Nonlinear polarization waves in a two-component Bose-Einstein condensate

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A two-component Bose-Einstein condensate whose dynamics is described by a system of coupled Gross-Pitaevskii equations accommodates waves with different symmetries. A first type of waves corresponds to excitations for which the motion of both components is locally in phase. For the second type of waves, the two components have a counterphase local motion. When the values of the inter- and intracomponent interaction constants are different, the long-wavelength behavior of these two modes corresponds to two types of sound with different velocities. In the limit of weak nonlinearity and small dispersion, the first mode is described by the well-known Korteweg-de Vries equation. In the same limit, we show that the second mode can be described by the Gardner equation if the values of the two intracomponent interaction constants are sufficiently close. This leads to a rich variety of nonlinear excitations (solitons, kinks, algebraic solitons, breathers) which do not exist in the Korteweg-de Vries description.

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I. INTRODUCTION

The two-component Gross-Pitaevskii (GP) equation describes the evolution of nonlinear excitations in various physical systems. Apparently, it first appeared in nonlinear optics under the name of "vector nonlinear Schrödinger equation" in order to describe self-interaction of electromagnetic waves with account of their polarization (see, e.g., [1–4]). When the values of the nonlinear constants are equal, the corresponding problem is completely integrable by the inverse scattering transform method [2] and many particular solutions have been found (see, e.g., [5] and references therein, and also Refs. [6,7] for an account of recent progress). Taking birefringence effects into account [8] leads to an even richer dynamics, as confirmed by experiments on propagation of light pulses in fibers [9]. Recently, the realization of spinor atomic Bose-Einstein condensates (BECs)—see the reviews [10,11] and references therein—as well as microcavity polariton condensates [12] arouse a surge of interest in vector solitons.

A specific feature of two-component condensates is the existence of two types of elementary excitations. In one mode both components move locally in phase; in the case of a small-amplitude potential flow this corresponds to the usual sound waves, which consist in density oscillations. In the other mode—denoted henceforth as the "polarization mode"—the two components move in counterphase in such a way that, in some situations, the total density remains constant in spite of the excitation of the relative motion of the condensate components. These two modes have different dispersion relations with different long-wavelength behaviors—i.e., different sound velocities—and, due to their different symmetries, different methods are required for their generation [13,14]. In some sense, this is analogous to the situation observed for the first and second sound in superfluid HeII: The second sound, which corresponds to a temperature (and entropy) wave, cannot be excited by oscillations of the container wall, contrarily to the usual density waves associated with the first sound (see, e.g., Ref. [15]).

In the present paper, we study the weakly nonlinear and weakly dispersive evolution of these two modes. Whereas in this limit the density waves are described by the standard Korteweg-de Vries (KdV) equation which accounts for quadratic nonlinearities, the polarization mode is much more peculiar. We show that, in typical experimental situations, its description requires one to take into account third-order nonlinearities and that the dynamics of the corresponding nonlinear polarization wave is then modeled by the Gardner equation. As is well known, this is a quite generic equation which arises when the coefficient of the quadratic nonlinear term is small and when the wave amplitude has the same order of magnitude as this coefficient. In particular, the Gardner equation (and also the modified KdV equation, which shares strong similarities with it) has been employed for describing internal waves in stratified fluids [16,17], lattice dynamics modeled by the discrete nonlinear Schrödinger equation [18], and quantum dynamics of condensates in optical lattices [19]. The Gardner equation admits a wide spectrum of nonlinear excitations [20] which can be generated by the flow of a fluid past an obstacle [21]. We expect that the phenomenology associated with this rich dynamics can be observed experimentally in the flow of a two-component BEC.

The paper is organized as follows. The main equations and the linear excitations of the system are presented in Sec. II; in this section we also discuss the degree of admixture between the density and the polarization waves. In Sec. III we derive the equations governing the weakly nonlinear and weakly dispersive dynamics of the system. When the values of the intraspecies interaction constants of the two components are close, we find that the polarization waves have to be described by the Gardner equation. In Sec. IV we display some solutions of the equations established in Sec. III and show that they are in good agreement with the ones deduced from a numerical integration of the full vector GP equation which governs the dynamics of the system. Finally, we present our conclusions and discuss experimental issues in Sec. V. Some technical points on the derivation of the nonlinear perturbation theory are given in Appendixes A–C.

II. MAIN EQUATIONS AND LINEAR WAVES

We consider a two-component condensate confined in a one-dimensional structure (e.g., a cigar-shaped trap in the case of an atomic BEC or a quantum wire embedded into an elongated cavity in the case of a polariton condensate). The condensate is described by a one-dimensional (1D) two-component order parameter $(\psi_+(x,t),\psi_-(x,t))$. In the case of an atomic BEC, the two-component order parameter may describe a two-species BEC such as realized by considering, for instance, ^{87}Rb in two hyperfine states [22], or a mixture of two elements [23], or different isotopes of the same atom [24]. In the case of a polariton condensate, the components ψ_+ and ψ_- account for the pseudospin of the polariton which consists in eigenstates of a system of interacting excitons and photons with spin projection ± 1. The dynamics of the system is modeled by a set of coupled Gross-Pitaevskii equations,

$$i\,\partial_t\psi_\pm + \tfrac{1}{2}\partial_{xx}\psi_\pm - [(\alpha_1\pm\delta)\,|\psi_\pm|^2 + \alpha_2\,|\psi_\mp|^2]\psi_\pm = 0, \quad (1)$$

written here under a standard nondimensional form. The parameter δ measures the difference between the intraspecies nonlinear interaction constants: $\alpha_1+\delta$ and $\alpha_1-\delta$ correspond to interactions between the particles described, respectively, by the components ψ_+ and ψ_- of the order parameter. It is supposed that the two species are labeled in such a way that $\delta>0$. The interspecies interaction constant is denoted by α_2. It may be either positive or negative. In the following we mostly concentrate on the case $\alpha_2>0$ but for completeness we give formulas valid for both signs of α_2 at the end of Sec. III A [Eqs. (21) and (22)], Sec. III B [Eqs. (24) and (25)], and Sec. III C [Eqs. (28) and (29)].

It is convenient to describe the dynamics of the condensate in terms of the field variables ρ, Φ, θ, and ϕ defined by [25]

$$\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} = \sqrt{\rho}\,e^{i\Phi/2}\,\zeta, \quad \zeta = \begin{pmatrix}\cos(\theta/2)\,e^{-i\phi/2}\\ \sin(\theta/2)\,e^{i\phi/2}\end{pmatrix}. \quad (2)$$

Here $\rho(x,t)=|\psi_+|^2+|\psi_-|^2$ denotes the total density of the condensate and $\Phi(x,t)$ corresponds to the velocity potential of its in-phase motion. The angle $\theta(x,t)$ is the variable describing the relative density of the two components [$\cos\theta = (|\psi_+|^2-|\psi_-|^2)/\rho$] and $\phi(x,t)$ is the potential of their relative (counterphase) motion. According to Eq. (2), the densities of the components of the condensate are given by

$$\begin{aligned}\rho_+(x,t) &= |\psi_+|^2 = \rho\cos^2(\theta/2),\\ \rho_-(x,t) &= |\psi_-|^2 = \rho\sin^2(\theta/2).\end{aligned} \quad (3)$$

Their phases are defined as

$$\varphi_+(x,t) = \tfrac{1}{2}(\Phi-\phi), \quad \varphi_-(x,t) = \tfrac{1}{2}(\Phi+\phi), \quad (4)$$

and their velocities are

$$v_+(x,t) = \partial_x\varphi_+, \quad v_-(x,t) = \partial_x\varphi_-. \quad (5)$$

It is also convenient to introduce a unit vector $\mathbf{S}$ representing the spinor ζ. One defines it as

$$\mathbf{S}(x,t) = \zeta^\dagger\,\boldsymbol{\sigma}\,\zeta, \quad (6)$$

where $\boldsymbol{\sigma} = (\sigma_x \quad \sigma_y \quad \sigma_z)^T$ is the Pauli matrices vector. In terms of the angles θ and ϕ and of the complex fields ψ_+ and ψ_-, the vector $\mathbf{S}$ reads

$$\mathbf{S}(x,t) = \begin{pmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{pmatrix} = \frac{1}{\rho}\begin{pmatrix}2\,\mathrm{Re}(\psi_+^*\,\psi_-)\\ 2\,\mathrm{Im}(\psi_+^*\,\psi_-)\\ |\psi_+|^2-|\psi_-|^2\end{pmatrix}. \quad (7)$$

The vector $\mathbf{S}$ can be called the *polarization vector* of the two-component condensate.

Substitution of Eq. (2) into Eq. (1) yields the system,

$$\begin{aligned}&\rho_t + \frac{1}{2}[\rho\,(U - v\cos\theta)]_x = 0,\\ &\Phi_t + \frac{\rho_x^2}{4\rho^2} - \frac{\rho_{xx}}{2\rho} - \frac{\cot\theta}{2\rho}(\rho\,\theta_x)_x + \frac{1}{4}\left(\Phi_x^2+\theta_x^2+\phi_x^2\right)\\ &\qquad + \rho\,(\alpha_1+\alpha_2+\delta\cos\theta) = 0,\\ &\rho\,\theta_t + \frac{1}{2}[\rho\,U\theta_x + (\rho\,v\sin\theta)_x] = 0,\\ &\phi_t + \frac{1}{2}U\,v - \frac{1}{2\rho\sin\theta}(\rho\,\theta_x)_x\\ &\qquad -\rho\,[(\alpha_1-\alpha_2)\cos\theta+\delta] = 0,\end{aligned} \quad (8)$$

where $U=\Phi_x$ and $v=\phi_x$ are, respectively, the mean and the relative velocities; hence the velocities (5) of the two components are equal to $v_\pm = (U\mp v)/2$. In what follows, we shall consider solitons and other nonlinear excitations corresponding to small deviations from a uniform quiescent condensate and, to simplify the treatment, we shall assume that the values of the nonexcited densities of the two components are equal. This can be realized by choosing either $\theta\to\pi/2$ or $\theta\to 3\pi/2$ when $|x|\to\infty$. To be definite, we shall consider the case $\theta\to\pi/2\equiv\theta_0$ since the other choice leads to similar results. The other parameters of the wave satisfy the boundary conditions,

$$\rho\to\rho_0, \quad U\to 0, \quad v\to 0 \quad \text{when} \quad |x|\to\infty. \quad (9)$$

In the uniform (i.e., constant density) state the time dependence of the two components of the order parameter is described by the multiplicative phase factors $\psi_\pm \propto \exp(-i\,\mu_\pm\,t)$, where the chemical potentials μ_+ and μ_- are given by

$$\begin{aligned}\mu_+ &= \frac{1}{8}(U-v)^2 + \rho_0\left[\alpha_2 + (\alpha_1-\alpha_2+\delta)\cos^2\frac{\theta_0}{2}\right],\\ \mu_- &= \frac{1}{8}(U+v)^2 + \rho_0\left[\alpha_2 + (\alpha_1-\alpha_2-\delta)\cos^2\frac{\theta_0}{2}\right].\end{aligned} \quad (10)$$

For future convenience we keep in Eq. (10) the general notation "θ_0" (even if we chose $\theta_0=\pi/2$).

We shall first consider linear waves propagating on top of a uniform background. Linearizing the system (8) with respect to the small variables $\rho'=\rho-\rho_0$, $\theta'=\theta-\theta_0=\theta-\pi/2$,

U, and v, one obtains

$$\begin{aligned} \rho'_t + \frac{\rho_0}{2}\,U_x &= 0,\\ U_t + (\alpha_1+\alpha_2)\,\rho'_x - \frac{1}{2\,\rho_0}\,\rho'_{xxx} - \rho_0\,\delta\,\theta'_x &= 0,\\ \theta_t + \frac{1}{2}\,v_x &= 0,\\ v_t + \rho_0\,(\alpha_1-\alpha_2)\,\theta'_x - \frac{1}{2}\,\theta'_{xxx} - \delta\,\rho'_x &= 0. \end{aligned} \tag{11}$$

One can notice that for $\delta = 0$ the system (11) splits into two pairs of independent equations: The first pair describes *density modes* (corresponding to oscillations of ρ and U) and the second pair describes *polarization modes* (corresponding to oscillations of θ and v). The distinction between these two modes is not strict when $\delta \neq 0$, but for convenience, we keep using the denominations "density modes" and "polarization modes" in what follows since the corresponding excitations are no longer hybridized in the limit $\delta \to 0$.

If we look for the solutions of the system (11) in the form of plane waves, i.e., assuming that ρ', U, θ', and v are proportional to $\exp[i(k\,x - \omega\,t)]$, then we readily get the dispersion laws,

$$\begin{aligned} \omega_d^2(k) &= \frac{\rho_0}{2}\big(\alpha_1 + \sqrt{\alpha_2^2+\delta^2}\big)k^2 + \frac{k^4}{4},\\ \omega_p^2(k) &= \frac{\rho_0}{2}\big(\alpha_1 - \sqrt{\alpha_2^2+\delta^2}\big)k^2 + \frac{k^4}{4}, \end{aligned} \tag{12}$$

for the density and the polarization modes, respectively. Expressions for the dispersion laws of linear waves propagating along binary condensates in which the background densities of the two components are not equal were found in Refs. [26,27].

In the long-wavelength limit $k \to 0$ we obtain the expressions for the density and polarization sound velocities $c_{d,p}^2 = \lim_{k\to 0}\omega_{d,p}^2(k)/k^2$:

$$\begin{aligned} c_d^2 &= \frac{\rho_0}{2}\big(\alpha_1 + \sqrt{\alpha_2^2+\delta^2}\big),\\ c_p^2 &= \frac{\rho_0}{2}\big(\alpha_1 - \sqrt{\alpha_2^2+\delta^2}\big). \end{aligned} \tag{13}$$

When $\delta \neq 0$, the degree of admixture between the density and the polarization waves can be evaluated by studying the dynamic structure factor $S(k,\omega)$ of the system. At zero temperature $S(k,\omega) = -\frac{1}{\pi}\,\Theta(\omega)\,\mathrm{Im}\,\chi(k,\omega)$, where Θ is the Heaviside step function, and $\chi(k,\omega)$ is the density response function (see, e.g., Refs. [28,29]). The susceptibility function $\chi(k,\omega)$ characterizes how the density of the system responds to a weak external scalar potential with wave vector k and frequency ω. In the presence of such a perturbation, using a trivial modification of Eq. (11) accounting for the effect of an external scalar potential, one obtains

$$\chi(k,\omega) = \frac{Z_d(k)}{(\omega + i\,0^+)^2 - \omega_d^2(k)} + \frac{Z_p(k)}{(\omega + i\,0^+)^2 - \omega_p^2(k)}, \tag{14}$$

where

$$\begin{aligned} Z_d(k) &= \frac{\rho_0}{2}\left(1 + \frac{\alpha_2}{\sqrt{\alpha_2^2+\delta^2}}\right)k^2\\ \text{and}\quad Z_p(k) &= \frac{\rho_0}{2}\left(1 - \frac{\alpha_2}{\sqrt{\alpha_2^2+\delta^2}}\right)k^2. \end{aligned} \tag{15}$$

This yields

$$\begin{aligned} S(k,\omega) = {} & \frac{Z_d(k)}{2\,\omega_d(k)}\,\delta[\omega - \omega_d(k)]\\ & + \frac{Z_p(k)}{2\,\omega_p(k)}\,\delta[\omega - \omega_p(k)]. \end{aligned} \tag{16}$$

One has $\int_{\mathbb{R}} \omega\, S(k,\omega)\, d\omega = [Z_d(k) + Z_p(k)]/2 = \rho_0 k^2/2$, in agreement with the f-sum rule [28,29]. In the case where $\delta = 0$, $Z_p(k)$ vanishes and the sum rule is exhausted by the peak at $\omega_d(k)$. As stated above, this means that when $\delta = 0$ the density fluctuations are completely described by the branch with dispersion $\omega_d(k)$. A further verification is that, in this case, the Feynman relation holds: $\omega_d(k) = k^2/(2\,S_k)$, where $S_k = \int_{\mathbb{R}} S(k,\omega)\,d\omega$ [30]. When $\delta \neq 0$ the relative contribution to the density fluctuation of each branch can be evaluated by computing in which proportion the two peaks in Eq. (16) contribute to S_k. The ratio of these two contributions to S_k is easily evaluated in the low- and large-k limits. Provided one does not go in the Manakov regime described below, one sees in each of these two limiting cases that, for small δ, the contribution to S_k of the mode with dispersion $\omega_p(k)$ is lower by a factor of order $(\delta/\alpha_2)^2$ than the contribution of the mode with dispersion $\omega_d(k)$. Hence, in the case of interest in the present work, where δ is small compared to α_2, one can legitimately denote the branch with dispersion $\omega_d(k)$ the "density modulation branch," and the one with dispersion $\omega_p(k)$ the "polarization modulation one." Note that the above analysis has been done in the case where $\alpha_2 > 0$. When $\alpha_2 < 0$, the roles of the density and of the polarization mode are exchanged, hence the subscripts "d" and "p" have to be permuted in Eqs. (12)–(16). Note also that in all the present work we suppose that

$$\alpha_1^2 > \alpha_2^2 + \delta^2, \tag{17}$$

which is the condition of modulational stability of the polarization mode (see, e.g., Ref. [31]).

As we can see from Eq. (13), in the Manakov regime where the nonlinear constants are equal (i.e., when $\delta = 0$ and $\alpha_1 = \alpha_2$), the polarization sound velocity vanishes. In this case the linear dispersion relation $\omega_p \simeq \pm\, c_p\, k$ can no longer be considered as correctly describing the dispersion relation in the long-wavelength limit and the dispersive effects cannot be considered as small, even when $k \to 0$. In the opposite configuration where the difference $\alpha_1 - (\alpha_2^2 + \delta^2)^{1/2}$ is large enough, the regime of linear dispersion becomes of great importance when the characteristic value of the wave vector k satisfies the condition,

$$k^2 \ll c_{d,p}^2. \tag{18}$$

In this case, for linear waves propagating in the positive-x direction, the dispersion laws (12) can be approximated by

$$\omega_{d,p}(k) \simeq c_{d,p}\, k + \frac{1}{8\, c_{d,p}}\, k^3. \tag{19}$$

Correspondingly, the wave amplitude, say, $\rho'(x,t)$, satisfies the linear equation,

$$\rho'_t + c_{d,p}\, \rho'_x - \frac{1}{8\, c_{d,p}}\, \rho'_{xxx} = 0, \tag{20}$$

where the last term describes a small dispersive correction to the propagation of pulses with constant sound speeds c_d or c_p.

If the amplitude ρ' is small but finite and such that the last term in Eq. (20) has the same order of magnitude as the leading nonlinear correction [typically, $\sim(\rho')^2$], then nonlinear effects cannot be omitted for correct description of the propagation of the pulse. This issue is addressed in the next section.

III. EVOLUTION EQUATIONS FOR WEAKLY NONLINEAR WAVES

A. Nonlinear density waves

We now take into account nonlinear effects in the propagation of a density pulse, with an accuracy up to second order in the field variables ρ', U, θ', and v. As shown in Appendix A, if, in a first stage, we neglect the dispersive effects, then the system (8) reduces to a single evolution equation [Eq. (A8)]. Dispersion can then be correctly taken into account by simply adding in this equation the dispersive term taken from Eq. (20). This is legitimate because the only other possible quadratic and dispersive term ($\sim\rho^{(1)}\,\rho^{(1)}_{\xi\xi}$) which one could consider including in our description is forbidden since it does not have the same symmetry with respect to the transformations $x \to -x$ and $t \to -t$ as the other terms. Physically, this term would correspond to a nonlinear damping absent in the conservative model (1). Thus, we arrive at the equation,

$$\rho'_t + c_d\, \rho'_x + \frac{3\, c_d \left(2\sqrt{\alpha_2^2+\delta^2} - |\alpha_2|\right)}{2\, \rho_0 \sqrt{\alpha_2^2+\delta^2}}\, \rho'\, \rho'_x - \frac{1}{8\, c_d}\, \rho'_{xxx} = 0, \tag{21}$$

where $\rho'(x,t) = \rho(x,t) - \rho_0$. Once the solution of Eq. (21) is found, the other field variables can be obtained using relations (A5) which we rewrite here for completeness with the final notations:

$$\begin{aligned} U(x,t) &= \frac{2\, c_d}{\rho_0}\, \rho'(x,t), \\ \theta'(x,t) &= \frac{\alpha_2 - \mathrm{sgn}(\alpha_2)\sqrt{\alpha_2^2+\delta^2}}{\rho_0\, \delta}\, \rho'(x,t), \\ v(x,t) &= \frac{2\, c_d \left[\alpha_2 - \mathrm{sgn}(\alpha_2)\sqrt{\alpha_2^2+\delta^2}\right]}{\rho_0\, \delta}\, \rho'(x,t). \end{aligned} \tag{22}$$

Equation (21) is the KdV equation for weakly nonlinear density waves. Although we only detailed the computation leading to Eqs. (21) and (22) in the case where α_2 is positive, we wrote the final results in a form which is also valid when α_2 is negative. In the latter case, formulas (13) defining the sound velocities c_d and c_p have to be exchanged.

In the limit $\delta \to 0$ we get the equation (valid for both signs of α_2),

$$\rho'_t + c_d^{(0)}\, \rho'_x + \frac{3\, c_d^{(0)}}{2\, \rho_0}\, \rho'\, \rho'_x - \frac{1}{8\, c_d^{(0)}}\, \rho'_{xxx} = 0, \tag{23}$$

with $c_d^{(0)} \equiv c_d(\delta = 0) = \{\rho_0(\alpha_1+\alpha_2)/2\}^{1/2}$, which reduces to the KdV equation for shallow Manakov solitons in the case where $\alpha_2 = \alpha_1$.

B. Nonlinear polarization waves: Quadratic nonlinearity

The equation of propagation of nonlinear polarization waves with account of quadratic nonlinearities can be obtained by a method similar to the one employed for obtaining the results presented in Sec. III A (see Appendix B). Starting from Eq. (B4) and reintroducing the dispersive effects as explained in Sec. III A, we arrive here also at a KdV equation,

$$\theta'_t + c_p\, \theta'_x + \mathrm{sgn}(\alpha_2)\, \frac{3\, c_p \left(\alpha_2^2 + 2\,\delta^2 - |\alpha_2|\sqrt{\alpha_2^2+\delta^2}\right)}{2\, \delta \sqrt{\alpha_2^2+\delta^2}}\, \theta'\, \theta'_x - \frac{1}{8\, c_p}\, \theta'_{xxx} = 0, \tag{24}$$

describing the dynamics of weakly dispersive and weakly nonlinear polarization waves. As stated in Appendix B, whereas in Sec. III A the KdV Eq. (22) has been written for the field variable ρ', it is here more convenient to derive the nonlinear perturbation theory using θ'. The other field variables are expressed in terms of θ' as follows:

$$\begin{aligned} \rho'(x,t) &= \frac{\rho_0\, \delta}{\alpha_2 + \mathrm{sgn}(\alpha_2)\sqrt{\alpha_2^2+\delta^2}}\, \theta'(x,t), \\ U(x,t) &= \frac{2\, c_p\, \delta}{\alpha_2 + \mathrm{sgn}(\alpha_2)\sqrt{\alpha_2^2+\delta^2}}\, \theta'(x,t), \\ v(x,t) &= 2\, c_p\, \theta'(x,t). \end{aligned} \tag{25}$$

The derivation of Eqs. (24) and (25) presented in Appendix B is only valid when $\alpha_2 > 0$, but we took care to write the final results (24) and (25) in a form which is also correct when $\alpha_2 < 0$. Note, however, that in the latter case, the definition of c_p has to be replaced by the one of c_d in Eq. (13).

In Eq. (24), contrarily to the case of the density waves exposed in Sec. III A, the coefficient of the nonlinear term vanishes in the limit $\delta \to 0$: when $\delta \ll |\alpha_2|$ we get

$$\theta'_t + c_p^{(0)}\, \theta'_x + \frac{9\, c_p^{(0)}\, \delta}{4\, \alpha_2}\, \theta'\, \theta'_x - \frac{1}{8\, c_p^{(0)}}\, \theta'_{xxx} = 0, \tag{26}$$

with $c_p^{(0)} \equiv c_p(\delta = 0) = \{\rho_0(\alpha_1-\alpha_2)/2\}^{1/2}$. This result is valid for both signs of α_2. It implies that when $|\theta'| \sim \delta \ll 1$ the level of accuracy accepted here is not sufficient: The cubic nonlinear terms [$\sim(\theta')^3$] neglected in the present treatment have the same order of magnitude as the quadratic term in the KdV Eq. (26). Thus, in the limit of small δ, we have to consider the next order of approximation.

C. Nonlinear polarization waves: Cubic nonlinearity

As advocated in Sec. III B, cubic nonlinearities become important when $\delta \sim |\theta'|$ is small: Their contributions can therefore be calculated from the system (8) with $\delta = 0$. The detailed calculations are presented in Appendix C. After reintroducing the dispersive effects according to the procedure exposed in Sec. III A, the result (C5) takes the form of the following modified KdV equation:

$$\theta'_t + c_p^{(0)}\,\theta'_x - \frac{3\,c_p^{(0)}\,(9\,\alpha_1 - \alpha_2)}{8\,\alpha_2}\,(\theta')^2\,\theta'_x - \frac{1}{8\,c_p^{(0)}}\,\theta'_{xxx} = 0. \quad (27)$$

If δ is small and $|\theta'| \sim \delta$, we also have to take into account the quadratic nonlinearity of Eq. (26) and corrections of order $\mathcal{O}(\delta^2)$ to the velocity of the polarization sound, which finally yields

$$\theta'_t + \left(c_p^{(0)} - \frac{\rho_0\,\delta^2}{8\,c_p^{(0)}\,\alpha_2}\right)\theta'_x + \frac{9\,c_p^{(0)}\,\delta}{4\,\alpha_2}\,\theta'\,\theta'_x - \frac{3\,c_p^{(0)}\,(9\,\alpha_1 - \alpha_2)}{8\,\alpha_2}\,(\theta')^2\,\theta'_x - \frac{1}{8\,c_p^{(0)}}\,\theta'_{xxx} = 0. \quad (28)$$

This is the Gardner equation describing the evolution of nonlinear polarization pulses in a two-component condensate in the limit where the intraspecies interaction constants are close [32]. This can be considered as an "intermediate" region when the quadratic and cubic nonlinearities make contributions of the same order of magnitude in evolution of the wave. In the limit of very small δ, when the quadratic nonlinearity effects can be neglected, the nonlinear polarization waves are correctly described by the modified KdV equation [Eq. (27)]. If instead δ is large, then the cubic nonlinearity effects are negligible and the evolution of nonlinear polarization pulses is described by the KdV equation [Eq. (24)].

Once the solution of the Gardner equation [Eq. (28)] has been found, the other field variables can be expressed in terms of θ' by the formulas,

$$\begin{aligned}
\rho'(x,t) &= \frac{1}{2\,\alpha_2}\left[\rho_0\,\delta\,\theta'(x,t) - 3\left(c_p^{(0)}\right)^2(\theta')^2(x,t)\right],\\
U(x,t) &= \frac{c_p^{(0)}}{\alpha_2}\left[\delta\,\theta'(x,t) - \frac{1}{2}\,(3\,\alpha_1 + \alpha_2)\,(\theta')^2(x,t)\right], \quad (29)\\
v(x,t) &= 2\,c_p^{(0)}\,\theta'(x,t),
\end{aligned}$$

which follow from Eqs. (25) and (C3). It is worth noticing that although during the derivation of the Gardner equation [Eq. (28)] we assumed the boundary condition $\theta' \to 0$ as $|x| \to \infty$, this equation remains valid for the description of the evolution of waves with boundary conditions $\theta' \to \theta_{1,2}$ as $x \to \pm\infty$ provided the values $\theta_{1,2}$ are small enough ($|\theta_{1,2}| \sim \delta \ll 1$). As we shall see, this additional freedom makes it possible to obtain new types of solutions of the vector GP equation [Eq. (1)].

IV. WEAKLY NONLINEAR WAVES IN A TWO-COMPONENT BEC

The system (1) admits several solutions with different properties depending on the signs and values of α_1, α_2, and δ. The possible solutions can also depend on the background distributions of the condensate densities and velocities. Our boundary conditions $\rho_\pm \to \rho_0/2$ when $|x| \to \infty$ exclude solutions such as dark-bright solitons for which the bright-soliton component has a vanishing density at $|x| \to \infty$. These solutions have already been studied in the literature (see, e.g., Refs. [3] and [4]) and have been observed in experiments [33–35]. Other nonlinear coherent patterns—such as dark-dark solitons, for instance (see Ref. [36])—also exist in situations where the background density does not vanish at infinity; we shall now present several new structures belonging to this class of solutions. To start illustrating our approach by a simple example, we first present the well-known dark-dark density solitons described in the limit of shallow solitons by the KdV equation [Eq. (21)].

A. Density KdV solitons

The evolution of density waves is described by the KdV equation [Eq. (21)] and its well-known soliton solution is given in this case by the formula,

$$\rho'(x,t) = -2\,\rho_0\,\frac{\sqrt{\alpha_2^2+\delta^2}}{2\sqrt{\alpha_2^2+\delta^2}-\alpha_2}\left(1 - \frac{V_s}{c_d}\right) \times \frac{1}{\cosh^2[\sqrt{2\,c_d\,(c_d - V_s)}\,(x - V_s\,t - x_0)]}, \quad (30)$$

where c_d is defined by the first part of Eq. (13) and x_0 denotes the initial location of the soliton's center at $t = 0$. From Eq. (22) we can see that at $x \to \pm\infty$ the polarization vector $\mathbf{S}$ lies in the (S_x, S_y) plane; it rotates towards the "southern hemisphere" of the $\mathbf{S}$ space when x goes from $-\infty$ to $+\infty$, and its total rotation angle in the (S_x, S_y) plane is equal to

$$\Delta\phi = 2\sqrt{2}\,\frac{\sqrt{\alpha_2^2+\delta^2}\left(\sqrt{\alpha_2^2+\delta^2}-\alpha_2\right)}{\delta\left(2\sqrt{\alpha_2^2+\delta^2}-\alpha_2\right)}\sqrt{1-\frac{V_s}{c_d}}. \quad (31)$$

This angle goes to zero as $\Delta\phi \propto \sqrt{1 - V_s/c_d}$ in the limit $V_s \to c_d - 0$ and as $\Delta\phi \propto (\delta/\alpha_2)$ in the limit $\delta \to 0$. As we see, the angle θ is the same at both sides of the soliton which means that the left and right asymptotic ratios of the densities of the two BEC components are equal. However, their relative phase changes across the soliton and this phase shift should be observable in interference experiments.

If $\delta = 0$, the solution (30) reduces to the shallow dark-dark soliton solution of the vector GP equation found in Ref. [27]. If in addition $\alpha_1 = \alpha_2$, then (30) reproduces the shallow Manakov dark soliton [2]. It is important to note that for $\delta = 0$ the polarization variable θ remains constant and that in this case $v \equiv 0$, so that the polarization vector $\mathbf{S}$ does not vary [and $\Delta\phi = 0$ in (31)]. We shall call the solution (30) a *density soliton* even for $\delta \neq 0$, when the polarization vector rotates according to Eq. (31), since in the limit $\delta \to 0$ it is a pure density mode.

B. Polarization KdV solitons

We now consider the polarization KdV solitons whose dynamics is described by Eq. (24). In this case the soliton solution is given by

$$\theta'(x,t) = -2\,\frac{\delta\sqrt{\alpha_2^2+\delta^2}}{\alpha_2^2+2\,\delta^2-\alpha_2\sqrt{\alpha_2^2+\delta^2}}\left(1-\frac{V_s}{c_p}\right) \times \frac{1}{\cosh^2[\sqrt{2\,c_p\,(c_p-V_s)}\,(x-V_s\,t-x_0)]}, \quad (32)$$

where c_p is defined by the second part of Eq. (13). Substituting this expression into Eq. (25) yields the evolution of the associated variables $\rho'(x,t)$, $U(x,t)$, and $v(x,t)$. In the present case, when x goes from $-\infty$ to $+\infty$, the total rotation of the azimuth ϕ of the polarization vector $\mathbf{S}$ is given by

$$\Delta\phi = -2\sqrt{2}\,\frac{\delta\sqrt{\alpha_2^2+\delta^2}}{\alpha_2^2+2\,\delta^2-\alpha_2\sqrt{\alpha_2^2+\delta^2}}\sqrt{1-\frac{V_s}{c_p}}. \quad (33)$$

This angle goes to zero as $\Delta\phi \propto \sqrt{1-V_s/c_p}$ in the limit $V_s \to c_p - 0$, but it diverges as $\Delta\phi \propto (\alpha_2/\delta)$ when $\delta \to 0$. This is a first indication that the polarization KdV soliton ceases to exist in this limit, which demonstrates the drastic difference between the density and the polarization nonlinear modes. The study of the amplitude of the polarization soliton (32) leads to the same conclusion: In the case where $\delta \ll |\alpha_2|$, the condition of applicability of the small amplitude approximation ($|\theta'| \ll 1$) reads

$$1-\frac{V_s}{c_p} \ll \frac{\delta}{|\alpha_2|}. \quad (34)$$

An ever stricter limitation follows from the condition that the soliton amplitude should be much smaller than the coefficient of the quadratic term in (26) so that we can indeed neglect the cubic nonlinearity terms. This imposes

$$1-\frac{V_s}{c_p} \ll \left(\frac{\delta}{\alpha_2}\right)^2. \quad (35)$$

Thus, the domain of applicability of the KdV approximation for describing nonlinear polarization waves is extremely small when $\delta \ll |\alpha_2|$. In this case we must take the cubic nonlinearity into account. As shown in Sec. III C this corresponds to describing the dynamics of the nonlinear wave by means of the Gardner equation [Eq. (28)].

C. Polarization Gardner solitons

The solution of the Gardner equation depends on the signs and on the values of the nonlinear interaction constants α_1, α_2, and δ. In what follows we suppose that $\alpha_1 > 0$, $9\,\alpha_1 - \alpha_2 > 0$, α_2 can be either positive or negative, and the condensate components are ordered in such a way that $\delta > 0$. It is worth noticing that, if $\alpha_2 > 0$, the inequality $9\alpha_1 - \alpha_2 > 0$ is stronger than the condition (17) necessary to the modulational stability of a uniform condensate. If instead $\alpha_2 < 0$, then the condition $9\alpha_1 - \alpha_2 > 0$ is fulfilled automatically and we return to the inequality (17) as the applicability condition of the present theory.

In the case $\alpha_2 > 0$, the soliton solution of the Gardner equation [Eq. (28)] is given by the formula (see, e.g., Ref. [20]),

$$\theta'(x,t) = \frac{\theta_1\,\theta_2}{\theta_1-(\theta_1-\theta_2)\cosh^2\left[\left(2\,c_p^{(0)}\,V\right)^{1/2}(x-V_s\,t)\right]}, \quad (36)$$

where

$$c_p^{(0)} = \sqrt{\frac{\rho_0\,(\alpha_1-\alpha_2)}{2}}, \quad V_s = c_p^{(0)} - \frac{\rho_0\,\delta^2}{8\,c_p^{(0)}\,\alpha_2} - V, \quad (37)$$

and

$$\frac{\theta_{1,2}}{2} = \frac{3\,\delta \pm \sqrt{(3\,\delta)^2+4\,\alpha_2\,(9\,\alpha_1-\alpha_2)\,V/c_p^{(0)}}}{9\,\alpha_1-\alpha_2}, \quad (38)$$

where the subscripts "1" and "2" correspond to the upper and lower signs, respectively; V is a free parameter defining the velocity and other properties of the soliton solution [note that V measures the soliton velocity in the reference frame moving with the polarization sound velocity, which is equal to $c_p^{(0)} - \rho_0\,\delta^2/(8\,c_p^{(0)}\,\alpha_2)$]. Substituting Eq. (36) into Eq. (29) yields the density and the polarization distributions of the polarization Gardner soliton.

At variance with the case of the polarization KdV soliton presented in Sec. IV B, the soliton amplitude θ_1 remains here finite in the limit $\delta \to 0$,

$$\theta_1\Big|_{\delta=0} = 4\sqrt{\frac{\alpha_2}{9\,\alpha_1-\alpha_2}\,\frac{V}{c_p^{(0)}}}, \quad (39)$$

and is small for small enough V. The angle of rotation of the polarization vector across the soliton solution is given by the expression,

$$\Delta\phi = 8\sqrt{\frac{2\,\alpha_2}{9\,\alpha_1-\alpha_2}}\arctan\sqrt{-\frac{\theta_1}{\theta_2}}, \quad (40)$$

which is finite for all δ and V.

The densities of each component of the soliton solution can be found from Eq. (3). Their behavior is represented in Fig. 1 together with the exact numerical solution of the vector GP equation. Analogous plots for the flow velocities of the two components are shown in Fig. 2.

If the parameter V satisfies the inequality $\rho_0\,\delta^2/(8\,c_p^{(0)}\,\alpha_2) \ll V$, then the soliton velocity can be approximated as $V_s \cong c_p^{(0)} - V$. If in addition we have $\rho_0\delta/(8c_p^{(0)}) \ll V \ll c_p^{(0)}$, then we can neglect the corrections of order $\sim\delta$ in the expressions for $\rho_\pm$ and substitute there the solution (36) with θ_1 equal to (39). Then, in particular, the densities at the center of each component of the soliton read

$$\rho_+^{\min} = \frac{\rho_0}{2}\left[1-4\sqrt{\frac{\alpha_2}{9\,\alpha_1-\alpha_2}}\sqrt{1-\frac{V_s}{c_p^{(0)}}} - \frac{12\,(\alpha_1-\alpha_2)}{9\,\alpha_1-\alpha_2}\left(1-\frac{V_s}{c_p^{(0)}}\right)\right], \quad (41)$$

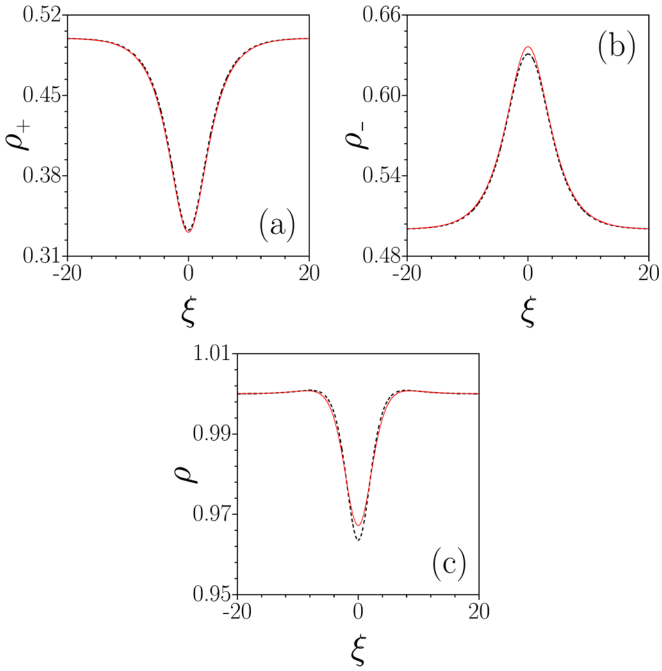

FIG. 1. (Color online) Dependence of (a) ρ_+, (b) ρ_-, and of (c) the total density ρ on $\xi = x - V_s t$ for the polarization Gardner soliton. The nonlinear interaction parameters are $\alpha_1 = 1$, $\alpha_2 = 0.6$, and $\delta = 0.05$, and the velocity parameter is $V = 0.03$ (in this case, the soliton velocity is $V_s \simeq 0.416$). The analytical results obtained within the Gardner description are shown by dashed black lines and the exact numerical results by red lines. All quantities are plotted in dimensionless units.

and

$$\rho_-^{\max} = \frac{\rho_0}{2}\left[1 + 4\sqrt{\frac{\alpha_2}{9\,\alpha_1 - \alpha_2}}\sqrt{1 - \frac{V_s}{c_p^{(0)}}} - \frac{12\,(\alpha_1 - \alpha_2)}{9\,\alpha_1 - \alpha_2}\left(1 - \frac{V_s}{c_p^{(0)}}\right)\right]. \qquad (42)$$

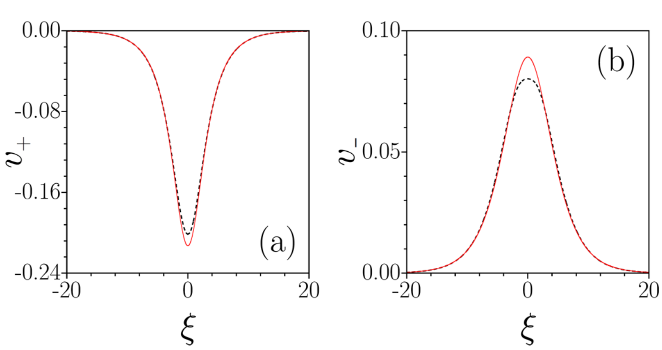

FIG. 2. (Color online) Dependence of (a) $v_+ = (U - v)/2$ and (b) $v_- = (U + v)/2$ on $\xi = x - V_s t$ for the polarization Gardner soliton. The curves are drawn for the same choice of parameters as in Fig. 1. Here also the analytical results obtained within the Gardner description are shown by dashed black lines and the exact numerical results by red lines. All quantities are plotted in dimensionless units.

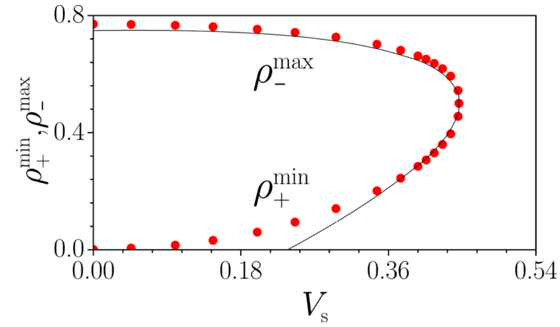

FIG. 3. (Color online) Densities $\rho_+^{\min}$ and $\rho_-^{\max}$ of each component at the center of the soliton as functions of the soliton velocity V_s. The nonlinear interaction parameters are the same as in Figs. 1 and 2: $\alpha_1 = 1$, $\alpha_2 = 0.6$, and $\delta = 0.05$. The analytical results (41) and (42) obtained within the Gardner description are shown by a black line and the exact numerical results by red dots. All quantities are plotted in dimensionless units.

For V_s close enough to $c_p^{(0)}$, these results are in good agreement with those obtained from the exact numerical integration of the vector GP equation [Eq. (1)] (see Fig. 3).

When $\alpha_2 < 0$, formula (36) for the soliton solution remains unchanged, but now in (38) the subscripts "1" and "2" correspond to the lower and upper signs, respectively, and α_2 should be replaced by $|\alpha_2|$ in the numerator of the expressions under the square roots in Eqs. (38)–(42).

D. Polarization algebraic solitons

In the examples of soliton solutions presented above, we have assumed for definiteness that both nonlinear constants were positive, i.e., $\alpha_1 > 0$ and $\alpha_2 > 0$, but the results could easily be extended to situations with $\alpha_1 > 0$ and $\alpha_2 < 0$, provided the conditions of modulation stability were not violated. We now consider another type of soliton solution of the Gardner equation, the so-called *algebraic soliton* (see, e.g., Ref. [20]) which only exists in the case where $\alpha_1 > 0$ and $\alpha_2 < 0$. The corresponding solution of Eq. (28) is given by

$$\theta'(x,t) = \theta_2 + \frac{\theta_1 - \theta_2}{1 + \frac{(c_p^{(0)})^2\,(9\,\alpha_1 - \alpha_2)}{8\,\alpha_2}\,(\theta_1 - \theta_2)^2\,(x - V_s\,t - x_0)}, \qquad (43)$$

where θ_2 is a free (nonzero) background density parameter which characterizes the solution. The other parameters are defined by

$$\theta_1 = \frac{12\,\delta}{9\,\alpha_1 - \alpha_2} - 3\,\theta_2, \qquad (44)$$

and

$$V_s = c_p^{(0)} - \frac{\delta^2}{8\,c_p^{(0)}\,\alpha_2} + \frac{9\,c_p^{(0)}\,\delta}{8\,\alpha_2} - \frac{3\,c_p^{(0)}\,(9\,\alpha_1 - \alpha_2)}{8\,\alpha_2}\,\theta_2^2. \qquad (45)$$

In this case, when the value of the background angle θ_2 is of order δ, the soliton velocity departs from the sound velocity by a value which is also of order δ, and, hence, the amplitude of the soliton is also of the same order of magnitude. Thus, the solution corresponds to a very small-amplitude and very wide soliton.

E. Kink solutions

Besides the soliton solutions, which are similar in many respects to the KdV solitons, the Gardner equation accommodates other types of solutions which are absent in the KdV description. In the present subsection we shall consider the kink solution, sometimes called "solibore" (see, e.g., Refs. [37] and [20]). It only exists in the case where $\alpha_1 > 0$ and $\alpha_2 < 0$ and if the background flow is modulationally stable. In this subsection we shall assume that these conditions are met. In terms of the solution of the Gardner equation [Eq. (28)] the kink relates flows with different values of the angle θ at left and right infinities:

$$\theta' \to \begin{cases} \theta_1 & (x \to +\infty), \\ \theta_2 & (x \to -\infty). \end{cases} \tag{46}$$

This replaces the boundary condition $\theta \to \pi/2$ presumed earlier, but the Gardner equation is still applicable provided the quantities $\theta_{1,2} - \pi/2$ are small enough. In the kink solution the asymptotic values θ_1 and θ_2 are related through

$$\theta_1 + \theta_2 = \frac{3\, c_p^{(0)}\, \delta}{8\, |\alpha_2|}. \tag{47}$$

For definiteness we suppose that $\theta_1 > \theta_2$; then the kink solution of the Gardner equation can be written as

$$\theta'(x,t) = \theta_1 - \frac{\theta_1 - \theta_2}{1 + \exp\left[\sqrt{\frac{9\alpha_1 - \alpha_2}{2|\alpha_2|}}\, (\theta_1 - \theta_2)\, (x - V_s\, t - x_0)\right]}, \tag{48}$$

where the kink velocity is equal to

$$V_s = c_p^{(0)} - \frac{\rho_0\, \delta^2}{8\, c_p^{(0)}\, \alpha_2} + \frac{c_p^{(0)}\, (9\,\alpha_1 - \alpha_2)}{16\, \alpha_2} \times \left[\left(\frac{6\,\delta}{9\,\alpha_1 - \alpha_2} + \theta_1\right)^2 - 3\,\theta_1^2\right]. \tag{49}$$

This solution is parametrized by the value of the angle θ_1 at right infinity ($x \to +\infty$); then θ_2 and V_s are defined by Eqs. (47) and (49). The explicit expression of the dynamical variables $\rho'(x,t)$, $U(x,t)$, and $\phi(x,t)$ can be found by substitution of Eq. (48) into Eq. (29). We have illustrated the behavior of the kink solution by the plots of Fig. 4. As one can see, this solution represents a two-fluid flow in which one component partially replaces the other: upstream the kink (on the left side) the density ρ_+ is greater than ρ_- and downstream the opposite situation occurs. In terms of the total density, the kink solution looks like an asymmetric dark soliton moving with velocity V_s. It is important to notice that the velocities of the two components are different and do not vanish at $x \to \pm\infty$. This means that there is a finite flux of one component into the region occupied by the other and the kink soliton describes the wave at the boundary between the two regions. In this sense, the kink soliton can be considered as a dispersive analog of a shock wave.

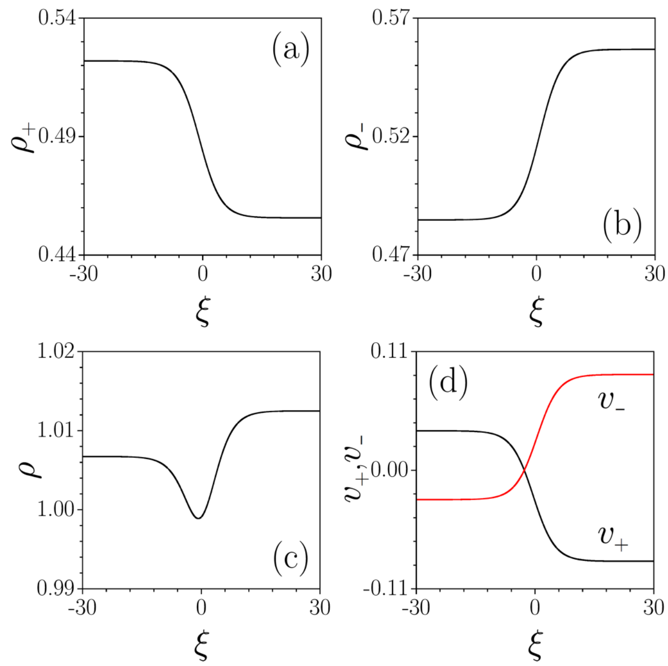

FIG. 4. (Color online) Densities (a) ρ_+, (b) ρ_-, (c) total density ρ, and (d) velocities $v_\pm = (U \mp v)/2$ of the two components as functions of $\xi = x - V_s\, t$. The nonlinear interaction parameters are $\alpha_1 = 1$, $\alpha_2 = -0.6$, and $\delta = 0.05$. The parameter θ_1 is equal to 0.05 and the corresponding kink velocity is $V_s \simeq 0.896$. The exact numerical solutions of the vector GP equation are not shown here because they cannot be distinguished from the analytical results derived within the Gardner description. All quantities are plotted in dimensionless units.

F. Breather solution

As a last example of the new wave structures supported by the vector GP equation, we shall present in this subsection the breather solution found in Refs. [38,39], which appears as a consequence of the corresponding solution of the Gardner equation. In our notations it only exists if $\alpha_2 > 0$ and can be written in the form,

$$\theta'(x,t) = -\frac{2}{c_p^{(0)}} \sqrt{\frac{2\,\alpha_2}{9\,\alpha_1 - \alpha_2}} \times \frac{\partial}{\partial x} \arctan \frac{\kappa\, \cosh p\, \cos \Theta_b - k\, \cos q\, \sinh \Phi_b}{\kappa\, \sinh p\, \sin \Theta_b + k\, \sin q\, \cosh \Phi_b}, \tag{50}$$

where k and κ are "wave numbers" of an envelope and a carrier wave, correspondingly:

$$k = \frac{3\, c_p^{(0)}\, \delta}{\sqrt{2\,\alpha_2\, (9\,\alpha_1 - \alpha_2)}} \times \frac{\sinh(2\,p)}{\cos^2 q\, \cosh^2 p + \sin^2 q\, \sinh^2 p}, \tag{51}$$

and

$$\kappa = \frac{3\, c_p^{(0)}\, \delta}{\sqrt{2\,\alpha_2\, (9\,\alpha_1 - \alpha_2)}} \times \frac{\sin(2\,q)}{\cos^2 q\, \cosh^2 p + \sin^2 q\, \sinh^2 p}, \tag{52}$$

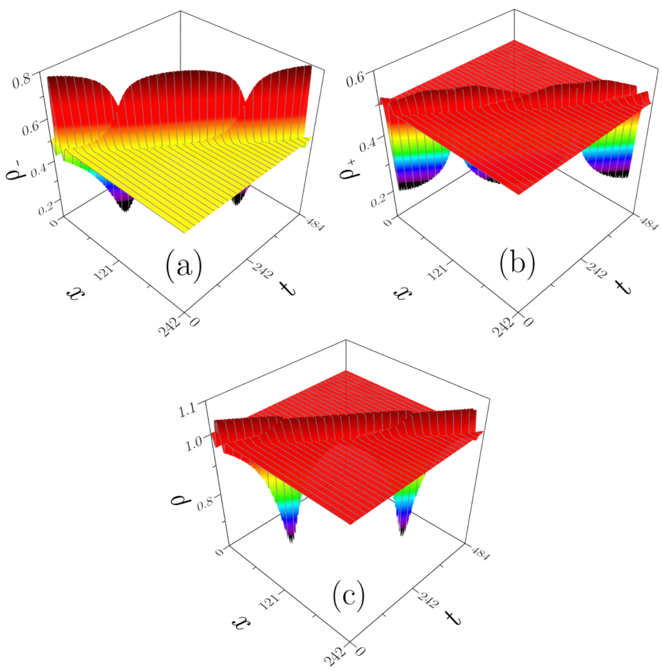

FIG. 5. (Color online) Plots of (a) ρ_+, (b) ρ_-, and of (c) the total density ρ as functions of x and t for a breather solution. The nonlinear interaction parameters are $\alpha_1 = 1$, $\alpha_2 = 0.6$, and $\delta = 0.4$. All quantities are plotted in dimensionless units.

p and q being free parameters. The velocities of the envelope and the carrier wave are given by the expressions,

$$\begin{aligned} V_b &= c_p^{(0)} - \frac{\rho_0\,\delta^2}{8\,c_p^{(0)}\,\alpha_2} - \frac{3\,\kappa^2 - k^2}{8\,c_p^{(0)}}, \\ V_i &= c_p^{(0)} - \frac{\rho_0\,\delta^2}{8\,c_p^{(0)}\,\alpha_2} - \frac{\kappa^2 - 3\,k^2}{8\,c_p^{(0)}}, \end{aligned} \tag{53}$$

while the phases are defined as

$$\Theta_b = k\,(x - V_b\,t) + \Theta_0, \quad \Phi_b = \kappa\,(x - V_i\,t) + \Phi_0. \tag{54}$$

Substitution of Eq. (50) into Eqs. (29) and (3) yields the densities of the two components of the breather solution of the vector GP equation within the Gardner approximation. Their behavior is illustrated in Fig. 5 and it corresponds to the nonstationary propagation of a nonlinear wave packet with an envelope of velocity V_b.

V. CONCLUSION AND DISCUSSION

In this paper, we have shown that there exists a region of values of the nonlinear constants for which the vector GP equation can be reduced to a single nonlinear evolution equation, the so-called Gardner equation. This equation describes the evolution of nonlinear polarization excitations which, in the linear limit, reduce to polarization sound waves which are disturbances that do not modify the total density of the condensate.

The Gardner equation accommodates new types of nonlinear waves (such as algebraic solitons, kinks, breathers) which have not yet been observed in two-component condensates. Since this equation is obtained as a weakly nonlinear and weakly dispersive approximation of the vector GP equation, it was important in the present work to verify if the new waves we have identified persist in the exact GP scheme. We performed this check numerically (cf. Figs. 1–4) and, indeed, the solutions of the Gardner equation remain stable with respect to decay to more elementary excitations when their dynamics is governed by the vector GP equation. In other words, although at the moment an analytic description of the wave pattern only exists in the small amplitude limit (i.e., in the Gardner scheme), similar large-amplitude wave patterns can be described as exact numerical solutions of the vector GP equation.

The Gardner equation is obtained in the limit where the difference δ between the intraspecies nonlinear interaction constants is small compared to the interspecies nonlinear interaction constant α_2. This limit is certainly reached in polaritonic condensates since in these systems $\delta = 0$. In this field α_1 is positive and one generally assumes that α_2 is negative, typically of order of $-\alpha_1/10$. However, the value of α_2 depends on the detuning between the photon and the exciton modes and may be positive, as demonstrated in Refs. [40–43]. One can thus hope to observe in exciton-polariton systems all the types of waves presented above: polarization algebraic solitons and breathers ($\alpha_2 > 0$), and also kinks ($\alpha_2 < 0$).

In the case of ultracold atomic vapors, if one considers, for instance, a spinor condensate of ^{87}Rb, the parameters α_1 and α_2 are both positive, with $\delta/\alpha_2 \simeq 2.76 \times 10^{-2}$ in the case where the spinor is formed by two hyperfine states $|F,m_F\rangle = |1,\mp 1\rangle$ and $|2,\pm 1\rangle$ (such as those studied in Refs. [44,45]) and $\delta/\alpha_2 \simeq 7.17 \times 10^{-3}$ in the case where the spinor is formed by the two hyperfine states $|1,-1\rangle$ and $|2,-2\rangle$ (such as those studied in Ref. [35]). Here again we are in a range of parameters where the requirement $\delta/\alpha_2 \ll 1$ is met. It is also important to fulfill the condition (17) of modulational stability. In the above cases the ratio $\alpha_1^2/(\alpha_2^2 + \delta^2)$ is, respectively, $\simeq$1.00005 and $\simeq$1.04. These values are barely larger than unity: Hence, although the criterion (17) is fulfilled, the hypothesis of weak dispersion is questionable. One can circumvent (or deliberately increase) this problem by tuning the interspecies scattering length by $\pm 10\%$ as demonstrated in Ref. [45] (see also Ref. [24] where the ^{85}Rb scattering length is modulated over a wide range of values in a ^{85}Rb-^{87}Rb mixture). In the case where the two components of the condensate consist in two isotopes of the same element with a small mass difference δm, then, for observing the effects of a difference δ in the intraspecies interactions, the condition $|\delta m/m| \ll |\delta/\alpha_2|$ should be satisfied. Since typically $|\delta m/m| \sim 0.01$, these effects are also observable in principle even in this type of two-component condensates. Hence, as polaritonic condensates, atomic condensates offer interesting prospects for observing the new nonlinear excitations proposed in the present work.

We expect that these new excitations can be experimentally generated by phase and density engineering methods or by the flow of a two-component condensate past an obstacle, as proposed in Ref. [46].

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APPENDIX A: NONLINEAR DENSITY WAVE

In the appendices we present some details of the derivation of the approximate evolution equations in the framework of the nonlinear perturbation theory. To simplify the presentation, we omit the dispersive effects at this stage (they are accounted for in the main text).

Expanding the system (8) up to second order in the variables $\rho' = \rho - \rho_0, U, \theta' = \theta - \theta_0, v$ and omitting the x derivatives of order higher than unity (dispersionless limit) yields

$$\begin{aligned}
&\rho'_t + \frac{\rho_0}{2}\,U_x + \frac{\rho_0}{2}\,(\theta'\,v)_x + \frac{1}{2}\,(\rho'\,U)_x = 0,\\
&U_t + (\alpha_1+\alpha_2)\,\rho'_x + \frac{1}{2}\,(U\,U_x + v\,v_x)\\
&\qquad -\rho_0\,\delta\,\theta'_x - \delta\,(\rho'\,\theta')_x = 0,\\
&\theta'_t + \frac{1}{2}\,v_x + \frac{1}{2\,\rho_0}\,v\,\rho'_x + \frac{1}{2}\,U\,\theta'_x = 0,\\
&v_t + \rho_0\,(\alpha_1-\alpha_2)\,\theta'_x + \frac{1}{2}\,(U\,v)_x\\
&\qquad +(\alpha_1-\alpha_2)\,(\rho'\,\theta')_x - \delta\,\rho'_x = 0.
\end{aligned} \tag{A1}$$

Within the standard perturbation theory, considering nonlinear density waves propagating in the positive-x direction, one introduces the stretched variables (see, e.g., Ref. [47]),

$$\xi = \epsilon^{1/2}\,(x - c_d\,t), \quad \tau = \epsilon^{3/2}\,t, \tag{A2}$$

and expands the field variables ρ', U, θ', and v in powers of ϵ:

$$\begin{pmatrix}\rho'\\ U\\ \theta'\\ v\end{pmatrix} = \epsilon\begin{pmatrix}\rho^{(1)}\\ U^{(1)}\\ \theta^{(1)}\\ v^{(1)}\end{pmatrix} + \epsilon^2\begin{pmatrix}\rho^{(2)}\\ U^{(2)}\\ \theta^{(2)}\\ v^{(2)}\end{pmatrix} + \cdots . \tag{A3}$$

The corresponding expansion of Eq. (A1) yields, at leading order, the consistent system of equations,

$$\begin{aligned}
-\,c_d\,\rho^{(1)}_\xi + \frac{\rho_0}{2}\,U^{(1)}_\xi &= 0,\\
-c_d\,U^{(1)}_\xi + (\alpha_1+\alpha_2)\,\rho^{(1)}_\xi - \rho_0\,\delta\,\theta^{(1)}_\xi &= 0,\\
-c_d\,\theta^{(1)}_\xi + \tfrac{1}{2}\,v^{(1)}_\xi &= 0,\\
-c_d\,v^{(1)}_\xi + \rho_0\,(\alpha_1-\alpha_2)\,\theta^{(1)}_\xi - \delta\,\rho^{(1)}_\xi &= 0.
\end{aligned} \tag{A4}$$

The determinant of this linear system is zero, and all variables can thus be expressed in terms of one of them, $\rho^{(1)}$, for instance,

$$\begin{aligned}
&U^{(1)} = \frac{2\,c_d}{\rho_0}\,\rho^{(1)}, \quad \theta^{(1)} = \frac{\alpha_2 - \sqrt{\alpha_2^2+\delta^2}}{\rho_0\,\delta}\,\rho^{(1)},\\
&v^{(1)} = \frac{2\,c_d\,\left(\alpha_2 - \sqrt{\alpha_2^2+\delta^2}\right)}{\rho_0\,\delta}\,\rho^{(1)}.
\end{aligned} \tag{A5}$$

These are the relations actually realized in the case of a linear density wave.

At next order in ϵ we get

$$\begin{aligned}
&-c_d\,\rho^{(2)}_\xi + \frac{\rho_0}{2}\,U^{(2)}_\xi\\
&\quad = -\rho^{(1)}_\tau - \frac{\rho_0}{2}\,(\theta^{(1)}\,v^{(1)})_\xi - \frac{1}{2}\,(\rho^{(1)}\,U^{(1)})_\xi,\\
&-c_d\,U^{(2)}_\xi + (\alpha_1+\alpha_2)\,\rho^{(2)}_\xi - \rho_0\,\delta\,\theta^{(2)}_\xi\\
&\quad = -U^{(1)}_\tau - \frac{1}{2}\left(U^{(1)}\,U^{(1)}_\xi + v^{(1)}\,v^{(1)}_\xi\right) + \delta\,(\rho^{(1)}\,\theta^{(1)})_\xi,\\
&-c_d\,\theta^{(2)}_\xi + \frac{1}{2}\,v^{(2)}_\xi\\
&\quad = -\theta^{(1)}_\tau - \frac{1}{2\,\rho_0}\,v^{(1)}\,\rho^{(1)}_\xi - \frac{1}{2}\,U^{(1)}\,\theta^{(1)}_\xi,\\
&-c_d\,v^{(2)}_\xi + \rho_0\,(\alpha_1+\alpha_2)\,\theta^{(2)}_\xi - \delta\,\rho^{(2)}_\xi\\
&\quad = -v^{(1)}_\tau - \frac{1}{2}\,(U^{(1)}\,v^{(1)})_\xi - (\alpha_1-\alpha_2)\,(\rho^{(1)}\,\theta^{(1)})_\xi.
\end{aligned} \tag{A6}$$

As was already observed at order $\mathcal{O}(\epsilon)$ [compare with Eq. (A4)], the left-hand side of the system (A6) has a vanishing determinant; hence the expressions at this side are linearly dependent. Therefore, the expressions in the right-hand side must also be linearly dependent with the same proportionality coefficients [which are the same as the ones already involved at the order $\mathcal{O}(\epsilon)$ and explicitly written in Eq. (A5)]. This condition yields the evolution equation,

$$\rho^{(1)}_\tau + \frac{3\,c_d\,\left(2\sqrt{\alpha_2^2+\delta^2} - \alpha_2\right)}{2\,\rho_0\sqrt{\alpha_2^2+\delta^2}}\,\rho^{(1)}\,\rho^{(1)}_\xi = 0. \tag{A7}$$

Returning to the original variables x and t, and writing $\rho' \simeq \epsilon\,\rho^{(1)}$ (which corresponds to the required level of accuracy) one obtains

$$\rho'_t + c_d\,\rho'_x + \frac{3\,c_d\,\left(2\sqrt{\alpha_2^2+\delta^2} - \alpha_2\right)}{2\,\rho_0\sqrt{\alpha_2^2+\delta^2}}\,\rho'\,\rho'_x = 0. \tag{A8}$$

APPENDIX B: NONLINEAR POLARIZATION WAVE: QUADRATIC NONLINEARITY

The treatment of nonlinear polarization waves taking into account nonlinear terms at quadratic order is similar to the one presented in Appendix A. We introduce here the stretched variables,

$$\xi = \epsilon^{1/2}\,(x - c_p\,t), \quad \tau = \epsilon^{3/2}\,t, \tag{B1}$$

and make use of the series expansions (A3). The corresponding expansion of Eq. (A1) yields at first order a system of equations similar to the system (A4) with c_d replaced by c_p, which in turn yields

$$\begin{aligned}
&\rho^{(1)} = \frac{\rho_0\,\delta}{\alpha_2 + \sqrt{\alpha_2^2+\delta^2}}\theta^{(1)}, \quad v^{(1)} = 2\,c_p\,\theta^{(1)},\\
&U^{(1)} = \frac{2\,c_p\,\delta}{\alpha_2 + \sqrt{\alpha_2^2+\delta^2}}\,\theta^{(1)}.
\end{aligned} \tag{B2}$$

These expressions are equivalent to Eq. (A5)—with c_d replaced by c_p—but it is now more convenient to express all variables in terms of $\theta^{(1)}$. Tedious calculations at second order, similar to those for the density wave, lead to the equation,

$$\theta^{(1)}_\tau + \frac{3\,c_p\left(\alpha_2^2 + 2\,\delta^2 - \alpha_2\sqrt{\alpha_2^2+\delta^2}\right)}{2\,\delta\sqrt{\alpha_2^2+\delta^2}}\,\theta^{(1)}\,\theta^{(1)}_\xi = 0. \quad \text{(B3)}$$

Returning to the original variables x and t, and writing $\theta' \simeq \epsilon\,\theta^{(1)}$ (which corresponds to the required level of accuracy) one obtains

$$\theta'_t + c_p\theta'_x + \frac{3\,c_p\left(\alpha_2^2 + 2\,\delta^2 - \alpha_2\sqrt{\alpha_2^2+\delta^2}\right)}{2\,\delta\sqrt{\alpha_2^2+\delta^2}}\,\theta'\,\theta'_x = 0. \quad \text{(B4)}$$

APPENDIX C: NONLINEAR POLARIZATION WAVE: CUBIC NONLINEARITY

For describing the polarization waves up to the third nonlinear order one follows a procedure different from the one presented in Appendixes A and B. A series expansion of the system (8) up to third order in the small variables ρ', U, θ', and v and up to first order in the derivatives of these quantities (we again postpone the inclusion of the dispersive effects in order to simplify the presentation) yields

$$\begin{aligned}
&\rho'_t + \frac{\rho_0}{2}\,U_x + \frac{\rho_0}{2}\,(\theta'\,v)_x + \frac{1}{2}\,(\rho'\,U)_x + \frac{1}{2}\,(\rho'\,\theta'\,v)_x = 0,\\
&U_t + (\alpha_1+\alpha_2)\,\rho'_x + \frac{1}{2}\,(U\,U_x + v\,v_x)\\
&\quad + \frac{1}{2\,\rho_0}\,\rho'_x\,(\theta'_x)^2 - \frac{(\rho'_x)^3}{2\,\rho_0^3} = 0,\\
&\theta'_t + \frac{1}{2}\,v_x + \frac{1}{2\,\rho_0}\,v\,\rho'_x + \frac{1}{2}\,U\,\theta'_x\\
&\quad - \frac{1}{2\,\rho_0^2}\,v\,\rho'\,\rho'_x - \frac{1}{2}\,v\,\theta'\,\theta'_x - \frac{1}{4}\,(\theta')^2\,v_x = 0,\\
&v_t + \rho_0\,(\alpha_1-\alpha_2)\,\theta'_x + \frac{1}{2}\,(U\,v)_x + (\alpha_1-\alpha_2)\,(\rho'\,\theta')_x\\
&\quad + \frac{1}{2\,\rho_0^2}\,(\rho'_x)^2\,\theta'_x - \frac{\rho_0}{2}\,(\alpha_1-\alpha_2)\,(\theta')^2\,\theta'_x = 0.
\end{aligned}$$

We now introduce the stretched variables,

$$\xi = \epsilon^{1/2}\left(x - c_p^{(0)}\,t\right), \qquad \tau = \epsilon^{5/2}\,t, \quad \text{(C1)}$$

with $c_p^{(0)} = \{\rho_0(\alpha_1-\alpha_2)/2\}^{1/2}$, and use the series expansions (A3) up to the third order in ϵ.

At order $\mathcal{O}(\epsilon)$ we get

$$\rho^{(1)} = 0, \quad U^{(1)} = 0, \quad \text{and} \quad v^{(1)} = 2\,c_p^{(0)}\,\theta^{(1)}. \quad \text{(C2)}$$

At next order we obtain the algebraic relations,

$$\begin{aligned}
\rho^{(2)} &= -\frac{3\left(c_p^{(0)}\right)^2}{2\,\alpha_2}\,(\theta^{(1)})^2, \quad v^{(2)} = 2\,c_p^{(0)}\,\theta^{(2)},\\
U^{(2)} &= -\frac{c_p^{(0)}\,(3\,\alpha_1+\alpha_2)}{2\,\alpha_2}\,(\theta^{(1)})^2.
\end{aligned} \quad \text{(C3)}$$

Finally, at order $\mathcal{O}(\epsilon^3)$, it is enough to consider only the equations verified by $\theta^{(3)}$ and $v^{(3)}$:

$$\begin{aligned}
&-c_p^{(0)}\,\theta^{(3)}_\xi + \frac{1}{2}\,v^{(3)}_\xi\\
&\quad = -\theta^{(1)}_\tau - \frac{1}{2\,\rho_0}\,\rho^{(2)}_\xi - \frac{1}{2}\,U^{(2)}\,\theta^{(1)}_\xi\\
&\qquad + \frac{1}{2}\,v^{(1)}\,\theta^{(1)}\,\theta^{(1)}_\xi + \frac{1}{4}\,(\theta^{(1)})^2\,v^{(1)}_\xi,\\
&-c_p^{(0)}\,v^{(3)}_\xi + \rho_0\,(\alpha_1-\alpha_2)\,\theta^{(3)}_\xi\\
&\quad = -v^{(1)}_\tau - \frac{1}{2}\,(U^{(2)}\,v^{(1)})_\xi - (\alpha_1-\alpha_2)\,(\theta^{(1)}\,\rho^{(2)})_\xi\\
&\qquad + \frac{\rho_0}{2}\,(\alpha_1-\alpha_2)\,(\theta^{(1)})^2\,\theta^{(1)}_\xi.
\end{aligned}$$

Again, the expressions in the left-hand side are linearly dependent and the compatibility condition for this system yields the evolution equation which, with account of Eqs. (C2) and (C3), can be written as

$$\theta^{(1)}_\tau - \frac{3\,c_p^{(0)}\,(9\,\alpha_1-\alpha_2)}{8\,\alpha_2}\,(\theta^{(1)})^2\,\theta^{(1)}_\xi = 0. \quad \text{(C4)}$$

Returning to the original variables x and t, and writing $\theta' \simeq \epsilon\,\theta^{(1)}$ (which corresponds to the required level of accuracy) one obtains

$$\theta'_t + c_p^{(0)}\,\theta'_x - \frac{3\,c_p^{(0)}\,(9\,\alpha_1-\alpha_2)}{8\,\alpha_2}\,(\theta')^2\,\theta'_x = 0. \quad \text{(C5)}$$

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