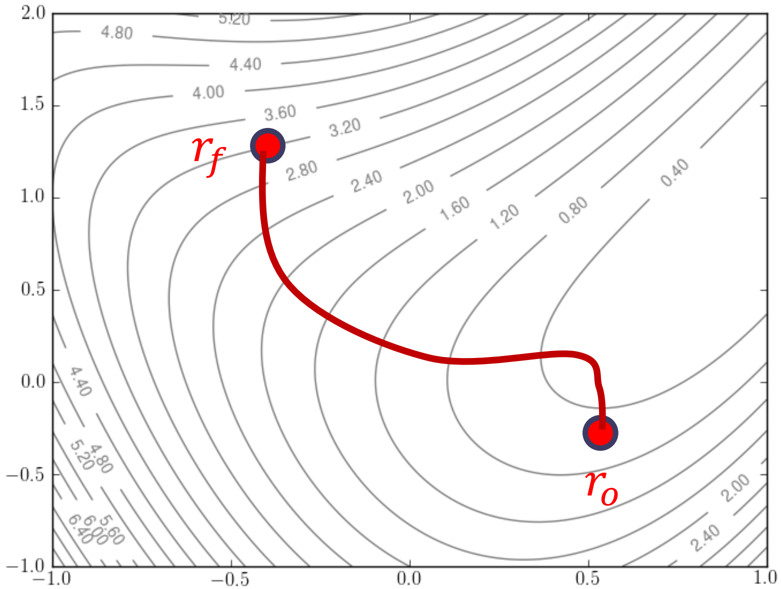
Swimming speed: v_s

Running speed: v_r

Aatish Bhatia

$$\min_{\theta}[t = t_r + t_s] \Rightarrow \frac{\sin\theta_s}{v_s} = \frac{\sin\theta_r}{v_r}$$

Dynamics of a classical point particle $[\boldsymbol{r} = (x, y)]$



$V(x, y)$

Kinetic energy : $T = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right)$

Potential energy: $\mathrm{V}(x, y)$

Lagrangian : $L = T - V$

$$S(\mathbf{r_f}, \mathbf{r_0}, t) = \int_0^t L\left(\mathbf{r}(t'), \dot{\mathbf{r}}(t')\right) dt'$$

(action)

minimize $S(\boldsymbol{r}', \boldsymbol{r}'', t)$ ⇒ $m\ddot{r} = -\nabla V(r)$

$$\text{Hamilton Jacobi:} \qquad \partial_t S = \frac{1}{2m} (\nabla S)^2 + V(\mathbf{r})$$

Control

- $X \equiv$ motor speed or position, chemical concentration, etc ...

- dynamics : $dX_t = a_t dt + \sigma dw_t$

control

white noise

running cost

- Cost function : $c[a(\cdot), w(\cdot), X_t, t] \equiv \int_t^T \ell(a_t, X_t)dt \; + \; c_T(X_T)$

final cost

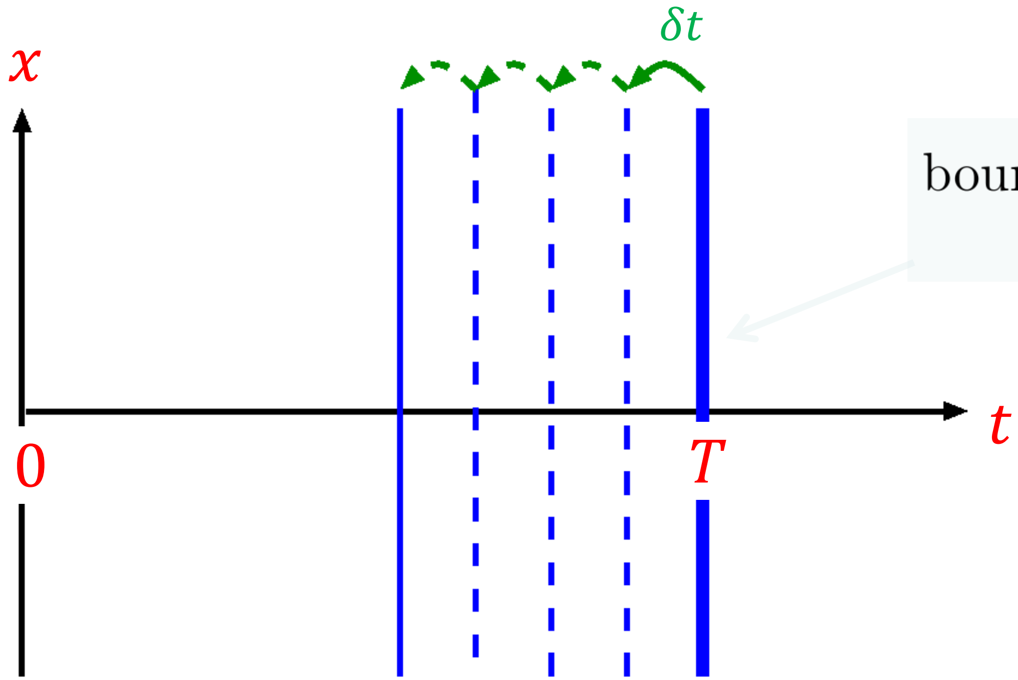
$$\text{e.g.} \quad \ell(a, X) \equiv \frac{\mu}{2} a^2 - V(X)$$

<u>Problem</u>

choose control $a(.)$ to minimize expected cost $\langle c(X_{t_0}) \rangle_{\text{noise}}$

Linear programming

- value function : $u(x,t) \equiv \min_{a(.)} \langle c(X_t) \rangle_{\text{noise}|X_t=x}$



boundary condition :
$u(x,T) = c_T(x)$

$$\partial_t u = \frac{1}{2\mu}(\partial_x u)^2 + V(x) - \frac{\sigma^2}{2}\partial_x^2 u$$
$$a(x,t) = -\partial_x u(x,t)$$

(Hamilton-Jacobi -Bellman)

Game “theory”

A simple game:

2 players

2 strategies

	Hawk	Dove
Hawk	(V-C)/2 , (V-C)/2	V,0
Dove	0,V	V/2, V/2

[Hawk and dove]

Differential games:

- Each players i characterized by a set of continuous “state variables” $\boldsymbol{x}^i = (x_1^i, x_2^i, \dots, x_n^i)$
- Each player solve an optimization problem
- The cost function of player i depend on $\boldsymbol{x}^i$ but also on all the $\boldsymbol{x}^{j \neq i}$

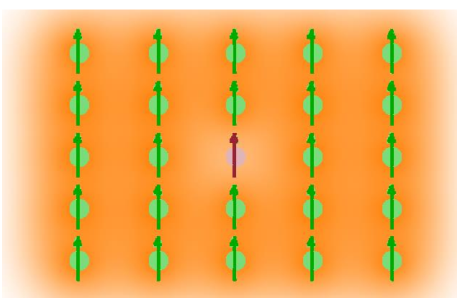
e.g.: - three persons trying to sell ice creams on a beach

- two competing companies trying to decide on the size of their marketing department.
- air heaters in different houses when the price of energy depends on total consumption.

Games with a large number of players :

- As the number of players increases, the study of such games becomes quickly intractable.
- However, for a very large number of « small » players, one can recover some degree of simplification through the notion of “mean field”.

<u>Weiss [mean field] theory of magnetization</u>



$$\hat{H} = -J \sum_{\langle i,j \rangle} s_i \cdot s_j \Rightarrow \hat{H}^{\mathrm{mf}}_{i_0} = -\underbrace{J\nu\langle s\rangle}_{B_{\mathrm{mf}}} \cdot s_{i_0}$$

\# of neighbors

Effective magnetic field

⇒ **Mean Field (differentiable) Games** *[Lasry & Lions (2006)]*

Mean Field Games

A mean field game paradigm : model of population dynamics

[Guéant, Lasry, Lions (2011)]

- N agents $i = 1, 2, \cdots, N \quad (N \gg 1)$
- state of agent $i \longrightarrow$ real vector $\mathbf{X}^i$ (here just physical space)

$$m(\mathbf{x}, t) \equiv \frac{1}{N} \sum_1^N \delta(\mathbf{x} - \mathbf{X}_t^i) \qquad \text{density of agents}$$

- agent's dynamic

$$d\mathbf{X}_t^i = \mathbf{a}_t^i dt + \sigma d\mathbf{w}_t^i$$

$d\mathbf{w}_t^i \equiv$ white noise
drift $\mathbf{a}_t^i \equiv$ control parameter

- agent tries to optimize (by the proper choice of $\mathbf{a}_t^i$) the cost function

$$\int_t^T d\tau \left[\frac{\mu}{2} (\mathbf{a}_\tau^i)^2 - V[m](\mathbf{X}_\tau^i, \tau) \right] + c_T(\mathbf{X}_T^i)$$

Mean Field Game = coupling between a (collective) stochastic motion and an (individual) optimization problem through a mean field $V[m](\mathbf{x},t)$

- Langevin dynamic $d\mathbf{X}_t^i = \mathbf{a}_t^i dt + \sigma d\mathbf{w}_t^i$ leads to a *(forward)* diffusion equation for the density $m(x,t)$

$$\begin{cases} \partial_t m + \nabla_{\mathbf{x}}(am) - \dfrac{\sigma^2}{2}\Delta_{\mathbf{x}} m = 0 \\ m(x,t\!=\!0) = m_0(x) \end{cases} \qquad \text{(Kolmogorov)}\ .$$

- Optimization problem, through linear programming, leads to a *(backward)* Hamilton-Jacobi-Bellman equation for the value function $u(\mathbf{x},t)$

$$\begin{cases} \partial_t u + \dfrac{1}{2\mu}\left(\nabla_{\mathbf{x}} u\right)^2 + \dfrac{\sigma^2}{2}\Delta_{\mathbf{x}} u = V[m](x,t) \\ u(x,t\!=\!T) = c_T(x) \end{cases} \qquad \text{(HJB)}\ .$$

- Kolmogorov coupled to HJB through the drift $a(x,t) = -\partial_x u(x,t)$

- HJB coupled to Kolmogorov through the mean field $V[m](x,t)$

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Long time limite and the « ergodic » state

Theorem [*Cardaliaguet, Lasry, Lions,Porretta (2013)*]

- No explicit time dependence: $V[m](\boldsymbol{x}, \not{t})$
- Long time limit for the optimization : $T \to \infty$
- … + other (technical) conditions ….

⇒

$\exists$ an *ergodic* state $(\bar{m}(\mathbf{x}), \bar{u}(\mathbf{x}))$ such that,

$$\text{for } 0 \ll t \ll T \qquad \left| \begin{array}{l} m(\mathbf{x}, t) \simeq \bar{m}(\mathbf{x}) \\ u(\mathbf{x}, t) \simeq \bar{u}(\mathbf{x}) \end{array} \right.$$

$$(\bar{m}, \bar{u}, \lambda) \text{ such that } \quad \left\{ \begin{array}{l} \lambda + \dfrac{1}{2\bar{\mu}} (\nabla_{\mathbf{x}} \bar{u})^2 + \dfrac{\sigma^2}{2} \Delta_{\mathbf{x}} \bar{u} = V[\bar{m}](x, t) \\ -\nabla_{\mathbf{x}} (\bar{m} (\nabla_{\mathbf{x}} \bar{u})) - \dfrac{\sigma^2}{2} \Delta_{\mathbf{x}} \bar{m} = 0 \end{array} \right. .$$

Recent, applications oriented, mean field game models

- Models for vaccination policies [Laetitia Laguzet, Ph.D. thesis, 2015]
- Price formation process in the presence of high frequency participant [Lachapelle, Lasry, Lehalle, Lions (2015)]
- Load shaping via grid wide coordination of heating-cooling electric loads [Kizilkale and Malhamé, (2015))

Part II
Quadratic Mean Field Games and the Non-Linear Schrödinger equation

The wo main avenues of research for MFG

- Proof of the internal consistency of the theory, and of the existence and uniqueness of solutions to the MFG equations
  [cf Cardaliaguet's notes from Lions collège de France lectures]
- Numerical schemes to compute exact solutions of the problem
  [eg: Achdou & Cappuzzo-Dolcetta (2010), Lachapelle & Wolfram (2011), etc …]

⇨

- Our (physicist) approach : develop a more "qualitative" understanding of the MFG (extract characteristic scales, find explicit solutions in limiting regimes, etc..).
- Facilitated for "*quadratic*" MFG thanks to the connection with Non-linear Schrödinger equation.

Quadratic mean field game & non-linear Schrödinger equation

Quadratic mean field games

- N agents, state $\mathbf{X}^i \in \mathbb{R}^n$ with Langevin dynamics $d\mathbf{X}^i_t = \mathbf{a}^i_t dt + \sigma d\mathbf{w}^i_t$
- cost function $\displaystyle\int_t^T d\tau \left[\frac{\mu}{2}(\mathbf{a}^i_\tau)^2 - V[m](\mathbf{X}^i_\tau, \tau)\right] + c_T(\mathbf{X}^i_T)$
- System of coupled pde's $[a(\mathbf{x},t) = -\nabla_{\mathbf{x}} u(\mathbf{x},t),\ m(\mathbf{x},t) \equiv \text{density of agents}]$

$$\begin{cases} \partial_t m + \nabla_{\mathbf{x}}(am) - \dfrac{\sigma^2}{2}\Delta_{\mathbf{x}} m = 0 \\ m(x, t{=}0) = m_0(x) \end{cases} \qquad \text{(Kolmogorov)} .$$

$$\begin{cases} \partial_t u - \dfrac{1}{2\mu}(\nabla_{\mathbf{x}} u)^2 + \dfrac{\sigma^2}{2}\Delta_{\mathbf{x}} u = -\nabla_{\mathbf{x}} V[m](x,t) \\ u(x, t{=}T) = c_T(x) \end{cases} \qquad \text{(HJB)} .$$

Mapping of quadratic mean field games to the non-linear Schrödinger equation

Quadratic mean field games

- N agents, state $\mathbf{X}^i \in \mathbb{R}^n$ with Langevin dynamics $d\mathbf{X}^i_t = \mathbf{a}^i_t dt + \sigma d\mathbf{w}^i_t$
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Transformation to NLS

- Cole-Hopf transform: $\Phi(\mathbf{x},t) = \exp\left(-\frac{1}{\mu\sigma^2}u(\mathbf{x},t)\right)$

$$-\mu\sigma^2\partial_t\Phi = \frac{\mu\sigma^4}{2}\Delta_{\mathbf{x}}\Phi + V[\mathbf{x},m]\Phi$$

- "Hermitization" of Kolmogorov: $\Gamma(\mathbf{x},t) \equiv m(\mathbf{x},t)\exp\left(u(\mathbf{x},t)/(\mu\sigma^2)\right)$ (i.e. $m(\mathbf{x},t) = \Gamma(\mathbf{x},t)\Phi(\mathbf{x},t)$)

$$\sigma^2\partial_t\Gamma - \frac{\sigma^4}{2}\Delta_{\mathbf{x}}\Gamma = \frac{\Gamma}{\mu}\underbrace{\left(\frac{\partial u}{\partial t} - \frac{1}{2\mu}(\nabla_{\mathbf{x}}u)^2 + \frac{\sigma^2}{2}\Delta_{\mathbf{x}}u\right)}_{V[\mathbf{x},m]\quad !!!}$$

$$\Longrightarrow \qquad \mu\sigma^2\partial_t\Gamma = \frac{\mu\sigma^4}{2}\Delta_{\mathbf{x}}\Gamma + V[\mathbf{x},m]\Gamma$$

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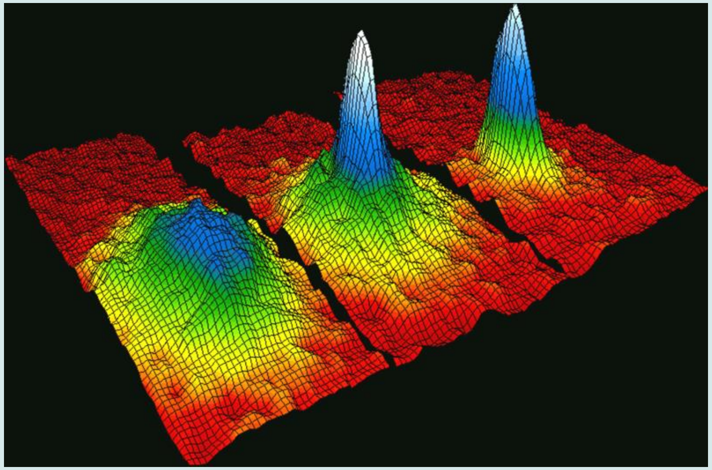
$$\Rightarrow \quad \mu\sigma^2\partial_t\Gamma = \frac{\mu\sigma^4}{2}\Delta_{\mathbf{x}}\Gamma + V[\mathbf{x},m]\Gamma$$

Bose-Einstein condensates



Rubidium atoms (170 nK)

At very low temperature (and sufficiently high density), systems of bosons "*condensates*"
→ **all particles (the rubidium atoms here) are in the same "quantum state" $\boldsymbol{\Psi}(\boldsymbol{x}, \boldsymbol{t})$**

- quantum mechanics of a particle of mass μ in potential $U_0(\mathbf{x})$

$$i\hbar\partial_t\Psi = -\frac{\hbar^2}{2\mu}\Delta_{\mathbf{x}}\Psi + U_0(\mathbf{x})\Psi \qquad \text{(Schrödinger)}$$

- Many particles with local interaction $V(\mathbf{x}-\mathbf{x}') = g\,\delta(\mathbf{x}-\mathbf{x}')$

$$\text{Mean field} \quad \Rightarrow \quad U_0 \to U_0(\mathbf{x}) + g|\Psi|^2$$

density of atoms

$$i\hbar\partial_t\Psi = -\frac{\hbar^2}{2\mu}\Delta_{\mathbf{x}}\Psi + U_0(\mathbf{x})\Psi + g|\Psi|^2\Psi$$

(Non-linear Schrödinger (or Gross-Pitaevskii) equation)

$$\Rightarrow \quad i\hbar\partial_t\Psi = -\frac{\hbar^2}{2\mu}\Delta_{\mathbf{x}}\Psi + U_0(\mathbf{x})\Psi + g|\Psi|^2\Psi$$

(Non-linear Schrödinger (or Gross-Pitaevskii) equation)

- MFG equations, specifying to $V[m](\mathbf{x}) \equiv U_0(\mathbf{x}) + gm(\mathbf{x},t)$

$$\begin{cases} \mu\sigma^2\partial_t\Gamma = \frac{\mu\sigma^4}{2}\Delta_{\mathbf{x}}\Gamma + U_0(\mathbf{x})\Gamma + g\,m\Gamma \\ -\mu\sigma^2\partial_t\Phi = \frac{\mu\sigma^4}{2}\Delta_{\mathbf{x}}\Phi + U_0(\mathbf{x})\Phi + g\,m\Phi \end{cases}$$

$m = \Psi\Gamma$

$$i\hbar\partial_t\Psi = -\frac{\hbar^2}{2\mu}\Delta_{\mathbf{x}}\Psi + U_0(\mathbf{x})\Psi + g|\Psi|^2\Psi$$

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$m = \Gamma\Phi$

Formal change $(\Psi, \Psi^*, \hbar) \to (\Phi, \Gamma, i\mu\sigma^2)$ maps NLS to MFG !!!

Why the excitement ?

- Man Field Games exist since 2005-2006, the Non-Linear Schrödinger equation since at least the work of Landau and Ginzburg on superconductivity in 1950.
- NSL applies to many field of physics : superconductivity, non-linear optic, gravity waves in inviscid fluids, Bose-Einstein condensates, etc..

  → huge literature on the subject
- We feel we have a good qualitative understanding of the “physics” of NLS, together with a large variety of technical tools to study its solutions.

[NB : Change of variable giving NLS known by Guéant, (2011)]

A case study: a quadratic mean field game in the strong positive coordination regime

To illustrate how this 'transfer of knowledge' works, consider a simple (but non-trivial) quadratic mean field game :

- $d = 1$
- Local interaction $V[m](x) = U_0(x) + g\, m$
- Strong positive coordination (large positive g)

(If it helps, think of it as a population dynamics model for a aquatic specie living in a river :

- $U_0(x) \equiv$ intrinsic quality of the location (e.g. for food gathering)
- g measure the protection from predator by other members of the group.
- $T =$ daylight duration, $m_0(x) =$ initial distribution in the morning, $c_T(x) =$ quality of shelter for the night)

Schrödinger vs Heisenberg representation and Ehrenfest relations

Quantum mechanics

- State of the system $\equiv$ wave function $\Psi(x,t)$
- Observables $\equiv$ operators: $\hat{O} = f(\hat{p}, \hat{x})$
- Average $\langle \hat{O} \rangle \equiv \int dx \Psi^*(x) \hat{O} \Psi(x)$
- Hamiltonian $\equiv$ $\hat{H} = \dfrac{\hat{p}^2}{2\mu} + V(x) = -\dfrac{\hbar^2}{2\mu}\Delta_x + V(x)$

$$\hat{x} \equiv x\times$$
$$\hat{p} \equiv i\hbar\partial_x$$

$$i\hbar\partial_t \Psi = \hat{H}\Psi \qquad \Rightarrow \qquad i\hbar\frac{d}{dt}\langle \hat{O} \rangle = \langle [\hat{H}, \hat{O}] \rangle$$

$$\Rightarrow \quad \begin{cases} \dfrac{d}{dt}\langle \hat{x} \rangle = \dfrac{1}{\mu}\langle \hat{p} \rangle \\ \dfrac{d}{dt}\langle \hat{p} \rangle = -\langle \nabla_x V(\hat{x}) \rangle \end{cases} \qquad \text{(Ehrenfest)}$$

Quadratic Mean Field Games

- Operators: $\hat{X} \equiv x\times \quad \hat{\Pi} \equiv \mu\sigma^2\partial_x \quad \hat{O} = f(\hat{\Pi}, \hat{X})$

- Average: $\langle \hat{O} \rangle(t) \equiv \int dx \Gamma(x,t)\hat{O}\Phi(x,t)$ $\quad m = \Gamma\Phi$

$$\Rightarrow \quad \text{if } \hat{O} = O(\not{\hat{\Pi}}, \hat{X}) \qquad \langle \hat{O} \rangle \equiv \int dx\, m(x) O(x)$$

$$\left(\langle \hat{1} \rangle \equiv \int dx\, m(x) = 1 \qquad \langle \hat{X} \rangle \equiv \int dx\, x m(x) \right)$$

- Hamiltonian $\equiv \quad \hat{H} = \frac{\hat{\Pi}^2}{2\mu} + V[m](x) = \frac{\mu\sigma^4}{2}\Delta_x + V[m](x)$

$$\begin{cases} +\mu\sigma^2\partial_t\Gamma = \hat{H}\Gamma \\ -\mu\sigma^2\partial_t\Phi = \hat{H}\Phi \end{cases} \quad \Rightarrow \quad \mu\sigma^2 \frac{d}{dt}\langle \hat{O} \rangle = \langle [\hat{H}, \hat{O}] \rangle$$

Exact relations

$$\text{Force operator}: \hat{F}[m_t] \equiv -\nabla_x V[m_t](\hat{X})$$

$$(V[m_t] = U_0 + g m_t \;\rightarrow\; \hat{F}[m_t] \equiv \underbrace{\hat{F}_0}_{-\nabla_x U_0} - g\,\nabla_x m_t)$$

$$\Sigma^2 \equiv \langle(\hat{X}^2\rangle - \langle\hat{X}\rangle^2) \qquad \Lambda \equiv (\langle\hat{X}\hat{\Pi} + \hat{\Pi}\hat{X}\rangle - 2\langle\hat{\Pi}\rangle\langle\hat{X}\rangle)$$

$$\begin{cases} \dfrac{d}{dt}\langle\hat{X}\rangle = \dfrac{1}{\mu}\langle\hat{\Pi}\rangle \\ \dfrac{d}{dt}\langle\hat{\Pi}\rangle = \langle F[m_t]\rangle \end{cases} \qquad \begin{cases} \dfrac{d}{dt}\Sigma^2 = \dfrac{1}{\mu}\Big(\langle\hat{X}\hat{\Pi} + \hat{\Pi}\hat{X}\rangle - 2\langle\hat{\Pi}\rangle\langle\hat{X}\rangle\Big) \\ \dfrac{d}{dt}\Lambda = -2\langle\hat{X}\hat{F}[m_t]\rangle + 2\langle\hat{\Pi}^2\rangle \end{cases}$$

$$\mathcal{E}_{\text{tot}}(t) \equiv \frac{1}{2\mu}\langle\hat{\Pi}^2\rangle + \langle U_0(\hat{X})\rangle + \langle\hat{H}_{\text{int}}\rangle \quad \equiv \quad \text{conserved quantity}$$

$$\langle\hat{H}_{\text{int}}\rangle \equiv \frac{g}{2}\int dx\, m_t(x)^2$$

Ergodic solution

Stationary non-linear Schrödinger

Let $\Psi_\mathrm{e}(x)$ the solution of the stationary NLS

$$\lambda\Psi_\mathrm{e} = \frac{\mu\sigma^4}{2}\Delta_x\Psi_\mathrm{e} + U_0(x)\Psi_\mathrm{e} + g\,|\Psi_\mathrm{e}|^2\Psi_\mathrm{e}$$

$$\text{Define}\quad \begin{cases} \Gamma_\mathrm{e}(x,t) \equiv \exp\left(+\frac{\lambda}{\mu\sigma^2}t\right)\Psi_\mathrm{e}(x) \\ \Phi_\mathrm{e}(x,t) \equiv \exp\left(-\frac{\lambda}{\mu\sigma^2}t\right)\Psi_\mathrm{e}(x) \end{cases}$$

$$\Rightarrow \quad \text{solution of} \quad \begin{cases} \mu\sigma^2\partial_t\Gamma = \frac{\mu\sigma^4}{2}\Delta_\mathbf{x}\Gamma + U_0(\mathbf{x})\Gamma + g\,m\Gamma \\ -\mu\sigma^2\partial_t\Phi = \frac{\mu\sigma^4}{2}\Delta_\mathbf{x}\Phi + U_0(\mathbf{x})\Phi + g\,m\Phi \end{cases}$$

with $m_\mathrm{e}(x) \equiv \Gamma_\mathrm{e}(x,t)\Phi_\mathrm{e}(x,t) = |\Psi_\mathrm{e}(x)|^2 = \text{const.}$

⇒ **Ergodic solution of the MFG problem**

Limiting case $U_0(x) \equiv 0$ (NB: $g > 0$)

In that case solution of stationary NLS known (bright soliton)

$$\Psi_e(x) = \frac{\sqrt{\eta}}{2} \frac{1}{\cosh\left(\frac{x}{2\eta}\right)}$$

$$\eta \equiv 2\mu\sigma^4/g$$

caracteristic length scale

"Strong coordination" regime

- meaning : variations of $U_0(x)$ on the scale η are small
- ergodic state

$$m_e(x) \simeq \frac{\eta}{4} \frac{1}{\cosh^2\left(\frac{x - x_{\max}}{2\eta}\right)}$$

$$x_{\max} = \mathrm{argmax}[U_0]$$

Ehrenfest relations & ergodic solution → most of the story (for strong positive coordination)

Generic scenario

1. Herd formation: extension = η, position $x_0 = < x >_{m_0}$ (very short time process)
2. Propagation of the group :
   - as a classical particle of mass μ in pot $U_0(x)$
   - initial position $x(0) = x_0$
   - final velocity $\dot{x}(T) = -\partial_x c_T(x(T))$
3. Herd dislocation near $t = T$ (again very short time process).

NB: boundary pb, rather than initial value pb → possibly more than one solution

Propagation phase in the large T regime (Cardialaguet)

Assuming $U_0(x)$ bounded and with a single maximum (at x_{max}), the only way not to be sent to ∞ as $T \to \infty$ is to spend most of the time near x_{max}, which is an unstable fix point.

Thus, propagation phase decompose into:

a. Start from x_0 with a total energy $U_0(x_{\mathrm{max}})$
b. Approach x_{max} following its stable manifold
c. Stay close to x_{max} as long as necessary
d. Move away from x_{max} following its unstable manifold
e. Arrive at T with final velocity $\dot{x}(T) = -\partial_x c_T(x(T))$

If more than one maximum, possible phase transition (discontinuous variation of the solution) as T increases, as the systems switches from one maximum to another.

Herd formation

First stage of dynamic = herd formation.

- **It takes place on a short time scale.**
- **Can we be more precise ?**

- Assume initial distribution $m_0(x)$ "featureless", i.e. well characterized by its mean x_0 and variance Σ^2
- Neglect U_0 during the herd formation phase

⇒ variational Ansatz :

$$\Gamma(x,t) = e^{-\gamma(t)/\sigma^2} \frac{1}{(2\pi\Sigma_{x_i}^2)^{1/4}} e^{-\frac{(x-x_0)^2}{4\Sigma^2}(1-\frac{\Lambda(t)}{\sigma^2})}$$

$$\Phi(x,t) = e^{+\gamma(t)/\sigma^2} \frac{1}{(2\pi\Sigma^2)^{1/4}} e^{-\frac{(x-x_0)^2}{4\Sigma^2}(1+\frac{\Lambda(t)}{\sigma^2})} ,$$

Action :

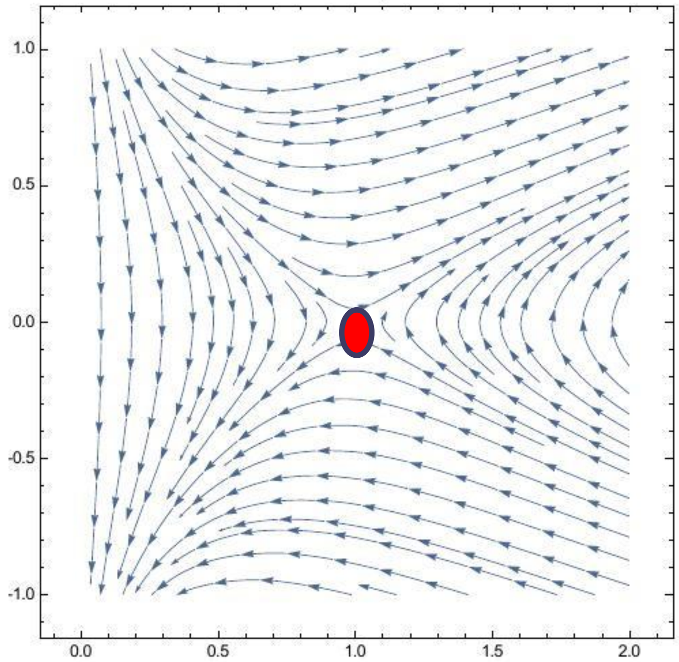
$$S[\Gamma(x,t),\Phi(x,t)] \equiv \int dt\, dx \left[\frac{\sigma^2}{2}(\partial_t\Phi\,\Gamma - \Phi\partial_t\Gamma) - \frac{\sigma^4}{2\mu}\nabla\Phi.\nabla\Gamma + U_0(x)\Phi\Gamma + \frac{g}{2}\Phi^2\Gamma^2\right]$$

$$\Rightarrow \begin{cases} \dot{\Sigma^2} = \dfrac{\Lambda}{\mu} \\ \dot{\Lambda} = -\dfrac{\sigma^4}{2\mu}\left(1-\dfrac{\Lambda^2}{4}\right)\dfrac{1}{\Sigma^2} + \dfrac{g}{2\sqrt{\pi}\Sigma} \end{cases}$$

$\Rightarrow$ hyperbolic fixed point : $\Lambda^* = 0 \quad \Sigma^* = \sqrt{\pi}\dfrac{\mu\sigma^4}{g}$

~ soliton scale η

Flow near the fix point



Large T : need to stay on stable and unstable manifold of the fixed point.

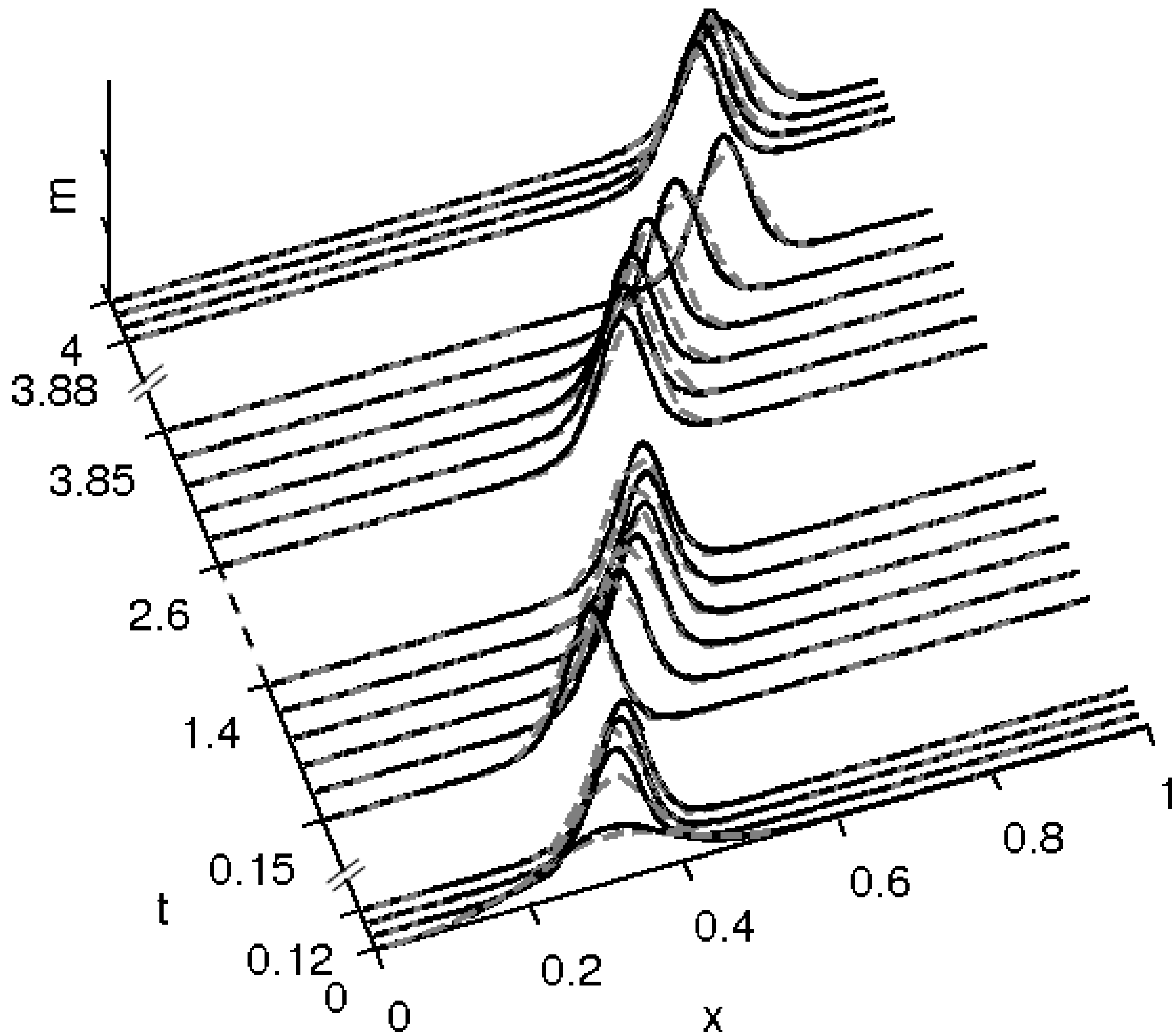
$$\frac{d^2}{dt^2}\Sigma^2 = \frac{g}{2\mu\sqrt{\pi}}\left(\frac{1}{\Sigma_*} - \frac{1}{\Sigma(t)}\right)$$

$$-(z_t - z_i) - \log\left(\frac{1-z_t}{1-z_i}\right) = \frac{t}{\tau^*}$$

$$z_t \equiv \frac{\Sigma}{\Sigma^*}$$

$$\tau^* \equiv 2\pi\sqrt{\frac{\mu\eta^3}{g}}$$

Comparison with numerical simulation



Conclusion

- Mean field games = new tool to study a variety of socio-economic problems
- Formal, but deep, relation between a class of mean field games and the Non-Linear Schrödinger equation dear to the heart of physicists
- Classical tools developed in that context (Ehrenfest relations, solitons, variational methods, etc ..) can be used to analyze mean field games
- Here: application to a simple population dynamics model

  → rather thorough understanding of this model
- It seems rather clear that the connection with NLS will eventually provide a good level of understanding for all quadratic mean field games

Two open (longer term) questions

- Quadratic mean field games represent a kind of paradigm of mean field game. How much is this true ?
  - Can we find realistic (application oriented) mean field games in that class ?
  - Is the qualitative behavior of quadratic mean field games generic ?

- Can fishes solve the MFG equations (even in their NLS form) ?