Propagation of a Dark Soliton in a Disordered Bose-Einstein Condensate

Nicolas Bilas[1] and Nicolas Pavloff[1]
[1]*Laboratoire de Physique Théorique et Modèles Statistiques, Université Paris Sud, bât. 100, F-91405 Orsay Cedex, France*
(Received 25 May 2005; published 23 September 2005)

We consider the propagation of a dark soliton in a quasi-1D Bose-Einstein condensate in presence of a random potential. This configuration involves nonlinear effects and disorder, and we argue that, contrarily to the study of stationary transmission coefficients through a nonlinear disordered slab, it is a well-defined problem. It is found that a dark soliton decays algebraically, over a characteristic length which is independent of its initial velocity, and much larger than both the healing length and the 1D scattering length of the system. We also determine the characteristic decay time.

DOI: 10.1103/PhysRevLett.95.130403 PACS numbers: 03.75.Lm, 05.60.Gg, 42.65.Tg

Phase coherent systems display wave mechanical properties distinct from those typically observed at macroscopic scale. In particular, transport in presence of disorder is strongly affected by interference effects, leading to weak or strong localization, as observed in many different fields (electronic or atomic physics, acoustics, electromagnetism). Our understanding of these effects have made great progresses over the last decades in the case of noninteracting linear waves. Some studies have considered the propagation of plane waves or bright solitons in a disordered region in the case of attractive interaction [see, e.g., the review [1] and the discussion below of the results of Ref. [2]], but almost nothing is known in the case of repulsive nonlinearity.

The field of Bose condensed atomic vapors allows new investigations of such phenomena in presence of repulsive or attractive interaction, in an intrinsically phase coherent system over which the experimental control is rapidly progressing. Such studies have begun with the observation of "fragmentation of the condensate" over a microchip [3]; random potentials have recently been engineered using an optical speckle pattern [4]; and it has also been proposed to implement disorder by using two different atomic species in an optical lattice [5].

In the present Letter, we study the transport properties of a quasi-one-dimensional (1D) Bose-Einstein condensate in presence of disorder and repulsive interaction. The configuration we study corresponds to the "1D mean field regime" [6], where the system is described by a 1D order parameter $\psi(x, t)$ depending on a single spatial variable: the coordinate x along the direction of propagation. $\psi(x, t)$ obeys the nonlinear Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + [U(x) + g|\psi|^2]\psi, \qquad (1)$$

where $U(x)$ is the random potential and g an effective coupling constant which reads $g = 2\hbar\omega_\perp a$ in the case of particles experiencing an effective repulsive interaction characterized by the 3D s-wave scattering length a ($a > 0$), and a transverse harmonic confinement with pulsation $\omega_\perp$ [7]. It is customary to define the oscillator length $a_\perp = (\hbar/m\omega_\perp)^{1/2}$ and $a_1 = a_\perp^2/2a$ [$-a_1$ is the 1D scattering length [7]]. Denoting by n_{1D} a typical value of $|\psi(x, t)|^2$, the 1D mean field regime corresponds to a situation where $1 \ll n_{1D} a_1 \ll (a_1/a_\perp)^2$. The first inequality ensures that the system does not get in the Tonks-Girardeau limit and the second that the transverse wave function is the ground state of the linear transverse Hamiltonian [6,8].

A particular issue specific to Eq. (1) is the very possibility to define a transmission coefficient. Since the equation is nonlinear, it is not possible in general to disentangle an incident and a reflected current in the region upstream the potential (in other words, a reflected atom will interact with the incident beam) [9]. A possible way for avoiding this problem is to change the transverse confining potential upstream the disordered potential so that, in this region, nonlinear effects become negligible [9]. However, even in this case, a technical difficulty arises because of multistability: several stationary solutions exist for a given input state [1,9]. Moreover, in the case of repulsive interaction we consider here, for extended enough disordered region, no stationary solution exists and the transmission coefficient can only be defined via a time average [10].

A way out of these difficulties consists in studying the propagation of a soliton in the system. This constitutes an intrinsically time-dependent problem, but the input and output states can be precisely characterized, and the transmission is simply defined by comparing the large time behaviors ($t \to \pm\infty$) of the solution. This route has been followed by Kivshar *et al.* [2] in the case of *attractive* nonlinearity ($g < 0$). In this case, a solitary wave is a bright soliton, characterized by two parameters: its velocity V and the number of particles N inside the soliton. The disordered potential was taken as

$$U(x) = g_{\rm imp} \sum_n \delta(x - x_n), \qquad \text{where } g_{\rm imp} = \frac{\hbar^2}{mb}. \qquad (2)$$

$U(x)$ describes a series of static impurities with equal intensity and random positions x_n. The x_n's are uncorrelated and uniformly distributed with mean density $n_{\rm imp}$. In this case $\langle U(x)\rangle = g_{\rm imp} n_{\rm imp}$ and $\langle U(x_1)U(x_2)\rangle - \langle U(x_1)\rangle \times \langle U(x_2)\rangle = (\hbar^2/m)^2 D\delta(x_1 - x_2)$, with $D = n_{\rm imp}/b^2$. From what is known in the case of linear waves, this type of potential is typical insofar as localization properties are concerned. In the weakly nonlinear regime $mV^2/2 \gg \hbar^2 N^2/(ma_1^2)$, it was found in Ref. [2] that the soliton velocity remains approximatively constant in the disordered region, whereas N shows an exponential decay similar to what occurs for a linear wave packet. In the opposite strongly nonlinear regime, it was found that the soliton behaves very differently, leading asymptotically to a configuration where both N and V become practically constant (independent of x).

In the present Letter, we consider the case of *repulsive* nonlinearity where the solitary waves are dark solitons. We find that the propagation of these solitons in a disordered potential is quite peculiar for the two following reasons: first, the strongly and weakly nonlinear cases cannot be considered as distinct because, in a given system, the number of particles forming the soliton cannot evolve independently from its velocity; and second, a dark soliton has a velocity bounded by the velocity of sound in the system. As a result, dark solitons behave differently from the bright ones studied in Ref. [2]: initially rather "nonlinear" solitons decay algebraically (and not exponentially), becoming eventually "linear." Besides, the length covered by the soliton in the disordered region is independent of its initial velocity.

Let us thus consider a dark soliton with initial velocity V, incident from the left on a disordered potential of type (2), with $x_0 = 0 < x_1 < \cdots$. This situation is illustrated in Fig. 1. The soliton is characterized by two parameters, its velocity V and the asymptotic background density $n_\infty = \lim_{x\to\pm\infty}|\psi(x,t)|^2$. Instead of n_∞, one can equivalently employ the chemical potential $\mu = gn_\infty$, the healing length $\xi = (a_1/n_\infty)^{1/2}$, or the speed of sound $c = \hbar/m\xi$. A dark soliton has a velocity $V \le c$, an energy

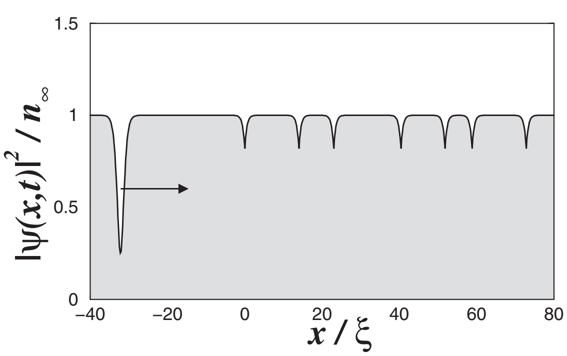

FIG. 1. Density profile of a dark soliton incident with velocity $V = c/2$ on pointlike repulsive obstacles with random positions corresponding to a potential $U(x)$ given in Eq. (2), with $\xi/b = 0.2$ and $n_{\rm imp}\xi \simeq 0.1$. The arrow represents the direction of propagation of the soliton.

$$E_{\rm sol} = \frac{4}{3}\mu\left(\frac{a_1}{\xi}\right)\left(1 - \frac{V^2}{c^2}\right)^{3/2}, \tag{3}$$

and consists in a density trough of typical extension $\xi(1 - V^2/c^2)^{-1/2}$, corresponding to a number of missing particles $\Delta N = 2(a_1/\xi)(1 - V^2/c^2)^{1/2}$. In the 1D mean field regime where (1) holds, $a_1 \gg \xi$ and ΔN is typically a large number, except in the limit where V is close to c. This occurs at velocities around $V_{\rm crit} = c[1 - (\xi/2a_1)^2]^{1/2}$. At such velocities $\Delta N \sim 1$ and the soliton has an extension $\sim a_1$.

We consider the case where the average separation between the impurities is much larger than the healing length ($n_{\rm imp}\xi \ll 1$) and the initial velocity of the soliton is not close to c. In this case, the scattering of the soliton from the impurities can be treated as a sequence of independent events. When the soliton encounters a single obstacle, it radiates phonons which form two counter propagating wave packets moving at velocity c. Accordingly, its energy decreases by an amount $\delta E_{\rm sol} = -E^+_{\rm rad} - E^-_{\rm rad}$, where $E^+_{\rm rad}$ ($E^-_{\rm rad}$) is the forward (backward) emitted energy. It was found in Ref. [11] that

$$E^\pm_{\rm rad} = \mu\left(\frac{\xi}{b}\right)^2 F^\pm(V/c), \tag{4}$$

where $F^\pm(v)$ is a dimensionless function defined for $v = V/c \in [0, 1]$ as

$$F^\pm(v) = \frac{\pi}{16v^6}\int_0^{+\infty} dy \frac{y^4(-v \pm \sqrt{1 + y^2/4})^2}{\sinh^2\left(\frac{\pi y\sqrt{1+y^2/4}}{2v\sqrt{1-v^2}}\right)}. \tag{5}$$

Equation (4) is a perturbative result valid in the limit $b \gg \xi$ and $V^2 \gg c^2(\xi/b)$. The first inequality ensures that the impurity only weakly perturbs the static background and the second that the scattering of the soliton by the impurity can be treated perturbatively. The soliton having lost energy during the collision, its velocity changes by an amount $\delta V = c\delta v$ which, from (3), is related to $\delta E_{\rm sol}$ via $v\delta v(1 - v^2)^{-1} = -\frac{1}{3}\delta E_{\rm sol}/E_{\rm sol}$.

Since $n_{\rm imp}\xi \ll 1$, one can go to the continuous limit considering the successive collisions as a sequence of random uncorrelated events. Over a length δx the solitons will experience $n_{\rm imp}\delta x$ such collisions. This leads to the following differential equation:

$$\frac{dv}{dx} = \frac{1}{4x_0}\frac{F^+(v) + F^-(v)}{v\sqrt{1 - v^2}}, \tag{6}$$

where $x_0 = a_1 b^2/(\xi^3 n_{\rm imp}) = a_1/(D\xi^3)$. Equation (6) can be solved analytically in the high velocity regime, when $v \to 1$. In this limit, $F^+(v) + F^-(v) \simeq \frac{4}{15}(1 - v^2)^{5/2}$ and, for a soliton of initial velocity $V_{\rm init}$ one obtains

$$\frac{V(x)}{c} = \left\{1 - \frac{1 - (V_{\rm init}/c)^2}{1 + [1 - (V_{\rm init}/c)^2]\frac{2x}{15x_0}}\right\}^{1/2}. \tag{7}$$

We compare in Fig. 2 the results of this approximate solution with the numerical solution of Eq. (6) in the cases $V_{\rm init}/c = 0.75$, 0.5, and 0.25. The agreement is very good, even for initial velocities which are not close to c.

The soliton is accelerated as it progresses through the disordered region (as seen in Fig. 2) because it radiates energy at each collision with an impurity. This increased velocity after a loss of energy is a typical feature of dark solitons which can be considered as particles with a negative kinetic mass which decreases with increasing energy [12] [see Eq. (3)]. One also notices in Fig. 2 that V saturates when it gets close to c, meaning that, in this regime, the rate of energy loss decreases. The reason for this phenomenon is that a dark soliton cannot have a velocity higher than c. As a result, when its velocity reaches this upper bound, the soliton cannot lose a large fraction of its energy, because this would lead, after the collision, to an unphysical value of V (larger than c). This phenomenon has an important consequence on the maximum distance L over which the soliton can travel in the disordered region. As seen in Fig. 2, L is very large and seems independent from the initial velocity of the soliton. In order to get a quantitative evaluation, we define L as being the length after which the soliton is a trough containing only one particle, i.e., the velocity $V(L)$ in Eq. (7) reaches the value $V_{\rm crit}$. In this limit the soliton can no longer be detected by standard imaging techniques, and for all practical purposes one can consider that it has totally decayed. From (7) one obtains

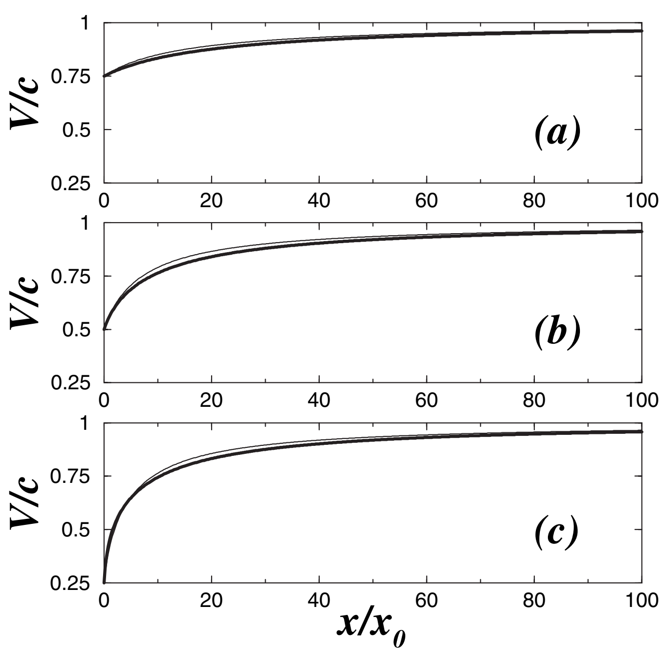

FIG. 2. Evolution of the velocity V of a dark soliton as a function of the distance x traveled in the disordered region. In each plot, the thick line corresponds to the numerical solution of Eq. (6) and the thin line to Eq. (7). Case (a) corresponds to an initial velocity $V_{\rm init} = 3c/4$, case (b) to $V_{\rm init} = c/2$, and case (c) to $V_{\rm init} = c/4$.

$$L = \frac{15x_0}{2}\left(\frac{2a_1}{\xi}\right)^2 = 30a_1\frac{(a_1/\xi)^2}{D\xi^3}. \qquad (8)$$

This confirms what was inferred from Fig. 2: a slow soliton will initially decay more rapidly than a fast one and altogether, the distance over which solitons can travel before completely decaying is independent of their initial velocity. As expected, L decreases for increasing disorder, the effect of the disorder being measured by the dimensionless parameter $D\xi^3$, i.e., by the two points correlation function of the random potential. Irrespective of the value of the parameter $D\xi^3$, we remark that L is large compared with a_1, since in the 1D mean field regime $a_1 \gg \xi$. Hence, a dark soliton covers quite a large distance in the disordered region before decaying.

The distance L is covered in a time τ which we now evaluate. It is important to realize that V is not the average velocity of the soliton, but its velocity between two obstacles: in vicinity of an impurity, the velocity of the soliton decreases if $g_{\rm imp} > 0$ and increases if $g_{\rm imp} < 0$. As a result, the asymptotic position of the soliton is shifted compared to what it would be in absence of obstacle. In the case of a single impurity, this shift Δ can be quite accurately evaluated by means of the "effective potential theory" as being [11]

$$\Delta = \int_{-\infty}^{+\infty} dx\left[1 - \frac{1}{\sqrt{1 - U_{\rm eff}(x)/mV^2}}\right], \qquad (9)$$

where $U_{\rm eff}$ is an effective potential which reads in the case of a pointlike impurity $U_{\rm eff}(x) = \frac{g_{\rm imp}}{2\xi}\cosh^{-2}(x/\xi)$. In the limit $V^2 \gg c^2(\xi/b)$ where Eq. (4) holds, the shift reads $\Delta \simeq -c^2\xi^2/(2bV^2)$. In the presence of multiple impurities, going to the continuous limit, one obtains that during a time δt the soliton covers a distance $\delta x = V\delta t + (n_{\rm imp}\delta x)\Delta$. Combining this relation with Eq. (6) one obtains a differential equation allowing to determine $v = V/c$ as a function of t:

$$\frac{dv}{dt} = \frac{c}{4x_0}\frac{F^+(v) + F^-(v)}{\sqrt{1 - v^2}}\frac{1}{1 - n_{\rm imp}\Delta}. \qquad (10)$$

In the limit $v \to 1$, this equation admits the analytical solution

$$t = \frac{15x_0}{c}\left\{G\left(v, \frac{n_{\rm imp}\xi^2}{2b}\right) - G\left(\frac{V_{\rm init}}{c}, \frac{n_{\rm imp}\xi^2}{2b}\right)\right\}, \qquad (11)$$

where

$$G(v, \alpha) = -\frac{\alpha}{v} + \frac{1 + 3\alpha}{4}\ln\left(\frac{1 + v}{1 - v}\right) + \frac{1 + \alpha}{2}\frac{v}{1 - v^2} \underset{v\to 1}{\simeq} \frac{1 + \alpha}{4(1 - v)}. \qquad (12)$$

We compared this approximate result with the numerical solution of (10) where Δ was evaluated through (9), and found that the accuracy of (11) is always very good, even for initial velocities not close to c [as was also the case for the approximate expression (7)].

The decay time τ of the soliton is the time at which $v = V_{\rm crit}/c \simeq 1 - \frac{1}{2}(\xi/2a_1)^2$:

$$\tau = \frac{L}{c}\left(1 + \frac{n_{\rm imp}\xi^2}{2b}\right). \qquad (13)$$

In this expression—as in (8)—we neglected a corrective term depending of the initial velocity, smaller by a factor $(\xi/a_1)^2$ than the leading term. τ is proportional to L/c, with a slight modification due to the shift induced at each scattering [13]: repulsive obstacles ($b > 0$) lead to an increased decay time since the soliton covers the distance L slightly more slowly that in the case of attractive obstacles.

Equation (13) can be given a simple physical interpretation (in a less rigorous setting) in the framework of the "effective potential approximation" [11]. In this approximation, solitons are considered as classical particles of mass $2m$ evolving in a potential $U_{\rm eff}$. One thus has $\langle m\dot{x}^2 + U_{\rm eff}(x)\rangle = \langle mV^2(x)\rangle$. The mean value of $U_{\rm eff}$ is the same as the one of U [14] and from Fig. 2, one sees that at leading order it is sensible to approximate $\langle mV^2(x)\rangle$ by $mc^2 = \mu$. One thus obtains $\langle \dot{x}^2\rangle \simeq c^2(1 - \langle U(x)\rangle/\mu)$. Finally, τ can be evaluated through the formula

$$\tau = \frac{L}{\langle \dot{x}\rangle} \simeq \frac{L}{\langle \dot{x}^2\rangle^{1/2}} \simeq \frac{L}{c}\left(1 + \frac{\langle U(x)\rangle}{2\mu}\right), \qquad (14)$$

which is identical to (13). Since formulas (8) and (14) depend only on simple characteristics of the random potential (the average and the two points correlation function), we expect them to be of very general validity, poorly affected by the specific potential present in the disordered region.

A final point to clarify is the effect of the random potential on the occurrence of superfluidity and Bose-Einstein condensation; i.e., is Eq. (1) truly applicable? In the strong disorder limit, a quantum phase transition occurs at $T = 0$ leading to a (nonsuperfluid) Bose glass phase [15] where the description of the system with a single order parameter $\psi(x, t)$ is inappropriate. However, in the case we consider here of an atomic vapor described as a weakly interacting Bose gas, it has been shown that a small amount of disorder does not drastically alter the properties of the system, but merely decreases the condensate and the superfluid fraction [16]. More precisely, based on the evaluations presented in Ref. [17] one can show that this effect is negligible provided $n_{\rm imp}\xi^3/b^2 \ll 1$, which is the case in the present study.

In conclusion, we have presented a description of the motion of a dark soliton in a disordered region. The soliton radiates energy when it encounters an obstacle. The repulsion between the particles has important consequences on the propagation of the dark soliton, whose salient features are all at variance with the one expected in the case of a linear wave packet or of a bright soliton: (i) the soliton is *accelerated* to the velocity of sound and disappears, (ii) its decay is algebraic, and (iii) the characteristic decay length and decay time are independent of the initial velocity of the soliton.

These results are generic and apply to many different fields (among which, optics in nonlinear fibers with positive group velocity dispersion) but the most promising experimental configurations seems to be achievable for a Bose condensed atomic vapor, either in a corrugated magnetic guide over a microchip [3], or in an elongated trap in presence of an optical speckle pattern [4].

We acknowledge support form CNRS and Ministère de la Recherche (Grant ACI Nanoscience 201). Laboratoire de Physique Théorique et Modèles Statistiques is Unité Mixte de Recherche de l'Université Paris XI et du CNRS, UMR 8626.

[1] S. A. Gredeskul and Y. S. Kivshar, Phys. Rep. **216**, 1 (1992).
[2] Y. S. Kivshar, S. A. Gredeskul, A. Sánchez, and L. Vázquez, Phys. Rev. Lett. **64**, 1693 (1990).
[3] J. Fortágh, H. Ott, S. Kraft, A. Günther, and C. Zimmermann, Phys. Rev. A **66**, 041604(R) (2002); A. E. Leanhardt *et al.*, Phys. Rev. Lett. **89**, 040401 (2002); J. Estève *et al.*, Phys. Rev. A **70**, 043629 (2004).
[4] J. E. Lye *et al.*, Phys. Rev. Lett. **95**, 070401 (2005); D. Clément *et al.*, cond-mat/0506638.
[5] U. Gavish and Y. Castin, Phys. Rev. Lett. **95**, 020401 (2005).
[6] C. Menotti and S. Stringari, Phys. Rev. A **66**, 043610 (2002).
[7] M. Olshanii, Phys. Rev. Lett. **81**, 938 (1998).
[8] D. S. Petrov, D. M. Gangardt, and G. V. Shlyapnikov, J. Phys. IV (France) **116**, 5 (2004).
[9] P. Leboeuf, N. Pavloff, and S. Sinha, Phys. Rev. A **68**, 063608 (2003).
[10] T. Paul, P. Leboeuf, N. Pavloff, K. Richter, and P. Schlagheck, cond-mat/0509446.
[11] N. Bilas and N. Pavloff, cond-mat/0501053 [Phys. Rev. A (to be published)].
[12] P. O. Fedichev, A. E. Muryshev, and G. V. Shlyapnikov, Phys. Rev. A **60**, 3220 (1999).
[13] The modification is small because $n_{\rm imp}\xi^2/b \ll 1$.
[14] This is because $U_{\rm eff}$ is the convolution of U with a normalized function; see [11].
[15] M. P. A. Fisher, P. B. Weichman, G. Grinstein, and D. S. Fisher, Phys. Rev. B **40**, 546 (1989).
[16] K. Huang and H. F. Meng, Phys. Rev. Lett. **69**, 644 (1992); S. Giorgini, L. Pitaevskii, and S. Stringari, Phys. Rev. B **49**, 12938 (1994).
[17] G. E. Astrakharchik and L. P. Pitaevskii, Phys. Rev. A **70**, 013608 (2004).