Mean Field Games in Crowd Dyanmcis

Konstantinos Koutsomitis
Faculté des Sciences d'Orsay, Université Paris-Saclay, 91400, Orsay, France

Supervisors: **Denis Ullmo, Cécile Appert-Rolland**

June 2025

Abstract

Pedestrian dynamics has seen a lot of advancements in recent years, in an environment of increasing interest to realistically model our complex societal systems. Among the models currently used, agent-based models and macroscopic descriptions have their own successes and failures when it comes to reproducing the full pedestrian behavior. Mean Field Games are a macroscopic framework, although directly derived from the microscopic dynamics of the agents, and have been proven to incorporate *anticipation*, which is a natural element in crowd dynamics. However, at their current stage, Mean Field Games in this context are derived from non-interacting microscopic dynamics, which are key factors in several situations, for instance *congestion*. Therefore, we aim to bridge realistic microscopic model with Mean Field Games, and as an initial step, the toy model of a weak Asymmetric Exclusion Process (WASEP) is considered. For that objective, a discrete, non-interacting version for the Mean Field Game had to be designed and tested, before being applied to a *two-body game* with exclusion. This kind of interaction will facilitate the connection between the models of different scales and the incorporation of congestion in the formalism.

Contents

1 Introduction

As our societies grow in complexity, social events and infrastructure accommodate larger and larger crowds. As a result, there is an increasing need to better understand pedestrian dynamics and their collective phenomena, which will allow for a safer and more efficient urban space planning, better prepared for evacuation in the case of a crisis. Studying crowd dynamics is not a new field, and several models have attempted to realistically describe them. However, the diversity of the pedestrian behavior under different density of crowds and surroundings, render a universal description a particularly challenging task. Many microscopic models have been proposed to simulate the dynamics of a crowd from an agent-based point of view and the most elaborate of them may include realistic inter-agent interactions. Nevertheless, they often fail to describe collective behavior in larger scales. On the other hand, macroscopic approaches traditionally attempt to model crowds as fluids and lose some realism in short scales, which can be significantly problematic in certain situations (under bottlenecks, high densities etc.).

Recently, Mean Field Games have been proposed to incorporate *long-range anticipation*[1], an element that all previous models fail to capture. Even though this game-theoretical approach involves a mean field approximation, introducing the mean density of agents, thus being inherently macroscopic, its final form crucially depends on the microscopic dynamics of the agents. Apart from the successful description of anticipative pedestrians, there is still a lot of room for enhancing its realism. This is the general objective of a broad project, in which our research group participates, this internship served as a first step towards connecting Mean Field Games with different microscopic models of increasing complexity.

The aim of the internship was to set up the framework with which Mean Field Games will be derived from microscopic models with interactive agents. In particular, there is a special interest for studying the case of interaction via exclusion on a discrete space, as with this form of interaction, we can hopefully get a microscopic analogue for congestion and incorporate it in a corresponding Mean Field Game. Therefore, the main part of the internship was focused on building and understanding the toolbox to be used in this case and by the end of the internship, a toy model will have been studied, *the 2-body game*.

2 Crowd Dynamics

Modeling the pedestrian dynamics can be separated in three levels. The *strategic* level, i.e. the destination of the pedestrians, the *tactical* level, which entails choosing the path towards the desired point of arrival and the *operational* level describing how the motion will take place during the route, through the interaction with the other pedestrians. Different models focusing on different scales of the system describe these levels differently, a first useful categorization would be in microscopic and macroscopic models.

2.1 Agent-based models

Pedestrians are intrinsically active agents, therefore a microscopic, agent-based approach is a natural attempt to describe their motion. Several such models have been introduced in the field, which aim to describe various microscopic behaviors.

Perhaps the simplest description of walkers is as grains[2]. Therefore, moving through a crowd, would correspond to moving in a granular medium which entails a specific phenomenology (hard-sphere-like collisions, accumulation of density in front of a moving object etc.). The granular model mainly finds applicability in very high densities, where physical interactions dominate. An interesting advancement is the introduction of *social forces*, which account for repulsive social interactions between agents that wish to sustain a personal space larger than their size [3]. Another, more elaborate model worth mentioning, which treats pedestrians as more active agents is a Time-to-Collision model, which entails a desired velocity, the outcome of an optimization process. The optimization is performed over a small time interval in the future (*short-sight anticipation*) and takes into consideration several factors, including a time-to-the-next-collision interaction, which drives away pedestrians that are about to collide[4]. In total, these models neglect anticipation effects at the operational level, dynamics of the agents purely rely on reacting to local interactions with the other agents and hence they are called *reactive.*

2.2 MFG on Crowd Dynamics

An experiment carried out by our group [1], involving an intruder passing through a static crowd, showed that pedestrian incorporate *long-range anticipation*, even at the operational level,a key factor that significantly alters their optimization process. In other words, they take into consideration the outcome of the optimization of others. This form of "competitive optimization" naturally calls for a game-theoretic approach. Therefore, instead of just grains, the pedestrian can be described as game-players, opting to minimize their discomfort (avoiding large density crowds or moving faster than necessary), while maximizing their personal benefit (reaching their desired destination, an exit of a building etc.). Since the players are numerous in a crowd and their motion stochastic the only feasible description is through performing a mean-field approximation, thus the agents interact only with an average field of density instead.

The resulting density profile from the experiment was compared to simulated data that correspond to the granular model, the Time-to-Collision model and Mean Field Games, which are displayed in fig.1. It can be easily seen that the results from the MFGs simulation best reproduce the density profile observed in the experiment, due to the anticipation element, the agents foresee the incoming obstacle and they step aside, instead of accumulating to the front of the passing obstacle.

However, Mean Field Games are still far from capturing the totality of macroscopic phenomena observed in crowds. There are several situations where anticipation is not the central feature that determines pedestrian motion, but more microscopic effects, where, for instance, granularity and collision avoidance, play a crucial role. Moreover, the geometrical features of the environment, such as doors or bottlenecks that can severely affect the motion of the crowd, are not always well incorporated due to the small-scale effects they introduce. This effect has to be taken into account in the dynamic and strategy of the agents, especially in situations where these spatial scales are comparable to the size of the agents and their separation length, namely in high densities. A typical example is congestion effects, i.e. the inability to reach high velocities in dense crowds,

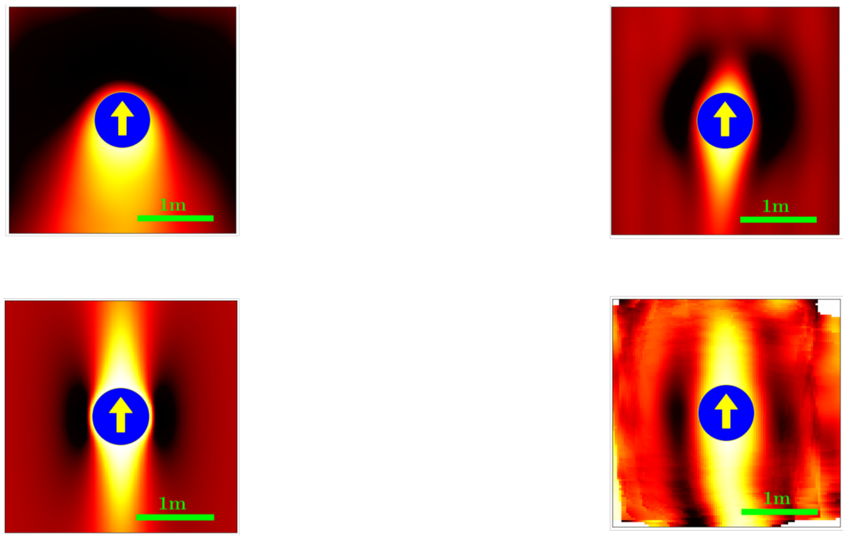


Figure 1: Density profiles of the crowd as the response to the intruder obtained from granular simulation (top left), Time-to-Collision model (top-right), Mean Field Games simulation (bottom left) and the experiment (bottom right) [1].

which are not incorporated in the Mean Field Games at the moment starting from the microscopic interaction, but rather from a macroscopic phenomenological approach [5]. However, so far the Mean Field Games descriptions for pedestrians have been derived from Langevin dynamics, which do not account for interactions between the agents. Therefore, it is necessary that new Mean Field Games be derived from more elaborate agent-based models and the connection between the microscopic interacting dynamics and their corresponding macroscopic description be better understood in order to enhance the realism of the framework.

3 Mean Field Games

Before we fully establish the framework of Mean Field Games, as they have been used in pedestrian dynamics, it is worth first mentioning some basic notions about Game Theory and Optimal Control.

3.1 Game Theory

Game Theory is the field of mathematics that studies the strategic interactions of agents-*players*. Formally, a game is a structure which contains a set of *players*, $\{i = 1, ..., N\}$, a set of *pure stragies*,$\{s_i\}$, and a set of payoff functions,$\{u_i\}$, the *utility* (or *cost*) functions [6]. Each player selects their strategic moves, with aiming only at maximizing their own utility (or minimizing their cost, respectively) by the end of the game.

A central notion in Game Theory is the *Nash equilibrium*. Let player i with a strategy s_i and let's denote s_{-i}, the set of strategies by all other players, except for the i-th player. The vector $\mathbf{s}^* = (\mathbf{s}_1^*, ..., \mathbf{s}_N^*)$ corresponds to a *Nash equilibrium* if:

$$u_i(s_i^*; s_{-i}^*) \geq u_i(s_i; s_{-i}^*) \quad \forall i$$

In other words, the Nash equilibrium corresponds to a state of the game, where any change in stragy of any player unilaterally would result to a loss of their utility.

A well known example of game with Nash equilibrium is the Prisoners' Dilemma. In this setting, two prisoners are presented with two options/strategies and they make their decision separately: to *cooperate* and stay silent or to *defect* and testify against the other.

A \ B	Cooperate	Defect
Cooperate	(5, 5)	(12, 1)
Defect	(1, 12)	(10, 10)

Table 1: Strategies of Prisoners' Dilemma, with their sentences in years.

Instead of maximizing their utility, here the prisoners wish to minimize their sentence. Regardless of the choice of prisoner B, prisoner A chooses to *defect*, since that minimizes their expected sentence. Similarly, prisoner B chooses to *defect*, as choosing to cooperate unilaterally would only increase their sentence. Therefore, the set of strategies (*defect*, *defect*), is the stationary point in the strategy space of the two players,the Nash equilibrium, since it is the strategy that results in the best outcome for each player considering that the other player is "playing Nash".

3.2 Optimal Control Theory

The field of Optimal Control corresponds to the study of all processes characterized by one or more parameters, namely *the control parameters*, which through their manipulation the minimization (or maximization) of a cost (gain) function can be attained.

The building blocks of an optimal control problem are:

- the state variable $\mathbf{X}(t)$, which fully describes the state of the system at time t.
- the control parameter $\mathbf{v}(t)$, which appears in the dynamical evolution of the state variable.
- a cost (or gain) function that depends on $\mathbf{X}$, t and functionally on the form of $\mathbf{v}$. It includes a running cost $L(\mathbf{X}, t)[\mathbf{v}]$ and a final cost $C_T(\mathbf{X})$ attributed to the state of the system at the end of the process:

$$c(\mathbf{X}, t)[\mathbf{v}] = \int_t^T \mathcal{L}(\mathbf{X}, \tau)[\mathbf{v}]d\tau + C_T(\mathbf{X}_T)$$

Since we seek to minimize the cost function, we are interested in its minimal value, namely the value function, $u(\mathbf{X}, t)$.

Noiseless & Stochastic Case

When the dynamics are deterministic, the optimization process is identical to the Principle of Least Action used in Analytical Mechanics. However, the optimal control is a very useful tool for modeling complex systems with stochastic dynamics. The most standard dynamics that can be considered are of Langevin type:

$$d\mathbf{X}_t = \mathbf{v}_t dt + \sigma d\boldsymbol{\xi}_t \tag{1}$$

where $\mathbf{v}_t dt$ is the drift term and $d\boldsymbol{\xi}_t$ a Gaussian white noise, $\mathbb{E}[d\xi] = 0$ and $\mathbb{E}[d\xi_\alpha d\xi_\beta] = \delta_{\alpha\beta} dt$.

Therefore, in the definition of the value function, since the state variable is a random variable, the quantity that needs to be minimized is the expectation value of the cost function,i.e.:

$$u(x,t) = \inf_{\mathbf{v}} \mathbb{E}\{c(\mathbf{X},t)[\mathbf{v}]\}$$

Bellman's Optimality Principle

The key point to determine the value function is the principle of Dynamic Programming, also known as Bellman's Optimality Principle:

> "An optimal policy has the property that, whatever the initial state and the initial decision, the remaining decisions must constitute an optimal policy with respect to the state resulting from the first decision."

In other words, according to this principle, an optimal policy will be the optimal one starting at any intermediate point of the trajectory. This allows us to slice the observation time interval and ensure that in all segments the optimization condition applies in the same way. A particularly useful slicing is the following:

$$u(\mathbf{X},t) = \inf_{\mathbf{v}} \mathbb{E}\{\int_t^T \mathcal{L}[\mathbf{v}](\mathbf{X},\tau)d\tau + C_T(\mathbf{X})\} \tag{2}$$

$$= \inf_{\mathbf{v}} \mathbb{E}\{\int_t^{t+dt} \mathcal{L}[\mathbf{v}](\mathbf{X},\tau)d\tau + \int_{t+dt}^T \mathcal{L}[\mathbf{v}](\mathbf{X},\tau)d\tau + C_T(\mathbf{X}_T)\} \tag{3}$$

Through linearity of the expectation value, we can identify the second term as the same value function, but starting from $\mathbf{X}_t + d\mathbf{X}_t$ at time $t + dt$. Taking dt, sufficiently small:

$$u(\mathbf{X},t) = \inf_{\mathbf{v}_t} \mathbb{E}\left[\mathcal{L}[\mathbf{v}]dt + u(\mathbf{X}_t + d\mathbf{X}_t, t+dt)\right] \tag{4}$$

Since the first moments of the random variable $d\mathbf{X}_t$ are known from the dynamics, $\mathbb{E}[d\mathbf{X}_t] = \mathbf{v}_t dt$ and $\mathbb{E}[d\mathbf{X}_t^2] = \sigma^2 dt$, we can expand the value function, up to terms of order dt:

$$u(\mathbf{X}_t + d\mathbf{X}_t, t+dt) \approx u(\mathbf{X},t) + \mathbf{v}_t dt \cdot \nabla u + \frac{\sigma^2}{2} dt \cdot \Delta u$$

and plug it in the previous equation:

$$u(\mathbf{X},t) = \inf_{\mathbf{v}_t}\left\{\mathcal{L}[\mathbf{v}]dt + u(\mathbf{X},t) + \partial_t u dt + \mathbf{v}_t dt\cdot\nabla u + \frac{\sigma^2}{2}dt\cdot\Delta u\right\}$$

$$\implies -\partial_t u = \inf_{\mathbf{v}_t}\{\mathcal{L}[\mathbf{v}] + \mathbf{v}_t\cdot\nabla u\} + \frac{\sigma^2}{2}\cdot\Delta u \tag{5}$$

Quadratic running cost

The running cost can take various forms and it usually contains the "free" part which depends only on the control parameter, and the "interaction" part between the agents. In a total analogy to the Lagrangian, a quadratic form of the free part is often used:

$$\mathcal{L}[\mathbf{v}] = \frac{1}{2}m\mathbf{v}^2 - V(\mathbf{X})$$

ensuring convexity of the total cost and therefore a unique solution for its minimum ($\mathbf{v*}$).

With this form, the optimization of eq.(5) reads:

$$\inf_{\mathbf{v}_t}\left\{\frac{1}{2}m\mathbf{v}_t^2 + \mathbf{v}_t\cdot\nabla u\right\} \implies \mathbf{v*}_t = -\frac{\nabla u}{m} \tag{6}$$

Substituting this result back to eq.(5), we arrive to the final form of the Hamilton-Jacobi-Bellman equation (HJB):

$$\partial_t u = \frac{1}{2m}(\nabla u)^2 - \frac{\sigma^2}{2}\Delta u + V(\mathbf{X}) \tag{7}$$

The HJB equation, equipped with the final condition $u(\mathbf{x},T) = C_T(\mathbf{X}_T)$, should be solved backwards.

3.3 Differential & Mean Field Games

How does the optimal control relate with game theory? There is a class of games, the *differential games*[7][8], which in contrast to turn-based games, evolve dynamically in continuous time and the available strategies to the players are a continuous set. Therefore, each player performs an optimal control while playing the game, with their strategy as the control parameter, aiming to maximize their utility. For instance, from eq.(6) in the quadratic game, the strategy each agent should follow to reach a Nash equilibrium, is the one that corresponds to the gradient descent on their value-function landscape.

However, the choice of strategies of the other players dynamically affects the game by altering the agent's utility through the interaction term. Typically, we consider $V[\hat{\rho}](\mathbf{X}_i)$, a function of the agent's state and a functional of the density of states of all the other players:

$$V[\hat{\rho}](\mathbf{X}_i) = g\hat{\rho} + V_0$$

where V_0 is the one-body term, which can model static obstacles of the environment ,g the strength of the interaction, taken negative to account for repulsion and $\hat{\rho}$ the empirical density:

$$\hat{\rho}(x,t) = \frac{1}{N}\sum_{j\neq i}\delta(x - \mathbf{X}_t^j)$$

This coupling leads to a *N-body game*, which is usually intractable, especially when the number of players increases. To overcome this obstacle, *Mean Field Games* were introduced[9][10], in which a mean-field approximation is used. In clear analogy to statistical physics, instead of including all the binary interactions among the N-agents, we consider that each agent interacts with a mean field created by the others. In our case, the empirical density, which is inherently a stochastic quantity, is replaced by its expectation value, $\rho = \mathbb{E}[\hat{\rho}]$, eliminating any stochasticity of the interaction term.

Considering Langevin dynamics ((1)) for each agent, the evolution of the average density follows the Fokker-Planck/Forward Kolmogorov equation.

$$\partial_t \rho_t(x) = -\nabla(\rho_t(x)\mathbf{v*}_t(x)) + \frac{\sigma^2}{2}\Delta\rho_t(x) \tag{8}$$

In total, the Mean Field Game (MFG) involves solving the following backward-forward systems of equations:

$$\begin{cases} \partial_t u = \frac{1}{2m}(\nabla u)^2 - \frac{\sigma^2}{2}\Delta u + V_0 + g\rho_t \\ u(x,T) = C_T(x) \end{cases} \tag{9}$$

$$\begin{cases} \partial_t \rho_t = -\nabla(\rho_t \mathbf{v*}_t) + \frac{\sigma^2}{2}\Delta\rho_t \\ \rho(x,0) = \rho_0(x) \end{cases} \tag{10}$$

with the optimization condition: $\mathbf{v*}_t = -\frac{\nabla u}{m}$.

3.4 Other applications of MFG

Mean-field games are a powerful tool in describing complex systems with a large number of interacting agents. A natural setting for such techniques is finance[11], where there is a high interest in modeling and predicting volatile markets, which get influenced by the strategic moves of the agents in the attempt to increase their profits. Similarly, in economics, different constituents of large economies (public sector, households, private firms, etc.) interact in such a manner that a game-theoretic approach is suitable for understanding macroeconomic phenomena[11]. Moreover, mean field games have been proven useful in epidemiology [12], when attempting to incorporate socio-economic factors which influence the social interactions and therefore the evolution of the epidemics. Another field in which mean-field games find increasing applicability is balancing the electricity grid, firstly, motivated by the need to create a system well-adapted to the intermittency of renewable energy sources, along with the general transition away from fossil fuels and conventional sources.

4 Walkers in an WASEP

Aiming to connect the macroscopic approach of Mean Field Games with more microscopic interactions, a suitable first candidate is the framework of *Weakly Asymmetric Simple Exclusion Processes (WASEP).* Such processes involve agents in a discrete space, where they can move stochastically, with a drift, to another site, only if it is not occupied by another agent. Exclusion was chosen as the interaction among the agents, due to its simplicity, it constitutes an appropriate first step into the regime of interacting agents and it has also been extensively studied. More specifically with this exclusion principle, we hope to further understand the case of interacting agents in MFGs and to model *congestion.* Being a macroscopic phenomenon,it is not yet understood, how congestion can be derived from microscopic models. Due to its nature,we hope that with exclusion, a natural microscopic analogue to congestion will emerge, due to the inability of the agents to pass through occupied sites. This connection can provide valuable insight into the general attempt of connecting interactive agents with MFGs.

4.1 Setting up the Discrete Model

With that goal in mind, the agents were placed on a one-dimensional lattice of L sites,with the possibility of hopping to adjacent sites or staying immobile, with periodic boundary conditions. Hence, the dynamics are determined by the choice of the hopping probabilities $P_{L/R}$. In that attempt, it was useful to use the existing knowledge for MFGs in the continuum, and for that reason the hopping probabilities that were chosen:

$$P^i_{R/L}(t) = \pm\left(\frac{v^i(t)L}{2} + \frac{\sigma^2 L^2}{2}\right)\Delta t \quad \text{with} \quad \Delta t \sim \Delta x^2 \tag{11}$$

First, it is worth underlying that only the drift term, i.e. corresponding to the difference of the two rates, was chosen time-dependent, as it plays the role of the control parameter. Their sum, which will drive diffusion, is kept constant. In other words, the agents choose the *asymmetry* of their hopping rates, toward left or right, but not the *frequency* of the hops. Starting with these dynamics, we can now derive the Mean Field Game. The scaling $\Delta t \sim \Delta x^2$ was chosen to ensure the convergence to the continuous dynamics we wanted to recover, i.e. a Langevin motion, from which the standard Mean Field Games have been derived.

Value function

The value function for at the site x_i at time t reads:

$$u(x_i, t) = \inf_{v^i} \mathbb{E}\{\int_t^T \mathcal{L}[v^i](x_i, \tau)d\tau + C_T(x_i)\} \quad \approxeq \quad \inf_{v^i} \mathbb{E}\{\sum_{\Delta t} \mathcal{L}[v^i](x_i, t)\Delta t + C_T(x_i)\}$$

$$\implies u(x_i, t) = \inf_{v^i}\{\mathcal{L}[v^i](x_i, \tau)\Delta t + u(x_{j|i}, t + \Delta t)\} \tag{12}$$

where $u(x_{j|i}, t + \Delta t)$, corresponds to the value function at time $t + \Delta t$ and site x_j, after having been at site x_i at time t. Looking at maximum one-site displacement per time interval Δt, we can express it as:

$$u(x_i, t) = \inf_{v^i} \{\mathcal{L}[v^i](x_i, \tau)\Delta t + P_R^i \Delta t \cdot u(x_{i+1}, t + \Delta t) + P_L^i \Delta t \cdot u(x_{i-1}, t + \Delta t) \\ + (1 - (P_R^i + P_L^i)\Delta t) \cdot u(x_i, t)\}$$

$$\implies u(x_i, t) = u(x_i, t + \Delta t) + \inf_{v^i} \Big\{ \mathcal{L}[v^i](x_i, t)\Delta t + \Delta t \frac{v^i L}{2}(u(x_{i+1}, t) - u(x_{i-1}, t)) \\ + \Delta t \frac{\sigma^2 L^2}{2} \big(u(x_{i+1}, t) + u(x_{i-1}, t) - 2u(x_i, t) \big) \Big\}$$

Taking a quadratic running cost $\mathcal{L}[v^i](x_i, t) = \frac{1}{2}m(v^i)^2 - V[\rho_i](x_i, t)$, we can implement the optimization condition:

$$\inf_{v^i} \{ \frac{1}{2}m(v^i)^2 + \frac{v^i L}{2}(u(x_{i+1}, t) - u(x_{i-1}, t))\} \implies v^* = -L\frac{u_{i+1} - u_{i-1}}{2m} \tag{13}$$

where for the simplicity of the notation, the time dependency has been omitted, since all quantities are calculated at time t.

Substituting the optimal control parameter into the main expression, we get:

$$u_i(t) = u_i(t + \Delta t) + \Delta t \left(-\frac{L^2}{8m}(u_{i+1} - u_{i-1})^2 + \frac{\sigma^2 L^2}{2}(u_{i+1} + u_{i-1} - 2u_i) - V[\rho_i] \right) \tag{14}$$

4.1.1 Connection with the continuous case

First of all, it has to be checked that the dynamics of eq.(11) under this scaling ($\Delta t \sim \Delta x^2$) converge to Langevin process. This can be seen by computing the first two moments of the displacement:

$$\lim_{\Delta t \to 0} \frac{\mathbb{E}[\Delta x_i]}{\Delta t} = P_R \Delta x_+ + P_L \Delta x_- = v_i$$

and

$$\lim_{\Delta t \to 0} \frac{\mathbb{E}[\Delta x_i^2]}{\Delta t} = P_R \Delta x_+^2 + P_L \Delta x_-^2 = \sigma^2$$

which correspond to the ones we would get from Langevin dynamics: $dx_i = v_i dt + \sigma d\xi_t$.

As far as the value function is concerned, by assigning to the site index i the spatial variable $x_i = \frac{i}{L}$, hence $\Delta x = \frac{1}{L}$, we can make the connection to the continuum limit:

$$\frac{u_i(t) - u_i(t + \Delta t)}{\Delta t} = -\frac{1}{2m}(\frac{u(x_i + 1/L) - u(x_i - 1/L)}{2 \cdot 1/L})^2 \\ + \frac{\sigma^2}{2}\frac{u(x_i + 1/L) + u(x_i - 1/L) - 2u(x_i)}{1/L^2}$$

By taking $L \to \infty$ ($\Delta x \to 0, \Delta t \to 0$, with $\Delta x = \sqrt{\Delta t}$), we recover:

$$\partial_t u = \frac{1}{2m}(\nabla u)^2 - \frac{\sigma^2}{2}\Delta u + V[\rho_i]$$

which is the continuous HJB equation, already discussed in sec.3.3. Therefore,it is verified that with the rates mentioned above, both the dynamics of the agents and the evolution of their value function, recover indeed the continuous limit of standard MFG.

4.2 A single walker

First, it is useful to focus on the non-interacting case, where $V[\rho] = 0$. Without interaction, the system is no longer a game, but it is reduced to independent agents performing their optimal control, only being affected by the same final cost and the same, statistically independent noise, hence we can consider only one. In addition, it is analytically solvable, just by applying a *Cole-Hopf transformation* on eq.(9), which provides a useful point of reference. More specifically, the non-linear HJB equation under the transformation $\Phi(u) = e^{u(x,t)/m\sigma^2}$ is transformed in a backwards homogeneous diffusion equation:

$$\begin{cases} -\partial_t \Phi = \dfrac{\sigma^2}{2}\Delta\Phi \qquad \text{with} \\ \Phi(x,T) = e^{-C_T(x)/m\sigma^2} \end{cases} \tag{15}$$

which can be easily solved by convoluting the final condition with the Green function at an inverse time coordinate $\tau = T - t$. Due to the periodic boundary conditions, the Green function corresponds to a superposition of simple diffusion kernels on the infinite line, at periodic positions, i.e.:

$$G_{per}(x,t;x_0) = \sum_{n=-\infty}^{\infty} G_0(x,t;x_0) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x+nL-x_0)^2}{2\sigma^2 t}}$$

Hence, the solution for Φ, reads:

$$\Phi(x,t) = \sum_{n=-\infty}^{\infty} \int dy \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} e^{-\frac{(x+nL-y)^2}{2\sigma^2(T-t)}} \cdot e^{-C_T(y)/m\sigma^2} \tag{16}$$

and

$$u(x,t) = -m\sigma^2 \log \Phi(x,t)$$

Having the analytical solution in the continuous case, we can use it as a reference point to check numerically the convergence to the correct continuous limit.

For the following results, all the parameters that appear in the model were of $\mathcal{O}(1)$, among which the final cost used was a smooth function with amplitude of $\mathcal{O}(1)$, with an attraction point at $x_i = \frac{L}{2}$:

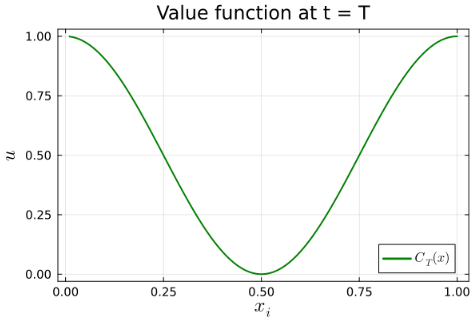


Figure 2: Final cost for the value function.

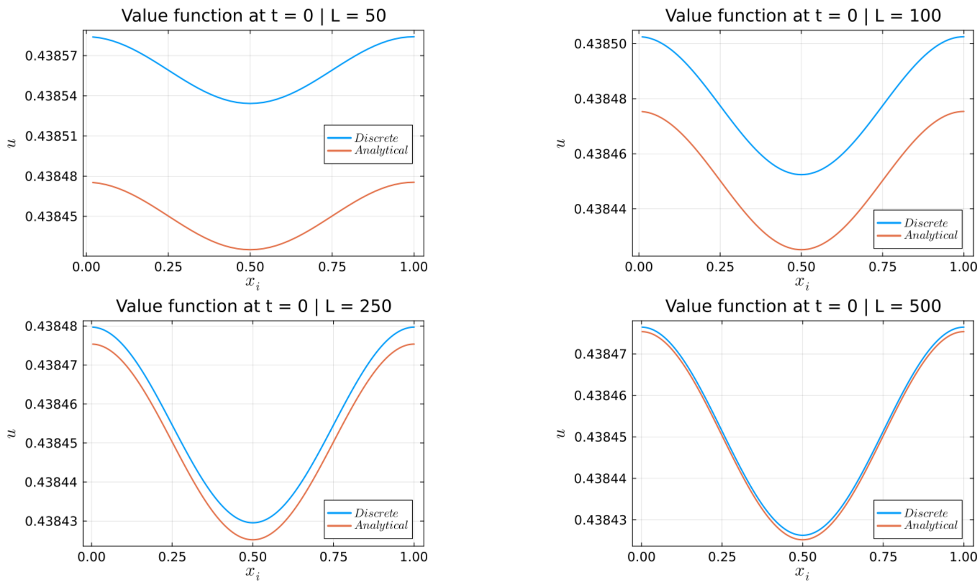


Figure 3: Value function back-propagated at the initial time, from the Discrete model (blue line) and the analytical continuous solution (orange line), for increasing number of sites L.

The convergence of the solution to the discrete model to the continuous solution can be seen in fig.3. As L increases ($\Delta x, \Delta t \to 0$), the discrete solution approaches the one obtained by the Cole-Hopf transformation. Having solved backwards the HJB and obtained the value function profile, the velocity field can be extracted. With this velocity, the density can be propagated forward according to eq.10. As it can be seen in fig.5, for the first half, the density profile is identical to the one driven purely by diffusion. This is due to the fact that the value function profile, and mainly the effect of the final cost, have been totally smoothened out by diffusion; therefore, the drift generated by its gradient is negligible compared to the $\mathcal{O}(1)$ diffusion coefficient, also seen in fig.4.

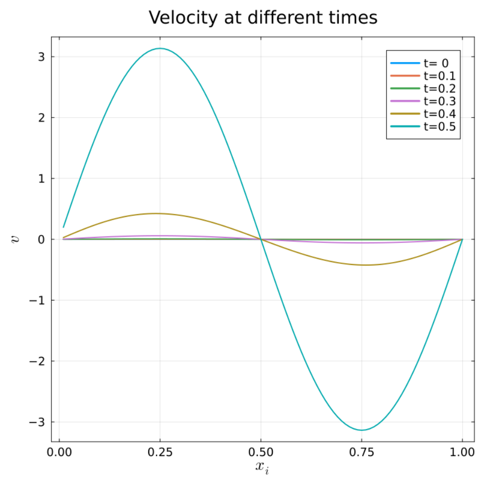


Figure 4: Velocity profile at different times.

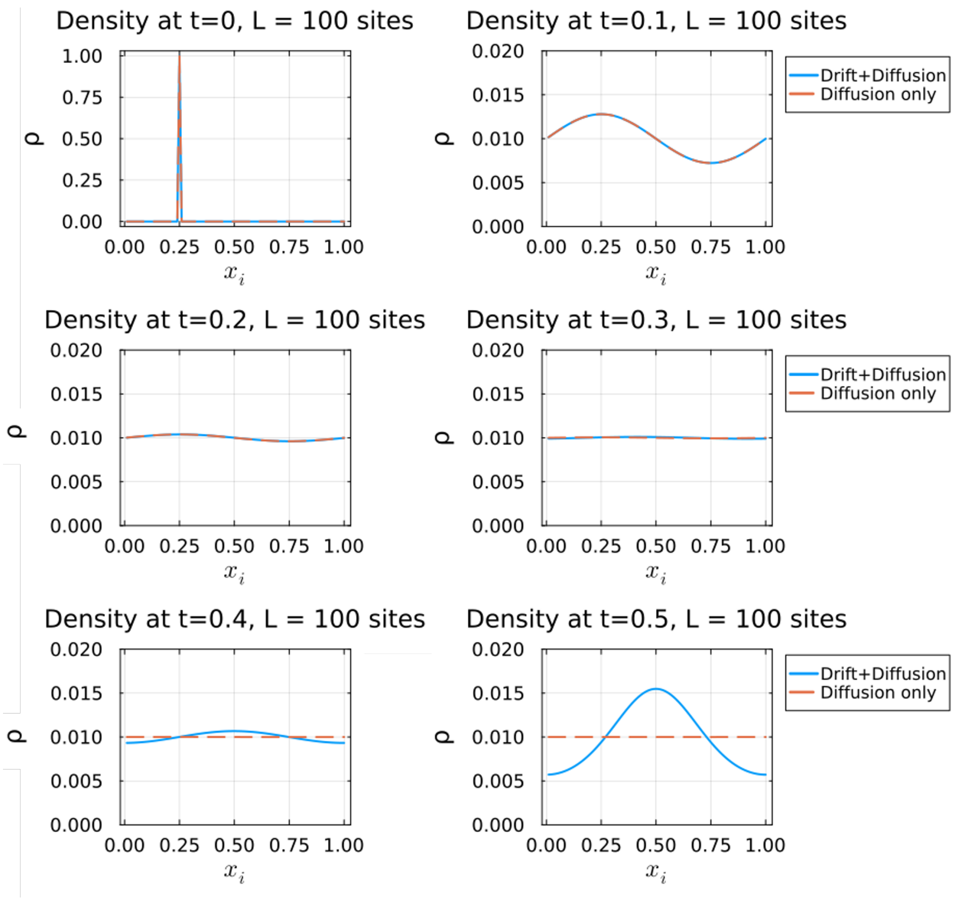


Figure 5: Density evolution under the strategic drift in the noisy environment (blue line) and one purely under diffusion (orange line).

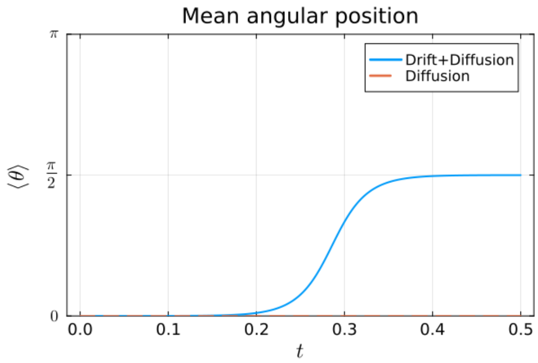
Only towards the end of the process the drift velocity starts to become significant. In this interval, the density profile, as shown in fig.5, firstly drifts towards the point of attraction $x_i = \frac{L}{2}$ and then narrows under the increasing velocity field. To further support the qualitative understanding of the system, we can look at the statistical properties of the density distribution. Due to the periodicity of the system (a ring of discrete sites), we map the position to an angle on the complex plane and get the first moments of the distribution on the unit circle.

$$z = e^{i\theta} \quad \text{with} \quad \theta = 2\pi x_i$$

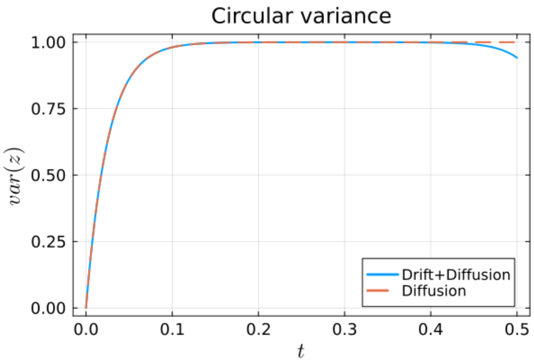
The mean angle is, then, independent of the initial position, hence an appropriate measure of the displacement of the mean value of the density distribution. The first two moments can be evaluated as following:

- $\langle z \rangle = \int dz \quad zP(z)$ and $\langle \theta \rangle = \arg(\langle z \rangle)$
- $var(z) = \int dz \quad |z - \langle z \rangle|^2 P(z) = 1 - |\langle z \rangle|^2$

In agreement with the density evolution in fig.5, it can be seen from fig.6a, the mean angular position, initialized at zero, does not move while diffusion dominates.Then,



(a) Mean angular position



(b) Circular variance of the position.

Figure 6: Statistical properties on the ring of the position distribution, starting from $x_0 = L/4$.

gradually the drift becomes significant, hence the distribution shifts its mean towards the point of attraction-the minimum of the final cost. When this is reached, the narrowing of the distribution takes place, which is reflected on its circular variance in fig.6b. Then, initial increase of the variance is evidently attributed to diffusion,which reaches a plateau, when the distribution has spread to the whole system, but only towards the end the variance drops under the effect of the increasing velocity field.

Monte Carlo simulation. With the same system, Monte Carlo simulations were also performed. The velocity field, extracted by solving the HJB equation, was used as an input in the hopping rates, with which the random walks were simulated. A histogram of the final positions of the walkers is shown in fig.7, which reproduces fairly well the final density profile, obtained by solving the HJB-Kolmogorov system.

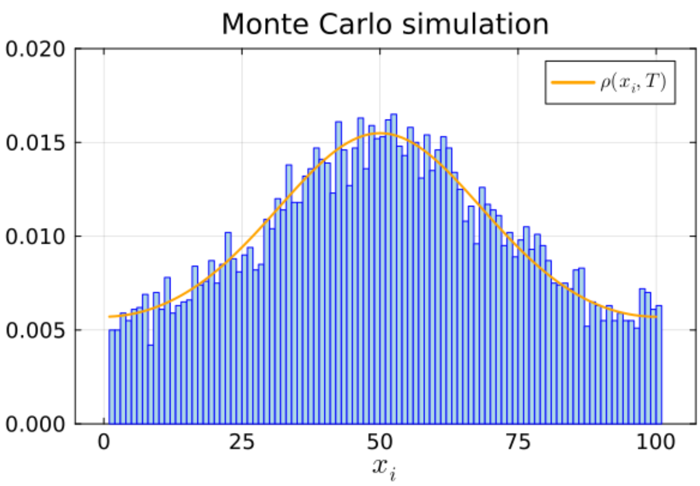


Figure 7: Histogram of the final position of the simulation walkers with the hopping rates of eq.(11)

4.3 Two-body game with exclusion

Having as a goal the N-body Mean Field Game with exclusion, an intermediate step is to treat the two-body game with the same dynamics. Specifically, by the end of the internship we plan to study the system of two agents on the discrete space with the kinetic dynamics of sec.4.1, but with the inability to jump to a site that is occupied by the other player. It is also planned to consider two zones: one interaction-free and one with exclusion, to see the difference in the optimal strategies that the players will follow. Exclusion in the two-body context is still an interesting case to study, as this interaction will clearly break the symmetry between the two players; there will be a clear separation between the strategies of the one preceding and the player left behind. However, this game poses an intrinsic challenge because of its stochastic nature. When each player optimizes their strategy, the state of the other player, which is stochastic due to the noise, will be crucial in the decision process. In this context, the fluctuations in each player's strategy will play an important role and should not be neglected. This situation may bring about the necessity to reformulate their utility to better match this stochastic case.

A direction which may be treated first in the remaining time of the internship is a deterministic version of the game. In particular, we can consider strategies of boolean variables, namely *go/not-go* moves, without noise. Then, a final cost can be chosen to differentiate the first and last to arrive to the end of the system. In this case, even though there is no stochasticity, how to perform the optimization is not trivial, since the strategies are no longer continuous variables. However, an iterative loop can still be constructed, where each player will minimize their personal cost given the output of the strategy of the other till convergence to the Nash equilibrium.

5 Conclusions & Prospects

During the work of this internship, a first attempt to modify the standard form of Mean Field Games has been presented. Derived from Langevin dynamics, the Mean Field Games do not account for interaction between the dynamics of the agents, constraining their ability to describe a wide variety of phenomena in crowd dynamics, where interactions play a dominant role. Congestion is a typical example of such phenomena and incorporating it is among the aims of this work.

More generally, Mean Field Games have to be derived from interacting microscopic dynamics to broaden their applicability. We focus on a discrete space and on Weakly Asymmetric Exclusion Processes; in that context a corresponding formulation of the dynamics through hopping rates and a discrete form of the Mean Field Games equations was one of the main parts of this internship. To ensure the validity of the discrete model, we checked its convergence to the continuous case (standard MFG) and its analytical solution. In order to further understand the dynamics on the lattice, the non-interacting case was examined, where we observe strong diffusion, and only towards the final time the strategies become non-trivial and the drift significant. In the meantime, by performing Monte Carlo simulations, we completed the toolbox that is planned to be used in the exclusion process.

By the end of the internship, the two-body game will be treated which we hope to provide useful insight for the effect of exclusion. Due to its challenge, the deterministic version of the game will may be first tackled, due to the short remaining time of the internship.

A more involved study of the two-body game will be carried out in the upcoming PhD thesis, before a full Mean Field Game derivation will be attempted. Later steps involve Mean Field Game derivations from other interacting models, more realistically used for pedestrian modeling [3][4]. At each stage, understanding the connection of the macroscopic MFG description with their corresponding agent-based model will be of primary importance and a physical interpretation of the interplay among the derived Mean Field Games will be needed. The goal is to integrate these Mean Field Games in the field of pedestrian dynamics in a multifaceted and well-understood way.

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